

THE PROBLEM OF TIME AND QUANTUM COSMOLOGY IN THE RELATIONAL PARTICLE MECHANICS ARENA

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Abstract

Most readers' principal interest in this article will be its reviews of the problem of time in quantum GR in Secs 11 and 16. Namely, that 'time' in GR and 'time' in ordinary Quantum Theory are mutually incompatible notions, which is problematic in trying to put these two theories together to form a theory of Quantum Gravity. This article also establishes relational particle models (RPM's) as useful models for many parts of the study of the problem of time and of quantum cosmology. It also unlocks RPM's, expositing how to understand the configuration spaces, classical dynamics and quantum mechanics of these in simple concrete examples. It then uses these to further our understanding of the histories, records, semiclassical, naïve Schrödinger interpretation and hidden time strategies toward resolving the Problem of Time, alongside consideration of Halliwell's combination of the first three. This article's relational whole-universe models are comparable to minisuperspace in amount of resemblance to full GR, but with a number of different resemblances including various midisuperspace-like ones which subsequently render RPM's appropriate as Problem of Time models. One point of view is that the best one can currently do as regards quantum gravity is to compare multiple such toy models, each of which allows for a different range of calculations that are too difficult to carry out for GR to be substantially completed. The current article is then the RPM counterpart of Ryan's book on minisuperspace models or Carlip's on $2 + 1$ GR, as regards each of these arenas being rendered open to detailed study. Other results covered include suitable variational techniques for relational physics, a novel critique of operator-ordering in quantum cosmology, reduced versus Dirac quantization, and qualitative issues concerning structure formation, uniform states and closed-universe effects in quantum cosmology. Finally, this article contributes to the debate of what is relationalism and background independence, which has remained of interest in theoretical physics from Newton versus Leibniz through to foundational issues in today's leading candidates of quantum gravity.

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Discussions welcome. It is my intent to update this every year or two in the manner of a 'Living Review'.

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Preface: this article as a Quantum Gravity program

This article builds up a Quantum Gravity program. I view Quantum Gravity as piecing together General Relativity (GR) and ordinary Quantum Theory. How to do this is far from simple and clear as these two individually successful branches of physics rest on what at least appear to be distinct and incompatible foundations. GR, as well as its aspect as a relativistic theory of gravity that renders Newtonian gravity compatible with relativity, has a second aspect as a freeing from absolute/background structure. Following on from this double nature, I argue that “Quantum Gravity” is a misnomer since only making reference to the former. I.e. a truer name would then be something like “Quantum theory of the relativistic gravitation–background independence gestalt”, which I abbreviate by QG with the G standing for ‘gestalt’ rather than just for ‘gravity’. The quantum relational particle mechanics (RPM) = ‘quantum background independence without SR’ (special relativity) models of the present article are then GR-like as regards this second aspect. I.e. a valid, albeit hitherto less considered alternative to other partial piecings together of theoretical physics programs such as the quantum relativistic theory not of gravity (alias QFT: Quantum Field Theory) [512, 627], or quantum nonrelativistic theory of Newtonian Gravity without background independence (see e.g. [302, 169]). There is value in multiple partial perspectives when the total perspective is intractably hard in the above sense; that is why to use of multiple toy models (see below). The current article is then a first review of how the partial features of QG that are present in RPM’s are of comparable value to the full problem as the partial features of, say, GR, QFT and Quantum Newtonian Gravity are.

In the present article, I make the following choices as compared to various other ventures in the literature.

- a) I choose a background independent/canonical approach [201, 400, 335] that pays careful attention to GR features rather than a ‘QM first’ covariant or perturbative strings approach [202, 272, 273, 121].
- b) I choose the simpler non-supersymmetric case. I do not necessarily choose the dimension of space, d , to be 3, though I want the theory to have the usual kind of degrees of freedom, so I require $d \geq 3$. 3- d space is then the simplest case that could possibly correspond to reality, and, as such, is in some ways preferable, with the capacity to extend such a scheme to $d \geq 4$ as an occasionally relevant bonus rather than something to specifically aim for or generalize to).
- c) For the moment, I choose an approach that rather more closely parallels geometrodynamics than the Ashtekar/loop quantum gravity (LQG) approaches. I make this choice partly because i) geometrodynamics has particularly solid conceptual foundations; ii) this article does, among other things, consider the possibility of making do without canonical transformations and phase space enlargements, particularly in programs intended to involve quantization. iii) The Problem of Time, which is the principal interest in this article, has been studied in greater breadth and depth in this setting (in part simply because the geometrodynamical setting has been around for longer). Moreover, the meaning of ‘relational’ and the study of background independence in physics as studied in this article will also be of interest to many practitioners of LQG, as well as being relevant toward a broad understanding of some of the features which M-theory (but not perturbative string theory) would be expected to have. Additionally, this article’s principal theme of the Problem of Time and an understanding of its various possible resolutions is an important one for both of these theories (and any other which purports to require background-independence). Finally, the above principal connections with other QG programs by no means exhausts the totality of such connections made in this article.
- d) I proceed via considering a toy model for geometrodynamics (it is less close to Ashtekar variables; other theories such as electromagnetism, Yang–Mills theory [246], the Husain–Kuchař model [330] and BF theory [246, 472, 586], serve as toy models for this). I have particular interest in midisuperspace-like aspects (i.e. features beyond minisuperspace, but within simpler models than actual midisuperspace: structure formation and nontrivial linear constraints), and in certain approaches to the Problem of Time (see below).
- e) This toy model is relational particle mechanics (RPM); this builds on how it has been shown [94, 21] that GR is relational in a Leibniz–Mach–Barbour sense (see [92, 79, 21] and the present article), which GR itself has also been shown to obey [94, 21]. I.e. RPM’s are background-independent in a similar (but simpler) way to how GR is.
- f) Within the possible RPM’s, I go for the simplest, so as to maximize tractability. Thus I choose
 - 1) to consider plain over mirror-image-identified configurations (though that possibly sins by toy-modelling ordinary geometrodynamics rather than affine geometrodynamics).
 - 2) To study distinguishable particles over (partly) indistinguishable ones (though that probably sins against being as Leibnizian as possible).
 - 3) I need scaled models as regards realistic Quantum Cosmology, semiclassicality and internal time (for some other purposes, pure-shape (i.e. scalefree) models are simpler, and these occur as subproblems within scaled models).
 - 4) Each of 1- d , 2- d and 3- d models have an increasing number of midisuperspace-like features. 2- d is the first scaled model to have a nontrivial linear constraint. Namely, the zero angular momentum constraint on the whole model universe. However, 1- d already has a notion of localization. (These two notions are interlinked in GR, where they are both related to spatial derivatives being nontrivial, but these notions are separate for RPM’s). However, for 2- d the associated quotiented-out group (analogue of the spatial diffeomorphisms in GR) is $SO(2)$, which is rather simple (e.g. Abelian, trivial orbit structure). Considering 3- d ’s $SO(3)$ thus apports a bit more. (Though it still in no way emulates a number of the more specific features of the diffeomorphisms, which lie entirely outside the scope of this article’s toy models). Finally, I note that 3- d is considerably less tractable than 2- d , so, for detailed considerations, I restrict myself to at most 2- d in the present article. To be clear, I consider 2- d RPM as a toy model of 3- d geometrodynamics because this analogy itself is very insensitive to the toy model’s d provided that it is > 1 .

5) RPM's also have much freedom in the form of their potential. No potential is in many ways the simplest possibility, but it lacks in realisticness and in normalizability. Multiple HO-like potentials are also simple to study (far more exactly tractable than more complicated cases, and now with nice boundedness properties on the QM states), though such potentials are also known to be atypical in their simpleness and properties. I consider it an improvement on the modelling to use the potential freedom so as to better match up with Quantum Cosmology – to have a slow heavy dynamics of RPM's scale. (This is the moment of inertia of the system, or, more usually, its square root.) It is in direct parallel with GR (semi)classical cosmological scale dynamics. Thus I do what I can to let this Cosmology–Mechanics analogy determine which potentials more complicated than HO's I am to consider. Another qualification of 'simple' is lacking in change of relative shape momenta (angular and/or relative distance momenta, see Sec 4). However, this is disappointing from the relational and emergence of dynamics perspectives, so I do what I can to get round this simpleness.

6) Zero angular momentum has the good fortune of simplifying mechanics from a fibre bundles/topological defects perspective. One gets very simple mathematics, particularly for 1- d models and 3 particles in 2- d for the above plain and distinguishable choices (\mathbb{S}^{N-2} and \mathbb{S}^2 mathematics that is very close to textbook QM of one particle in 3- d and straightforward generalizations of these).

This simpleness is a triumph, because then one can do and check many Problem of Time calculations (which would *not* make sense if done for atoms or molecules, say, due to the absolutist underpinnings in these latter cases). Thus the current project concerns taking very old and well-known mathematics and giving it a new interpretation by which it makes sense in a model Quantum Cosmology context to which many Problem of Time schemes and a number of other Quantum Gravity subtleties allegedly apply.¹ And it probes a distinct set of features from minisuperspace, having the abovementioned midisuperspace-like attributes while not having fully GR-inherited potentials or kinetic term indefiniteness (as well as not having counterparts of some field theoretic and/or diffeomorphism-specific features). This is rather in the spirit of the Problem of Time reviews of Kuchař [400] and Isham [335]; the present article could be viewed as a framework to improve on those by exploiting the newly opened-up RPM toy model arena.

As well as minisuperspace and inhomogeneous perturbations thereabout [296], other toy models of comparable complexity to those in the present article are Carlip's 2 + 1 gravity work [159], the parametrized particle [400], parametrized field theories [400] [add original references], strong gravity [331, 514, 240, 14], the Montesinos–Rovelli–Thiemann model [473] and the bosonic string as a toy model of geometrodynamics [407, 400, 389].

My specific RPM models permit dilational hidden time, emergent semiclassical time, timeless approaches (Naïve Schrödinger Interpretation, Conditional Probabilities Interpretation, Records Theory), Histories Theory and perhaps some observables-based approaches. Excluded approaches include superspace time (indefinite kinetic metric based) and various diffeomorphism-related approaches. I also see there to be sizeable difficulties with internal time and observables-type approaches. In the case of hidden time, I have used my models to develop counterarguments; study of observables-type approaches for my models has only just started. Of the approaches I favour and RPM's are capable of investigating, RPM's were largely selected due to their capacity for investigation of timeless approaches, and constructed to be compatible with the Semiclassical Approach. One further goal then is to piece together Records Theory, the Semiclassical Approach and Histories Theory. I provide an arena for systematic study of the Semiclassical Approach's usually omitted terms: back-reaction, nonadiabaticity, averaged terms and terms that bring in higher derivatives. I also provide a wider range of dilational hidden times. I further develop more on Records Theory, including notions of distance and some considerations of notions of information. I consider the histories–records–Semiclassical Approach combination along the lines of Halliwell [292, 294, 295] for RPM's. This remains work in progress. Principles underlying various approaches include the indefiniteness-scale-heavy physics alignment, and the relation between physical propositions and Problem of Time strategies.

This article also provides a considerable treatment of the nature of relationalism, which different authors (in particular Barbour and Rovelli) have taken to mean rather different things. I provide a number of new particularities of the variational principles of dynamics for the specifically-relational setting. RPM's additionally exhibit a number of aspects of the operator-ordering problem, including new midisuperspace ones and a connection between operator-ordering and the absolute versus relational debate. I also give useful qualitative treatment of uniform states and structure formation.

Prerequisites for understanding this article. Principles of Dynamics as in Lanczos [415], GR as in Wald [622], QM as in e.g. Landau and Lifschitz [416], and elementary Geometry and Geometrical Methods as in e.g. Nakahara [486]. A strong familiarity with real linear Methods of Mathematical Physics is also desirable, e.g. Courant and Hilbert [185].

Questions. The text is studded with these. They should lead to a number of papers and projects. Stars indicate which are expected to be harder to provide answers for.

¹However, 4 particles in 2- d is a tougher problem with less standard mathematics – it is \mathbb{CP}^2 , whose isometry group is $SU(3)$ rather than some $SO(p)$ group as occurs for N particles in 1- d or 3 particles in 2- d and which requires somewhat (albeit not that much) harder Methods of Mathematical Physics. 4 particles in 2- d is currently the frontier of my research, with only a few starting notes in the present article. The extension to N particles in 2- d is only slightly harder (unlike in 3- d , where increasing particle number very quickly leads to serious mathematical difficulties).

1 Introduction

This article approaches the Problem of Time and other foundational and conceptual issues in Quantum Gravity and Quantum Cosmology via use of simple concrete examples of the following relational² particle models (RPM's).

1) **Scaled RPM** alias Euclidean RPM (abbreviated to ERPM) alias Barbour–Bertotti 1982 theory [92] (after its first proponents and in distinction of their different 1977 theory [91]) is a mechanics in which only relative times, relative angles and relative separations are meaningful. In the physics literature, it was further studied or featured in reviews in [75, 98, 539, 583, 400, 79, 81, 83, 249, 250, 18, 19, 372, 24, 30, 33, 35, 36, 37, 40]. It has additionally been substantially discussed in the philosophy of physics literature (see footnote 3 for references).

2) **Pure-shape RPM** alias Similarity RPM (abbreviated to SRPM) alias Barbour 2003 theory [84] is a mechanics in which only relative times, relative angles and ratios of relative separations are meaningful. In the physics literature, it has been further studied in [85, 18, 20, 24, 25, 30, 31, 50, 34, 38, 41, 90, 53].

RPM's occur within the following relational program.³

1.1 Plausible underlying postulates for this article's relational program

Relationalism 0) **Relational physics is to be solely about the relations between tangible entities**

This is taken to be footnote 3's universalization to all physics of footnote 3's statement of relationalism for mechanics.

I use '**tangible entities**' in place of 'material objects' to make it clear that this is open to fields and 'force mediators' as well as 'matter building blocks', as befits modern physics. The key properties of tangible entities are, rather, as follows.

Relationalism 1) [Mach] They **testably act and are actable upon**.

Relationalism 2) [Leibniz] Any such which are **indiscernible are held to be identical**.

Note 1) That which is not testably acting or actable upon is held to be, rather, a non-entity. [This is still held to be a type of entity as regards being able to philosophize about it or mathematically represent it, just not a *physical* entity; absolute space is an obvious archetypal example.]

Note 2) As regards Relationalism 2), if one is doing physics, relationalism posits that physical indiscernibility *must* trump multiplicity of mathematical representation; this multiplicity still exists mathematically, but the true mathematics corresponding to the physics in question is the equivalence class spanning that multiplicity. One would then only wish to attribute physical significance to computations of tangible entities which succeed in being independent of the choice of representative

²These are relational in Julian Barbour's sense of the word rather than e.g. Carlo Rovelli's. See Appendix 2.D and Secs 11, 14, 17 for more about the latter and for a comparison. This article principally uses Barbour's sense of the word, and some generalizations thereof, as default.

³The historical background for this relational program (which is only occasionally relevant to this article and hence relegated to this footnote) is the absolute or relational motion debate. [I use 'relational' here rather than 'relative' here to avoid confusion with Einstein's relativity, which also features prominently in the present article. Philosophers of physics have also taken to using 'substantival' instead of 'absolute'.] This debate has been around since the inception of Newtonian Mechanics.

In brief, Newton's [487] traditional formulation of mechanics defined 'true' motion to be relative to an absolute space that is all-pervasive, infinite, invisible and can not be acted upon. Newton also considered motion to occur *in* time. His notion of time was additionally an absolute one. I.e., external, continuous, and uniformly flowing as 'needed to transform kinematic geometry into physical dynamics' [76]. (In fact, much of this conception of time was not new to Newton, c.f. Gassendi [247], Barrow [99] and Ptolemy [521].) The virtue of Newton's scheme is that it worked well in practise, nature immediately vindicating its practical consequences in detail.

However, Newton's formulation of mechanics has been argued to be bankrupt from a philosophical perspective. Absolute space does not comply with Leibniz's famous *identity of indiscernibles* [7], by which e.g. our universe and a copy in which all material objects are collectively displaced by a fixed distance surely share all observable properties and thus are one and the same. Also, Leibniz already considered time not to be a separate external entity with respect to which things change. Bishop Berkeley [120] and Mach [436] (see also e.g. [76, 81, 89] for further discussion) added to these arguments, e.g. Mach holding absolute space to be a non-entity on account of its not being actable upon. The alternative is for mechanics to be **relational**: spatial properties are to be entirely about the relations between material objects, and 'time to be abstracted from change' (Mach's notion). Berkeley additionally considered 'time as succession of ideas in our minds'. Historically, there was a lack in alternative theories with such features, though the comparatively recent RPM's of Barbour–Bertotti (1982) and Barbour (2003) that are the subject of this article do now serve as such.

Furthermore, Leibniz, Bishop Berkeley and Mach's arguments are philosophically compelling enough that they ought to apply to not just mechanics but to physics as a whole. As argued in Secs 1.10 and 2.6, GR is also relational, and indeed relationalism is one foundation from which GR can be derived, for all that this was not the historical route to GR. For, while Einstein was interested in 'Machian issues', the way in which he viewed these does not coincide with the relational program's view [632, 76, 94], nor did his historical route to GR [222, 223] constitute a *direct* implementation of Machian ideas.

See e.g. [75, 81, 584, 82, 83, 321, 517, 556, 89, 152, 216, 548, 577] as regards arguments for (and against) the value of Barbour and Bertotti's 1982 theory toward the absolute or relative motion debate. Relationalism would appear to have a good case for use in whole-universe situations, both from a philosophical perspective and because GR and at least some further GR-like theories do implement it. (These are good theories for whole-universe situations on a number of further grounds.)

of the equivalence class; the obvious archetype of this is gauge theory. An important lesson from there is that a set of part-tangible and part-non entities is often altogether more straightforward to represent mathematically.

Note 3) The word ‘testable’ also requires caution, as to whether it means now or ever. [Better data can split two hitherto indiscernible models. But ‘ever’ could well be regarded as containing part-non-entities such as the infinite future or the attainability of experimental/observational perfection. On the other hand, some uses of ‘ever’ are based on principle, e.g. the indiscernibility of the universe from itself under a uniform translation does not depend on how accurately we know the universe (unless this somehow provided evidence for a new paradigm of physics in which absolute position was a tangible entity). I do not resolve these dilemmas; I just make the reader aware that uses of the identity of indiscernibles vary as regards now or ever, and as regards observability or principle.]

Note 4) I do not take this SSSec’s principles too seriously; it is the standard use of configuration space, and the less standard Relationalism 4), 5), 6) and 8) perhaps supplemented by Relationalism 3) and 7) that are the actually mathematically-implemented starting point of this approach. As such, Relationalism 0) to 2) for now being vague or loose relative to Relationalism 3) to 8) is not a problem for the work actually done in this article, but only for this present SSSec, which represents a tentative *underlying* rather than a rigorous foundational *underlying*.

In this article’s sense of relational, one first treats instantaneous configurations and time, and does so separately for each due to the great conceptual heterogeneity between them. One largely accepts that the mathematics will be in general be most straightforward for mixtures of part-tangible- and part-non-entities, and one builds from there.

1.2 Configuration space

As regards the instantaneous configuration entities of a given theory, which I denote by Q^C , a key further concept is the *configuration space* \mathfrak{q} , i.e. the space of all the possible instantaneous configurations (each of the points in configuration space represents one state of the instantaneous configuration, i.e. a set of one value per Q^C).

Example 1) Particle positions are instantaneous configurations, for which Lanczos [415] provides a clear exposition of the configuration space notion.

Example 2) Inter-particle relative separations are instantaneous configurations (I give their configuration spaces in Sec 2).

Example 3) The values at each point of continuous extended objects (whether fields or geometrical objects that have values everywhere in space, including objects that have their own separate notion of space (NOS) of extent such as strings or membranes).

I denote the general NOS by Σ_p for $p \leq d$ the dimension of space, and I coordinatize it by $x^{\mu p}$. However, in this article all specific examples of this are for $p = d$ (field theories), in which case I drop the p subscripts, or for $p = 0$, in which case Σ_0 is just a point corresponding to the particle position, which point ceases to possess any internal coordinatization. I denote integration over whatever NOS is appropriate by $\int_{\Sigma_p} d\Sigma_p$. I take this to collapse to a multiplicative 1 for particle mechanics and other finite theories.

I denote finite theory cases’ instantaneous configurations by Q^C . For field theory cases, I denote the instantaneous configurations by $Q^C(x^\mu)$. Thus the typeface object Q^C is a portmanteau of the upright space-dependent case for field theories and the slanty case for finite theories. That is the default notation in this article as regards typeface, straight and slanty symbols. Such portmanteaux serve as a shorthand covering both cases at once to encompass both this article’s RPM models and the geometrodynamical formulation of GR that they are closely analogous to. The C in use above is a suitable multi-index, over particle and/or continuous extended object species.⁴

N.B. configuration space is not in general just a set, being, rather, further equipped with topological space, metric space and geometrical structure. A broad example of the last of these is Finsler metric geometry $\mathfrak{q} = \langle \mathfrak{S}, \mathcal{F} \rangle$ for \mathfrak{S} a topological manifold and \mathcal{F} the metric function.⁵ (Semi-)Riemannian geometry is then a common subcase of this which is usually written in terms of the metric \mathcal{M}_{AB} itself: $\mathfrak{q} = \langle \mathfrak{S}, \mathcal{M} \rangle$. Some of the relational program’s examples are in fact infinite-dimensional generalizations of such geometries. (But I still refer to such by the usual finite-dimensional geometries’ names, i.e. I elevate those names to be finite and field-theoretic portmanteaux of the usual finite notions).

Relationalism 3) [Barbour] concerns \mathfrak{q} **enjoying some kind of privileged primary status**. This article develops various possible variants of this in more detail (see Secs 4 and 16 in particular).

⁴I use $P[Q^A]$ as the portmanteau for $P(Q^A)$ for finite theories and $P(x^{\mu p}; Q^A)$ for infinite theories, where round brackets indicate function dependence, square brackets indicate functional dependence and $(;)$ is Isham’s notation for mixed function-functional dependence, with the former prior to the semi-colon and the latter after it. I use braces for actual brackets. I then use a special font (Large mathfrak) for such portmanteaux that come integrated over their corresponding notion of space (the action \mathfrak{S} is a such); these are therefore pure functionals in terms of what they depend on, but differ in the portmanteau way in terms of what features on the computational side of the equation.

⁵I generally use $\langle \rangle$ to denote ‘space of’. I use lower-case Greek indices for spatial indices 1 to d . I use bold font as shorthand for configuration space indices and underline for spatial indices.

1.3 The aim of building relational actions and then using the Principles of Dynamics

Relationalism 4) [Barbour] Aim to build up **relational actions** from natural compound objects derived from the Q^C .

Note 1) In making use of actions, we are working with, and drawing upon, the Principles of Dynamics. Many of the steps below follow from this as natural consequences of having an action, rather than of relationalism per se. But the particular actions in question embody relationalism, and the some of the subsequent Principles of Dynamics workings assist in enforcing that. This valuing and making specific use of the Principles of Dynamics differs from Mach's own thinking, and is due to Barbour (who called this additional part of his thinking a 'supplementary Cartesian element' [95]).

Note 2) We shall actually be developing a novel conceptual understanding of the Principles of Dynamics as best befits the relational perspective, whilst also demonstrating the practical equivalence of this and more standard formulations of the Principles of Dynamics (see Appendices 2.A and 4.C).

The compound objects readily include NOS derivatives (whenever the NOS is nontrivial) and contracted objects. From what the reader knows about actions, one might expect velocities to be required too, and these involve an incipient notion of time, so we need to discuss that next before making progress with the construction of actions (though the manner of that progress may well be unexpected to the reader).

1.4 Temporal relationalism and actions implementing it

1.4.1 Temporal relationalism

Relationalism 5) [Leibniz] A physical theory is **temporally relational** if there is **no meaningful primary notion of time for the system as a whole** (e.g. the universe) [92, 94, 25].

The very cleanest implementation of this is by using actions that are

i) **manifestly parametrization irrelevant (MPI)**, and

ii) **free of extraneous time-related variables**. [E.g. one is not to involve external absolute Newtonian time, or the dynamical formulation of GR's 'lapse' variable, see Sec 2.6.2.]

1.4.2 A preliminary, more conventional view on constructing temporally relational actions

The reader may now clamour that the preceding SSSec does not help with obtaining a notion of velocity since it denies rather than provides a primary notion of time for the system as a whole. A simple way of getting round this is to point out that MPI actions are trivially equivalent to **manifestly reparametrization-invariant (MRI)** ones. [For MRI, any choice of (monotonic) parameter will do, so that choice is empty, and any such choice is equivalent to making no choice by not parametrizing i.e. using the MPI action.] But MRI *has* a parameter λ , so that, for all that it is a non-entity, one can interpret it as a *label time* non-entity, and write down

$$\text{velocity} := d(\text{configuration variable})/d(\text{label time}) \text{ i.e. } dQ^A/d\lambda. \quad (1)$$

One can then straightforwardly build the kinetic term

$$T := ||\circ Q||_{\mathbf{M}}^2/2 = M_{AB} \circ Q^A Q^B/2. \quad (2)$$

I am assuming for now that this is the more standard physics for which this is homogeneous quadratic in the velocities, which assumption I lift later on in this SSec. M_{AB} is the configuration space metric portmanteau of $M_{AB}(Q^C)$ for finite theories and $M_{AB}(Q^C(x^\mu))$ for infinite theories (this is assuming *ultralocality*, i.e. no derivative dependence, collapsing the object from a functional to a mere function, which mathematical simplicity holds over the entirety of the standardly accepted fundamental theories of physics). Its determinant is M and its inverse is N^{AB} . $||\cdot||_{\mathbf{M}}$ denotes the corresponding \mathbf{M} -'norm'. I sometimes use the notation \mathcal{M} instead, in cases in which the configuration space is explicitly curved. $\circ := D/D\lambda$, the portmanteau of $d/d\lambda$ in the finite case and $\partial/\partial\lambda$ in the field-theoretic case.⁶ One can then form the Lagrangian portmanteau $[L[\dot{Q}^A, Q^A]]$ of the Lagrangian $L(Q, \dot{Q})$ and the *ultralocal* Lagrangian density $L(\dot{Q}; Q)$ for field theories], and thus, integrating over λ and the NOS, the relational action. Compliance with MRI does make the Lagrangian portmanteau in question *look* somewhat unusual, i.e. not be of difference-type form $L = T - V$, but rather of product form $L = 2\sqrt{TW}$. For now we take on trust that $W = E + U$ for $U = -V$, where V is the potential energy portmanteau: potential $V(Q^A)$ for finite theories and potential density

⁶This is intended as type of dot symbol; I often use large preceding derivative symbols rather than overhead derivative symbols for graphical ease of accommodation of the subtleties of configurational relationalism and of heavy-light splits by hanging suffixes on the large-symbol versions. I use δ/δ for functional derivatives, D/D as ordinary, partial derivative portmanteau, ∇/∇ as partial, functional derivative portmanteau, D/D as ordinary, functional derivative portmanteau. I hang cov, abs suffices on these for covariant and absolute derivatives, using D for ∂ and \mathcal{D} for δ in these contexts. Finally, I use Δ for Laplacians (suitably suffixed to say of which type), \mathbb{D} for measures and oversized δ for variations.

$V(x^\mu; Q^A(x^\mu))$ for field theories. Thus it assumed to be independent of the velocities (again a mathematical simplicity that happens to be in accord with the standardly-accepted fundamental physics), and time-independent (which makes good sense for fundamental whole-universe setting as opposed to the setting for dissipative subsystems or approximate modelling e.g. involving friction). This time-independence of the potential, alongside time-independence of the kinetic metric is part of what is covered by the no extraneous time-like objects part of the given mathematical implementation of temporal relationalism. E is the energy-like portmanteau: total energy E for finite theories and some kind of total energy density for field theories ('density' here includes a Jacobian factor). [U and W are, for now, to just be considered as the notational tidying defined above; we will justify this trust in Sec 1.4.11.] The MRI relational action is thus

$$\mathbf{S}_J^{\text{MRI}} = \int d\lambda \int_{\Sigma_p} d\Sigma_p L_J^{\text{MRI}} = 2 \int d\lambda \int_{\Sigma_p} d\Sigma_p \sqrt{TW}, \quad (3)$$

where the J stands for Jacobi since the finite case of this is Jacobi's principle.

1.4.3 The relational dethroning of the primary notion of velocity

A more manifestly relational manoeuvre, however, is to acknowledge that if there is no meaningful notion of time, then conventional physics' definition of velocity,

$$\text{velocity} := d(\text{configuration variable})/d(\text{some notion of time}), \quad (4)$$

reads that there is consequently no meaningful primary notion of velocity! Subsequently, nothing defined in terms of velocities such as kinetic terms, Lagrangians or canonical momenta, is guaranteed to meaningfully exist either. What *does* have tangible physical content, however, are the

$$d(\text{configuration variable}) \text{ i.e. } dQ^A \quad (5)$$

themselves, so that velocities are replaced by differentials of the configuration variables. In other words, changes *in time* are meaningless in relational physics, the tangible content residing, rather, among the changes of one configuration variable with respect to another,

$$d(\text{configuration variable 1})/d(\text{configuration variable 2}) \text{ i.e. } dQ_1/dQ_2. \quad (6)$$

1.4.4 The subsequent relational view on constructing temporally relational actions

As a follow-up of this dethroning of velocity from the realm of relational physics, the usual homogeneous quadratic kinetic term is supplanted by $ds^2/2$; given that I am taking the latter's paradigm to be primary, it is better that I recast this as the diki kinetic term being supplanted by ds^2 – the square of the arc element that corresponds to the kinetic metric,

$$(\text{kinetic arc element})^2 := ds^2 := ||dQ||_{\mathbf{M}}^2 = M_{AB} dQ^A dQ^B. \quad (7)$$

I note that this is a purely geometrical quantity. The action is, in complete generality, to be homogeneous linear in the $d(\text{configuration variable})$ so as to comply with MPI/MRI; in the homogeneous quadratic case this amounts to taking the square root. Then expression under the action's integral in the MPI action is of the form $d\tilde{s}$, where the tilde indicates a rescaling by some weight that is homogeneous of degree zero in the dQ [usually taken to be independent of them to match the conventional approach to fundamental physics having velocity-independent potential, though connecting these two will take a bit of work: B) below]. I use

$$(\text{physical arc element}) := d\tilde{s} := 2\sqrt{W}ds = 2\sqrt{W}||dQ||_{\mathbf{M}} \quad (8)$$

and then the action is

$$\mathbf{S}_J^{\text{MPI}} := \int \int_{\Sigma_p} d\Sigma_p d\tilde{s}. \quad (9)$$

Note 1) Such assembly of an action may be subject to some limitations, whether from implementing physical or philosophical principles (such as in Sec 1.6), or from purely mathematical simplicity postulates such as 'no derivatives higher than first (or, in a few special cases,⁷ second)'.

Note 2) In parallel to the Lagrangian portmanteau, ds and $d\tilde{s}$ are in fact portmanteaux of arclength elements in the finite case and full-blown Riemannian elements (no signature restriction implied) in the field-theoretic case.

Note 3) the above particular action coming from the homogeneous quadratic physics only assumption is of *Jacobi type*. In the case of mechanics itself, the NOS integration collapses to 1 and the above is precisely the Jacobi Action Principle, which is well-known and established to be cleaner in a number of ways than Euler–Lagrange's (alongside having a number of relational whole-universe features and applications).

Note 4) The MPI formulation does not have a primary concept of Lagrangian – it has the arc element instead. In other words, it is a '**geodesic principle**' (see Appendix 2.B for more detail).

⁷These include GR (see Secs 1.10 and 2.6).

Note 5) The above two notes, and how the integrand being a product and not a difference like the conventional Lagrangian does *not* unacceptably alter the physics resulting from the action, are explained below.

Note 6) The MRI or MPI action of the above form for homogeneous quadratic \mathbf{T} or $\mathrm{d}s^2$ physics is the Jacobi-type action principle; in the finite case of mechanics, it is the Jacobi action principle itself. This is more commonly encountered in the MRI form, the I note that Jacobi himself did use the MPI geodesic principle form.

1.4.5 More general temporally relational actions

$$\mathbf{S}_{\text{JS}}^{\text{MPI}} = \int \int_{\Sigma_p} \mathrm{d}\Sigma_p \mathrm{d}\tilde{s} = \int \int_{\Sigma_p} \mathrm{d}\Sigma_p \mathrm{d}\lambda \mathbf{L} = \mathbf{S}_{\text{JS}}^{\text{MRI}} \quad (10)$$

complies with temporal relationalism, and has mutual compatibility, by

$$\mathbf{L} \left[\frac{\mathrm{DQ}}{\mathrm{D}\lambda}, \mathbf{Q} \right] \mathrm{d}\lambda = \frac{\mathrm{d}\tilde{s}}{\mathrm{d}\lambda} \mathrm{d}\lambda = \mathrm{d}\tilde{s} = \frac{\mathrm{d}s}{\mathrm{d}\mu} \mathrm{d}\mu = \mathbf{L} \left[\frac{\mathrm{DQ}}{\mathrm{D}\mu}, \mathbf{Q}, \right] \mathrm{d}\mu . \quad (11)$$

[This uses the structural parallel and Euler's theorem for homogeneous functions, and takes λ and μ to be any two monotonically-related parameters; also, by this equivalence, I henceforth drop MPI and MRI labels.] The 'square root of homogeneous quadratic' Jacobi-type case is then an obvious subcase of this. Moreover, the above generalization includes passage from Riemannian geometry (no signature connotation intended) to Finsler geometry (no nondegeneracy connotation intended; \mathbf{L} is cast in the role of \mathcal{F}). It is still a geodesic principle, just for a more complicated notion of geometry. In fact, the Jacobi-type action more commonly arises in the literature (see e.g. [326, 415] not from the desire to be relational but rather from the desire to provide the natural mechanics for a given geometry. Synge's work starting with [596] generalized this to the more general geometry above [415]. The geometry in question enters by playing the role of configuration space geometry. In their honour, I denote the geometrically-natural construction of a mechanics given a metric geometry by $\text{JS} : \langle \mathbf{s}, \mathcal{F} \rangle \rightarrow \mathbf{S}_{\text{JS}}^{\text{MPI}}$.

1.4.6 Temporally relational formulation of the notion of conjugate momentum

The standard definition of conjugate momentum is

$$\mathbf{P}_A := \nabla \mathbf{L} / \nabla \dot{\mathbf{Q}} . \quad (12)$$

Then computing this from the MRI form of the Jacobi action gives

$$\mathbf{P}_A = \sqrt{\mathbf{W}/\mathbf{T}} \mathbf{M}_{\text{AB}} \circ \mathbf{Q}^A \quad (13)$$

However, manifest relationalism has been argued to have no place for velocities or Lagrangians, so either the above definition of momentum, or the conceptual role played by momentum, need a make-over. In this case, it turns out that the notion of conjugate momentum itself survives, by there being an equivalent manifestly-relational form for the defining formula,

$$\mathbf{P}_A := \nabla \mathrm{d}\tilde{s} / \nabla \mathrm{d}\mathbf{Q}^A . \quad (14)$$

Then computing this gives

$$\mathbf{P}_A = \sqrt{2\mathbf{W}\mathbf{M}_{\text{AB}}} \mathrm{d}\mathbf{Q}^B / \|\mathrm{d}\mathbf{Q}\|_{\mathbf{M}} , \quad (15)$$

which is manifestly within the span of notion (6).

1.4.7 Outline of the Dirac method

Next, one inspects to see if there are any inter-relations among these momenta due to the form of the action: (*primary constraints* [209]). Then variation of the action with respect to the base objects provides some evolution equations and perhaps some *secondary constraints* [209]. One then finds the evolution equations. Finally, one checks whether even more constraints may arise by the requirement that the evolution equations propagate the constraints; this is best done at the level of Poisson brackets (or Dirac brackets if required by the presence of so-called *second class* constraints).

1.4.8 Quadratic constraint: momenta as 'direction cosines'

The MRI implementation of temporal relationalism leads to constraints via the following general argument of Dirac [209] (as straightforwardly modified by me to concern differentials rather than velocities, and MPI instead of MRI actions). Since MPI actions are homogeneous of degree 1 in the differentials, the k momenta arising from these are homogeneous of degree 0 in the differentials. Hence the momenta are functions of at most $k-1$ independent ratios of differentials. Thus the momenta must have at least 1 relation between them (which is by definition a *primary constraint* [209]).

Moreover, for the above Jacobi-type action, there is precisely one such constraint (per relevant NOS point). This is due to

A) the square-root form of the action, by which the momenta are much like direction cosines and ‘their squaring to 1’ property by which they are not all independent. Here, instead, these square (using the \mathbf{M} matrix) to $2W$. Thus, the kinetic term in terms of momenta (a legitimately relational object since the momenta are), which is half of the above square, is equal to W . B) in cases with nontrivial space of extent, due to the particular *local* ordering of having the square root (and sum over A , B implicit in the kinetic arc element or kinetic term) *inside* the integral-over-space-of-extent sign as opposed to outside it. All-in-all, one has an energy-type constraint which is quadratic and not linear in the momenta,

$$\mathcal{Q}_{\text{quad}} := \mathbf{N}^{\text{AB}} \mathbf{P}_A \mathbf{P}_B / 2 + W = 0 . \quad (16)$$

1.4.9 Evolution equations

Again, the usual form of the evolution equations makes reference to Lagrangians, times and velocities:

$$\frac{D}{d\lambda} \left\{ \frac{\nabla dL}{\nabla \dot{Q}^A} \right\} = \frac{\nabla dL}{\nabla Q^A} . \quad (17)$$

There is however a manifestly relational version of this available,

$$d\{\nabla d\tilde{s} / \nabla dQ^A\} = \nabla d\tilde{s} / \nabla Q^A . \quad (18)$$

Then, computing it out,

$$\frac{\sqrt{2W}}{\|\mathbf{dQ}\|_{\mathbf{M}}} d \left\{ \frac{\sqrt{2W}}{\|\mathbf{dQ}\|_{\mathbf{M}}} dQ^A \right\} + \Gamma^A_{BC} \frac{\sqrt{2W}}{\|\mathbf{dQ}\|_{\mathbf{M}}} dQ^B \frac{\sqrt{2W}}{\|\mathbf{dQ}\|_{\mathbf{M}}} dQ^C = \frac{\nabla W}{\nabla Q_A} , \quad (19)$$

where Γ^A_{BC} are the configuration space Christoffel symbols.

1.4.10 Emergent Jacobi time

Both the conjugate momenta and the evolution equations are simplified if one uses

$$* := \frac{D}{D\mathfrak{t}^{\text{em}(J)}} := \sqrt{\frac{W}{I}} \circ := \frac{1}{N} \circ := \frac{1}{\dot{I}} \circ = \frac{D}{DI} . \quad (20)$$

Note 1) Here, $\mathfrak{t}^{\text{em}(J)}$ is to be interpreted as an emergent time, the portmanteau of position-independent time t for finite systems and position-dependent time $t(x^\mu)$ for infinite systems; I term it the **Jacobi emergent time**. It, and modifications of it, have long appeared in Barbour’s work (e.g. [92, 79]). Though I have also seen it in earlier Russian literature [6] (see also Sec 2.6.3), so I leave open the question of who its first proponent was.

Note 2) Inverting so as to obtain an expression for the emergent Jacobi time itself,

$$\mathfrak{t}^{\text{em}(J)} - \mathfrak{t}^{\text{em}(J)}(0) = \int \|\mathbf{dQ}\|_{\mathbf{M}} / \sqrt{2W} . \quad (21)$$

This implements

Relationalism 6) [Mach] **Time is to be abstracted from change.**

Relationalism 7) [Leibniz–Barbour] **THE MOST PERFECT time is to be abstracted from THE TOTALITY OF change** (my phrasing, and which is implemented here via *all* of the \mathbf{Q} are involved via the W and the $d\mathbf{Q}$).

Note 3) In (20), the first step defines a shorthand, the second is the MRI computational formula: the third relates the lapse N to it, the fourth recasts lapse as a truer velocity of the instant (VOTI) notion \dot{I} and the fifth reacts this as the even truer differential of the instant (DOTI) notion dI , by which the instant I itself is identified with the emergent time $\mathfrak{t}^{\text{em}(J)}$ that labels that instant.

Note 4) The above generalized lapse notion N is a portmanteau of N for finite theories and $N(x^\mu)$ for field theories. Similarly, the above generalized instant notion I is a portmanteau of I for finite theories and $I(x^\mu)$ for field theories.

The momenta in terms of $*$ are then just

$$\mathbf{P}_A = \mathbf{M}_{AB} * \mathbf{Q}^B , \quad (22)$$

while the evolution equations are

$$D_{\text{abs}}^2 \mathbf{Q}^A = * * \mathbf{Q}^A + \Gamma^A_{BC} * \mathbf{Q}^B * \mathbf{Q}^C = \nabla W / \nabla Q_A . \quad (23)$$

Note 5) This is but a **parageodesic equation** with respect to the kinetic metric (meaning it has a forcing term arising from the W), but is clearly going to be a true geodesic equation with respect to the physical (tilded) metric [see Appendix 2.B].

Note 6) Of use in practical computations, one can supplant one of the evolution equations (finite case) or one per space point (infinite case) with the Lagrangian form of the quadratic ‘energy-type’ constraint,

$$M_{AB} * Q^A * Q^B / 2 + W = 0 . \quad (24)$$

Note 7) As first remarked upon in the subcase of mechanics [79], the above reveals $\mathfrak{t}^{\text{em}(J)}$ to be a recovery on relational premises of the same quantity that is more usually assumed to be the absolute external Newtonian time, t^{Newton} ; there is a conceptually-similar recovery in the general case which covers a further number of well-known notions of time (see Sec 2).

Note 8) Another form for (23) is

$$*P_A = \nabla W / \nabla Q^A \quad (25)$$

Note 9) As $\mathfrak{t}^{\text{em}(J)}$ has been cast as an expression solely in terms of the Q^A and dQ^A ,

$$d(\text{configuration variable})/d(\text{emergent Jacobi time}) = dQ^A/d\mathfrak{t}^{\text{em}(J)} \quad (26)$$

do have tangible physical content, through being a type of (4) and not of (6). Thus in the relational approach the above equations entirely make sense from a temporally relational perspective, unlike in the absolute Newtonian counterparts of them that have the same mathematical form but pin an absolute time interpretation on the times present. The emergent time is provided by the system. For now, it furthermore gives the appearance of being provided by the whole of the subsystem. These last two sentences fit Mach’s own conception of time as per footnote 3. Also, the usually-assumed notion of time as an independent variable is un-Leibnizian and un-Machian. However, though it is to be overall abstracted from motion, once this is done it is a convenient choice for (emergent) independent variable.

Note 10) We shall from now on work with MPI and MRI forms held to be interchangeable, with preference for writing MPI ones. We will likewise present specific equations in terms of $*$ in the geodesic/Lagrangian picture (the geodesic picture is in terms of configuration variables and their differentials in parallel to how the Lagrangian picture is in terms of configuration variables and their velocities).

1.4.11 Inter-relating the Jacobi product and Euler–Lagrange difference actions

The more well-known difference alias Euler–Lagrange-type actions are, rather,

$$\mathbf{S}_{\text{EL}} = \int d\mathfrak{t} \int_{\Sigma_p} d\Sigma_p L = \int d\mathfrak{t} \int_{\Sigma_p} d\Sigma_p \{T_{\mathfrak{t}} - V\} . \quad (27)$$

Here $T_{\mathfrak{t}}$ is the kinetic energy formulated in terms of $D/D\mathfrak{t}$ derivatives for \mathfrak{t} conventionally an external notion of time (absolute time for mechanics, coordinate time for GR, which is a label time because GR is already-parametrized). See e.g. [415] or Sec 2.11 for how (3) \Rightarrow (27) in the case of mechanics by Routhian reduction, and [22] or Sec 2.2.1 for (27) \Rightarrow (3) by the emergence of some lapse-like quantity.⁸ Various advantages for product-type actions over difference-type actions stem from this as regards consideration of whole-universe fundamental physics, which is the setting for Quantum Cosmology.

The above justifies the identification of W as the combination of well-known physical entities $E - V$.

1.5 Configurational relationalism and its indirect implementation

Relationalism 8) [my extension and part-reformulation of Barbour] The action is to be **configurationally relational** as regards a group of transformations if a \mathfrak{g} -transformed world-configuration is indiscernible from one that is not. I.e., one is to build into one’s theory that a certain group of transformations \mathfrak{g} acting upon the theory’s configuration space \mathfrak{q} are to be irrelevant, i.e. physically meaningless [92, 94, 16, 21, 25, 27, 35], transformations. More specific detail of the \mathfrak{q} – \mathfrak{g} pairing is postponed until after we have seen details of specific examples, to Sec 2.9.

One way to implement this is to use not ‘bare’ configurations and their composites as above, but rather their arbitrary- \mathfrak{g} -frame-corrected counterparts.⁹ This is the only known way that is sufficiently widespread for a relational program to underlie the whole of the classical fundamental physics status quo of GR coupled to the Standard Model. The corrections are with respect to auxiliary variables \mathbf{g}^Z [a portmanteau of the finite case g^Z and the position-dependent case $g^Z(x^\mu)$ in field theory] that are paired with the infinitesimal generators of \mathfrak{g} .

Originally the \mathbf{g}^Z employed were in the form of multiplier coordinate corrections to the velocities

$$\dot{Q}^A - \sum_Z \mathfrak{g}_{\mathbf{g}^Z}^{\rightarrow} Q^A . \quad (28)$$

⁸Such mathematics conventionally appears in the mechanics literature under the name of the *parametrization procedure*. Namely, one may adjoin the original notion of time’s time variable to the configuration space $\mathfrak{q} \rightarrow \mathfrak{q} \times \mathbb{T}$ by rewriting one’s action in terms of a label-time parameter $\lambda \in \mathbb{T}$ (see e.g. [415]). However, in the relational context in which Barbour and I work, one rather at this stage adopts 1) above.

⁹I adopt the *passive*, as opposed to the *active*, point of view of transformations. That is the opposite of how Barbour conceives configurational relationalism (see e.g. spatial relationalism in [92] and Sec 2), which places the primary importance on the constituent objects of the universe. Moreover, these two a priori distinct conceptualizations turn out to be mathematically equivalent representations [174], at least in the context that is main subject of this article. Thus, within this context, I use “one way” and “the only way” to encompass both the passive *and* the active.

Here $\overrightarrow{\mathfrak{g}}$ denotes the infinitesimal group action. However, this formulation is not of consistent with manifest temporal relationalism since the multiplier corrections spoil this. Nor is it relational insofar as it is formulated in terms of meaningless label-time velocities. Nor is it an arbitrary- \mathfrak{g} frame approach. The next SSec amends these things, and then the rest of the present SSec applies just as well within this amendment.

The above also seems to be taking a step in the wrong direction as regards \mathfrak{g} being irrelevant in passing from the already-frame-redundant \mathfrak{q} to the principal fibre bundle¹⁰ over \mathfrak{q} , $P(\mathfrak{q}, \mathfrak{g})$. For, from the perspective of just counting degrees of freedom, this can be locally regarded as the product space $\mathfrak{q} \times \mathfrak{g}$, which has more degrees of freedom, as opposed to passing to the quotient space $\mathfrak{q}/\mathfrak{g}$, which has less. However, we shall also resolve this contention below.

Moreover, the essential line of thought of this SSec is the *only* known approach to configurational relationalism that is general enough to cover the Einstein–Standard Model presentation of Physics.¹¹

Note 1) This SSec’s notion of configurational relationalism is my [16] portmanteau that spans both of

1) **spatial relationalism**, as in the traditional mechanics setting and the (geometro)dynamical formulation of GR, and which was Barbour’s notion prior to this extension.

2) **Internal relationalism** [as in electromagnetism, Yang–Mills theory and the associated scalar and fermion gauge theories]. Note 2) This wide range of cases is afforded by correspondingly wide ranges of \mathfrak{q} and \mathfrak{g} . (Although adopting a \mathfrak{q} may carry connotations of there being some underlying NOS with which that is compatible. Also, given a \mathfrak{q} , there are consistency limitations on what variety of \mathfrak{g} can then have – see Sec 2.9). For scaled RPM, \mathfrak{g} is the Euclidean group of translations and rotations, while for pure-shape RPM it is the similarity group of translations, rotations and dilations.

Note 3) This extension and partial reformulation of Barbour’s spatial relationalism shows that it and the conventional notion of gauge theory (usually internal but also applicable to spatial concepts) bear a very close relation. Gauge theory is indeed Leibnizian and indeed an example of how such can usually only be implemented indirectly by mathematics of a mixed set of tangible entities and non-entities (respectively true dynamical degrees of freedom and gauge degrees of freedom).

Finally, I here propose to extend the meaning of configurational relationalism to have a second clause: ‘**no extraneous space structures**’. This matches temporal relationalism’s second clause of no extraneous time variables, and also serves to exclude the Nambu–Goto string action (which, without this extra criterion, is indeed includable under the previous literature’s formulation of relationalism). On such grounds, theories in fixed-background [333, 334, 541, 154, 160, 604, 585, 64, 215, 335, 372, 209] contexts are ‘less relational’ (or only relational in a weaker sense). It is, rather, with background-independent M-theory (or at least some limiting classical action for this) that the most purely relational program would be expected to make contact (see Sec 2.10.3). N.B. How generally this clause should be imposed requires further discussion in Sec 2.9.

1.6 Combining temporal and configurational relationalism

Typically, the potential term V is manifestly a good \mathfrak{g} -scalar but the kinetic term T has correction terms due to ‘ \mathfrak{g} -transformations and differentials not commuting’. Consider for now theories whose configuration space carry kinetic arc elements that are homogeneous quadratic and whose associated metric has at most dependence on the Q^A [i.e. a (semi)Riemannian as opposed to Finslerian or even more general geometry]. Then one can render the action MRI by supplanting the previous T by

$$T := ||\circ_{\mathfrak{g}}\mathfrak{q}||_{\mathbf{M}}^2/2 \text{ for } \circ_{\mathfrak{g}}Q^A := \circ Q^A - \sum_Z \overrightarrow{\mathfrak{g}}_{\mathfrak{g}^Z} Q^A. \quad (29)$$

The auxiliary variable in use here is now interpreted as the velocity corresponding to a cyclic coordinate; doing so requires taking into account the variational subtleties laid out in Appendix 2.A. I term the ensuing variational principle JBB[$P(\langle \mathfrak{S}, \mathcal{F} \rangle, \mathfrak{g})$], the BB standing for Barbour–Bertotti. [As a map, JBB = JS \circ \mathfrak{g} -bundle, for \mathfrak{g} -bundle: $\langle \mathfrak{S}, \mathcal{F} \rangle \times \mathfrak{g} \longrightarrow P(\langle \mathfrak{S}, \mathcal{F} \rangle, \mathfrak{g})$.] On relational grounds, however, I prefer the equivalent MPI form, the argumentation behind which never involves the label-time non-entity or the subsequently-dethroned notion of velocity.

Here, instead of the above, I supplant the previous configurational-rationally unsatisfactory arclength (density) ds by the \mathfrak{g} -corrected

$$ds_{\text{JBB}} = ||d_{\mathfrak{g}}\mathfrak{q}||_{\mathbf{M}} \text{ for } d_{\mathfrak{g}}Q^A := dQ^A - \sum_Z \overrightarrow{\mathfrak{g}}_{\mathfrak{g}^Z} Q^A. \quad (30)$$

The auxiliary variable in use here is now interpreted as the differential corresponding to a cyclic coordinate (which has its own close parallel to the abovementioned variational subtleties, also laid out in Appendix 2.A).

Note 1) One can then repeat the preceding SSec’s treatment of conjugate momenta, quadratic constraint, evolution equations and the beginning of the treatment of emergent time ‘by placing \mathfrak{g} suffices’ on the d , I , τ^{em} and $*$ (and, if one ever makes use of them, the $\circ = D/D\lambda$ and N ; I also find it helpful in considering the formulae inter-relating these to place a \mathfrak{g} suffix on the T as a reminder of the \mathfrak{g} -dependence residing within it).

¹⁰This being a bona fide bundle may require excision of certain of the degenerate configurations. Recall also that a principal bundle means that the fibres and the structure group coincide [486].

¹¹For mechanics in 1- and 2- d [25], I have found that working directly on the least redundant/most relational configuration space provides workable relational theories. Moreover, these coincide with the restriction to those dimensions of Barbour’s theories as formulated in arbitrary \mathfrak{g} -frame form and then reduced (Sec 3).

Note 2) $\mathfrak{t}_g^{\text{em(JBB)}}$ is no longer a relationally-satisfactory notion of time due to manifest configurational non-relationalism. As such, I term this a \mathfrak{g} -dependent proto-time. I also name it, and its eventual \mathfrak{g} -independent successor, after JBB rather than just after Jacobi, to reflect the upgrade to a configurational-relationally nontrivial context.

1.6.1 Linear constraints: enforcers of configurational relationalism

The novel feature in the variational procedure is, rather, that variation with respect to each \mathfrak{g} -auxiliary produces one independent secondary constraint. In this article's principal examples, each of these uses up *two* degrees of freedom. (That this is the case is testable for in a standard manner [209], and amounts to these constraints being first- and not second-class.) Thus one ends up on the quotient space $\mathfrak{q}/\mathfrak{g}$ of equivalence classes of \mathfrak{q} under \mathfrak{g} motions. This resolves the abovementioned controversy, confirming the arbitrary \mathfrak{g} -frame method both to indeed implement configurational relationalism and to be an indirect implementation thereof. I denote these linear constraints by \mathcal{L}_{inz} :

$$0 = \nabla L / \nabla \dot{c}^Z \text{ (or } \nabla \tilde{d} \tilde{s} / \nabla d c^Z) := \mathcal{L}_{\text{inz}} = P_A \frac{\delta}{\delta d c^Z} \{ \vec{\mathfrak{g}}_{d\tilde{c}} Q^A \}, \quad (31)$$

where the last form manifestly demonstrates these being purely linear in the momenta. The variational procedure behind obtaining these is justified in Sec 2.11.

One then applies the Dirac procedure to be sure that the constraints breed no further constraints (see Sec 2.9 for what the consequences are if they do).

1.6.2 Outline of ‘best matching’ as a procedure

Barbour’s term ‘best matching’ amounts to

- Best Matching 1) construct an arbitrary \mathfrak{g} -frame corrected action.
- Best Matching 2) Vary with respect to the \mathfrak{g} -auxiliary g^Z to obtain the linear constraint $\mathcal{L}_{\text{inz}} = 0$.
- Best Matching 3) Solve the Lagrangian form of $\mathcal{L}_{\text{inz}} = 0$ for g^Z . (This is often an impasse.)
- Best Matching 4) Substitute this back in the action to obtain a new action.
- Best Matching 5) Elevate this new action to be one’s primary starting point.

Note 1) Best Matching 2) to 4) can be viewed as a minimization (or, at least, as an extremization). One is searching for a minimizer to establish the least incongruence between adjacent physical configurations (see Secs 2 and 14 for more).

Note 2) Best Matching 2) to 5) can be viewed as a configuration space reduction procedure (see Secs 2 and 3 for more).

Note 3) See Fig 2 for a graphical demonstration.

1.6.3 Emergent Jacobi–Barbour–Bertotti time

As a distinct application of Best Matching 3), substitute its solution into the \mathfrak{g} -dependent proto-time $\mathfrak{t}_g^{\text{em(JBB)}}$ to render it configurational-relationally acceptable. [Albeit, this is at the cost of being technically implicit, subject to threat of nonuniqueness and of threat of not-in-practise solvable extremization.] I.e. the JBB emergent time is

$$\mathfrak{t}^{\text{em(JBB)}} - \mathfrak{t}^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}^{\text{relational}}}{\text{extremum } \mathfrak{g} \in \mathfrak{g}} \left(\int \|d_g \mathfrak{Q}\|_{\mathbf{M}} / \sqrt{W} \right). \quad (32)$$

Likewise (though there are strong relational arguments by which I view these as less conceptually useful),

$$N := \underset{\text{of } \mathbf{S}^{\text{relational}}}{\text{extremum } \mathfrak{g} \in \mathfrak{g}} \left(\sqrt{T_g / W} \right) =: \dot{\mathfrak{t}}. \quad (33)$$

1.7 Direct implementation of configurational relationalism

Configurational relationalism can also be implemented, at least in low dimensional RPM’s, by being directly constructed to be \mathfrak{g} -invariant [25, 35]. One can view this as working directly on the relational configuration space. There is now no need for any arbitrary \mathfrak{g} -frame variables, nor then do any linear constraints arise nor are these to be used as a basis for reduction to pass to a new action. Here, instead, one’s action is already directly \mathfrak{g} -invariant i.e. on $\mathfrak{q}/\mathfrak{g}$ by construction; this is reflected by the more complicated form taken by the kinetic arc element/kinetic term. This approach [25, 35] builds on work of Kendall [368] on the spaces of shapes and the ‘cones’ [35] over these so as to include scale, by then using the Jacobi–Synge approach to build a natural mechanics from a metric geometry. I.e. $\mathbf{JS}(\text{shape space})$ and $\mathbf{JS}(C(\text{shape space})) = \mathbf{JS}(\text{relationspace})$ where C stands for ‘the cone over’ (a notion explained in Sec 3).

Note 1) This *direct relationspace construction* (Sec 3.13) can be viewed as a gestalt of both the geometrization of mechanics that motivated Jacobi and Synge themselves and of the establishment of a temporally relational theory as above.¹²

¹²I use *relationspace* as the portmanteau of relational space for scaled RPM and shape space for pure-shape RPM, i.e. each case’s non-redundant configuration space.

Note 2) This provides a second foundation for RPM's that is independent of, but the output from which is coincident with, Barbour's work [25, 35]. In other words, it coincides with what arises from Barbour's formulation upon performing reduction (see also Sec 3.13).

Note 3) This formulation possesses emergent time as per (20), quadratic constraint (16), no linear constraints, evolution equations in form (19) and energy constraint (24) [the last two now have no \mathbf{g} subscripts].

Note 4) The end of Sec 3 establish that the direct relationalspace implementation coincides with the configuration space reduction procedure 'Best Matching 2) to 5)' in the case of 1- and 2- d RPM's for which both are explicitly calculable. By this, emergent JBB time is itself a J time for a more reduced configuration space geometry (though by this stage I consider BB to deserve sufficient credit, so I use 'emergent JBB time' whenever configurational relationalism has been taken into account, no matter by which means).

This article's notion of relationalism is further developed in Sec 2.9, Appendix 2.C and Secs 6, 11, 14 and 17. It is mostly (generalizations of) Barbour's relationalism, though it is also compared with Rovelli's notion of relationalism as per footnote 2 with and Louis Crane's notions of relationalism also in Secs 11, 14 and 16.

1.8 Scaled and pure-shape RPM examples

In both cases, one has N particles in dimension d is $\mathbf{q} = \{\mathbb{R}^d\}^N = \mathbb{R}^{dN}$. One then implements temporal relationalism by building a MRI/MPI Jacobi-type [415] action. For scaled RPM [92, 75, 98, 539, 583, 400, 79, 81, 433, 83, 249, 250, 94, 16, 372, 18, 19, 20, 24, 25, 30, 35, 36, 37, 40], let the group of irrelevant motions \mathfrak{g} is the d -dimensional Euclidean group of translations and rotations, explaining this theory's alias being 'Euclidean RPM' (ERPM). Thus only relative times, relative angles and relative separations are meaningful. E.g. for 3 particles in dimension $d > 1$, scaled RPM is a dynamics of the (scaled) triangle that the 3 particles form. This configurational relationalism is implemented indirectly (Sec 2.2) by introducing auxiliary variables that represent arbitrary Eucl(d)-frame corrections. The resulting action, which can be written as¹³

$$\mathbf{S}_{\text{JBB}}^{\text{ERPM}} = 2 \int d\lambda \sqrt{T_{\text{JBB}}^{\text{ERPM}} W} = \sqrt{2} \int \sqrt{W} ds_{\text{JBB}}^{\text{ERPM}} \quad (34)$$

$$\text{for } T_{\text{JBB}}^{\text{ERPM}} = \|\odot_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}}^2 / 2 \text{ or } ds_{\text{JBB}}^{\text{ERPM}} = \|d_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}} \quad (35)$$

$$\text{and } \odot_{\underline{A}, \underline{B}} \mathbf{q}^I := \dot{\mathbf{q}}^I - \dot{\underline{A}} - \dot{\underline{B}} \times \mathbf{q}^I \text{ or } d_{\underline{A}, \underline{B}} \mathbf{q}^I := d\mathbf{q}^I - d\underline{A} - d\underline{B} \times \mathbf{q}^I. \quad (36)$$

For pure-shape RPM, instead, now the group of irrelevant motions \mathfrak{g} is the d -dimensional similarity group of translations, rotations and dilatations [84, 18, 20, 24, 25, 30, 31] explaining this theory's alias as Similarity RPM (SRPM). Thus it is a theory only of relative times, relative angles and ratios of relative separations, to which I refer to as **pure shape** (taken to exclude size). The action for it is

$$\mathbf{S}_{\text{J-A}}^{\text{SRPM}} = 2 \int d\lambda \sqrt{T_{\text{J-A}}^{\text{SRPM}} W} = \sqrt{2} \int \sqrt{W} ds_{\text{J-A}}^{\text{SRPM}}. \quad (37)$$

$$\text{for } T_{\text{J-A}}^{\text{SRPM}} = \|\odot_{\underline{A}, \underline{B}, \underline{C}} \mathbf{q}\|_{\mathbf{m}}^2 / 2I \text{ or } ds_{\text{J-A}}^{\text{SRPM}} = \|d_{\underline{A}, \underline{B}, \underline{C}} \mathbf{q}\|_{\mathbf{m}}^2 / I \quad (38)$$

$$\text{and } \odot_{\underline{A}, \underline{B}, \underline{C}} \mathbf{q}^I := \dot{\mathbf{q}}^I - \dot{\underline{A}} - \dot{\underline{B}} \times \mathbf{q}^I + \dot{\underline{C}} \mathbf{q}^I \text{ or } d_{\underline{A}, \underline{B}, \underline{C}} \mathbf{q}^I := d\mathbf{q}^I - d\underline{A} - d\underline{B} \times \mathbf{q}^I + d\underline{C} \mathbf{q}^I. \quad (39)$$

Furthermore, in 1- or 2- d , RPM can be cast in direct relationalspace/reduced form [25, 35] (the two are proven equivalent in Sec 3 and jointly denoted by 'r'). Here, e.g. for ERPM,

$$\mathbf{S}_{\text{r}}^{\text{ERPM}} = 2 \int d\lambda \sqrt{T_{\text{r}}^{\text{ERPM}} W} = \sqrt{2} \int \sqrt{W} ds_{\text{r}}^{\text{ERPM}} \quad (40)$$

$$\text{for } T_{\text{r}}^{\text{ERPM}} = \|\odot \mathbf{Q}\|_{\mathcal{M}(\mathbf{Q})}^2 / 2, \text{ or } ds_{\text{r}}^{\text{ERPM}} = \|d\mathbf{Q}\|_{\mathcal{M}(\mathbf{Q})} \quad (41)$$

Here, $\mathcal{M}_{\text{AB}}(Q^C)$ are in-general-curved configuration space metrics, which are established in Sec 3 to be e.g. the usual metric on \mathbb{S}^{N-2} for N -particle 1- d SRPM, and the Fubini-Study metric on \mathbb{CP}^{N-2} for N particle 2- d SRPM.

The primary quadratic energy constraint is, for the indirectly-formulated versions of scaled and pure-shape RPM respectively, the 'energy constraints'

$$\mathcal{E} := n^{i\alpha j\beta} P_{i\alpha} P_{j\beta} / 2 + V = E, \quad \mathcal{E} := I n^{i\alpha j\beta} P_{i\alpha} P_{j\beta} / 2 + \mathbf{V} = \mathbf{E}. \quad (42)$$

The quadratic constraint is still an energy constraint of form (16) built from inverses of the above-mentioned metrics.

For RPM's, the zero total momentum, zero total angular momentum and zero total dilational momentum constraints

$$\underline{\mathcal{P}} := \sum_{I=1}^N \underline{\mathbf{p}}_I = 0, \quad \underline{\mathcal{L}} := \sum_{I=1}^N \mathbf{q}^I \times \underline{\mathbf{p}}_I = 0, \quad \underline{\mathcal{D}} := \sum_{I=1}^N \mathbf{q}^I \cdot \underline{\mathbf{p}}_I = 0 \quad (43)$$

¹³I use upper-case Latin indices for particle labels 1 to N . I use \mathbf{q}^I for particle labels, with corresponding masses m_I and conjugate momenta $\underline{\mathbf{p}}_I$. The configuration space metric $m_{I\alpha J\beta} = m_I \delta_{IJ} \delta_{\alpha\beta}$ with inverse $n^{I\alpha J\beta} = \delta_{IJ} \delta_{\alpha\beta} / m_I$. For scaled RPM, I use T, W, U, V, E , and sans serif versions thereof for pure-shape RPM's counterparts that have different physical dimensions. Thus the *pseudo-energy* \mathbf{E} has dimensions of energy $\times I$ and \mathbf{V}, \mathbf{U} and \mathbf{W} likewise, while \mathbf{T} has dimensions of energy/ I .

from variation with respect to \underline{A}^μ , \underline{B}^μ and C (so that only the former two constraints occur in scaled RPM).

The emergent time is given by (e.g. for ERPM)

$$t^{\text{em(JBB)}} - t_0^{\text{em(JBB)}} = \underset{\text{of } \mathbf{S}_{\text{JBB}}^{\text{ERPM}}}{\text{extremum } \underline{A}, \underline{B} \in \text{Eucl}(d)} \int \sqrt{T/W} d\lambda = \int \|\underline{d}_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}} / \sqrt{2W} . \quad (44)$$

1.9 Principal motivation: GR is a relational theory

Further motivation¹⁴ for the relational scheme are the following arguments that the geometrodynamical and conformogeometrodynamical formulations of GR are also relational [94, 21, 27, 47, 14, 491, 492, 15, 16, 17, 21, 26]. There are furthermore results about *not just casting, but deriving*, GR from less than the usual number of assumptions [94, 21, 26].

The Barbour-type indirect formulation of RPM's (9,29) makes these parallels particularly clear. In the geometrodynamical counterpart of this, $\mathbf{q} = \text{Riem}(\Sigma)$ – the space of Riemannian 3-metrics on some spatial manifold of fixed topology Σ (taken to be compact without boundary for simpleness). The group of irrelevant motions \mathbf{g} is $\text{Diff}(\Sigma)$, i.e., the diffeomorphisms on Σ . This configurational relationalism is implemented indirectly by introducing auxiliary variables that represent arbitrary- $\text{Diff}(\Sigma)$ corrections. Temporal relationalism is implemented by building a MRI/MPI action [94, 16] [this is similar to the Baierlein–Sharp–Wheeler action ([71], see also Sec 2.6.3) but now properly combining the temporal and configurational relationalisms, see again the preceding Sec reference]. The relational formulation of geometrodynamics is valuable in providing guidance in yet further investigations of alternative conceptual foundations for GR [92, 79, 94, 21], and as regards addressing the Problem of Time in Quantum Gravity (see Sec and Part III).¹⁵

$$\mathbf{S}_{\text{GR}}^{\text{relational}} = 2 \int d\lambda \int d^3x \sqrt{h} \sqrt{T_{\text{GR}}^{\text{relational}} \{\text{Ric}(h) - 2\Lambda\}} = \sqrt{2} \int d\lambda \int d^3x \sqrt{h} \sqrt{\text{Ric}(h) - 2\Lambda} \text{ds}_{\text{GR}}^{\text{relational}} \quad (45)$$

$$\text{for } T_{\text{GR}}^{\text{relational}} := \|\odot_{\text{F}} \mathbf{h}\|_{\mathcal{M}}^2/4 \text{ or } \text{ds}_{\text{GR}}^{\text{relational}} := \|\underline{d}_{\text{F}} \mathbf{h}\|_{\mathcal{M}}/2 \text{ and } \odot_{\text{F}} h_{\mu\nu} := \dot{h}_{\mu\nu} - \mathcal{L}_{\dot{\text{F}}} h_{\mu\nu} \text{ or } \underline{d}_{\text{F}} h_{\mu\nu} := dh_{\mu\nu} - \mathcal{L}_{\underline{d}\text{F}} h_{\mu\nu} . \quad (46)$$

In this case, \mathbf{q} is the space $\text{Riem}(\Sigma)$ of Riemannian 3-metrics on a fixed spatial topology Σ , and \mathbf{g} is the corresponding 3-diffeomorphism group, $\text{Diff}(\Sigma)$. So this action is a $\text{JBB}[P(\langle \text{Riem}(\Sigma), \mathcal{M} \rangle, \text{Diff}(\Sigma))]$ for a particular potential proportional to $\text{Ric}(h)$ [see 2.6 for more such constructions].

MPI/MRI gives [209] GR's Hamiltonian constraint

$$\mathcal{H} := \mathcal{N}_{\mu\nu\rho\sigma} \pi^{\mu\nu} \pi^{\rho\sigma} - \sqrt{h} \{\text{Ric}(h) - 2\Lambda\} = 0 , \quad (47)$$

which parallels (16). Also, variation with respect to the auxiliary $\text{Diff}(\Sigma)$ -variables \mathbf{F}^μ gives the GR momentum constraint

$$\mathcal{M}_\mu := -2D_\nu \pi^\nu_\mu = 0 . \quad (48)$$

The zero total dilational momentum constraint (43 iii) closely parallels GR's well-known maximal slicing condition [424],

$$\pi := h_{\mu\nu} \pi^{\mu\nu} = 0 . \quad (49)$$

Additionally, much like generalizing maximal slicing to constant mean curvature (CMC) slicing [652] turns on a ‘York time’ variable [652, 654, 397, 400, 335]

$$t^{\text{York}} := \frac{2}{3} h_{\mu\nu} \pi^{\mu\nu} / \sqrt{h} = c(\lambda \text{ coordinate label time alone, i.e. a spatial hypersurface constant}) , \quad (50)$$

one can think of the passage from pure-shape RPM to scaled RPM as involving an extra ‘Euler time’ variable [18, 20, 22, 35]

$$t^{\text{Euler}} := \sum_i \underline{\mathbf{R}}^i \cdot \underline{\mathbf{P}}_i . \quad (51)$$

This is all underlied for both GR and RPM's by scale–shape splits, the role of scale being played by \sqrt{I} or I for RPM's and by such as the scalefactor a or \sqrt{h} in GR. In both cases it is then tempting to use the singled-out scale as a time variable but this runs into monotonicity problems. These are avoided by using as times the quantities conjugate to (a function of) the scale, such as t^{York} or t^{Euler} . Moreover, GR in the conformogeometrical formulation on CMC slices [652, 654] can be reformulated [49] as a relational theory in which \mathbf{g} consists both of $\text{Diff}(\Sigma)$ and a certain group of conformal transformations (see Sec 2.7.5).

¹⁴As well as this and the aforementioned relevance to the absolute versus relative motion debate, yet further motivation is furnished by how configurational relationalism is closely related [16, 27, 285] to gauge theory. See also Sec 6 for some QM motivation.

¹⁵The spatial topology Σ is taken to be compact without boundary. $h_{\mu\nu}$ is a spatial 3-metric thereupon, with determinant h , covariant derivative D_μ , Ricci scalar $\text{Ric}(h)$ and conjugate momentum $\pi^{\mu\nu}$. Λ is the cosmological constant. Here, the GR configuration space metric $\mathcal{M}^{\mu\nu\rho\sigma} = \{h^{\mu\rho}h^{\nu\sigma} - h^{\mu\nu}h^{\rho\sigma}\}$ is the undensitized inverse DeWitt supermetric with determinant \mathcal{M} and inverse $\mathcal{N}_{\mu\nu\rho\sigma}$ itself the undensitized DeWitt supermetric $h_{\mu\rho}h_{\nu\sigma} - h_{\mu\nu}h_{\rho\sigma}/2$. The densitized versions are \sqrt{h} times the former and $1/\sqrt{h}$ times the latter. To represent this as a configuration space metric (i.e. with just two indices, and downstairs), use DeWitt's 2 index to 1 index map [201]. $\dot{\text{F}}^\mu$ is the velocity of the frame; in the manifestly relational formulation of GR, this cyclic velocity plays the role more usually played by the shift Lagrange multiplier coordinate. $\mathcal{L}_{\dot{\text{F}}}$ is the Lie derivative with respect to $\dot{\text{F}}^\mu$.

GR has an analogue of emergent JBB time,

$$t^{\text{em(JBB)}}(x^\mu) - t_0^{\text{em(JBB)}}(x^\mu) = \underset{\text{of } \mathbf{S}_{\text{GR}}^{\text{relational}}}{\text{extremum } F^\mu \in \text{Diff}(\Sigma)} \int ||d_F \mathbf{h}||_{\mathcal{M}} / \sqrt{\text{Ric}(h) - 2\Lambda} . \quad (52)$$

This represents the same quantity as the usual spacetime-assumed formulation of GR's *proper time*, and, in the (predominantly) homogeneous cosmology setting, the cosmic time. We shall also see how (Secs 12 and 13) an approximation to it also coincides with GR's own version of emergent semiclassical (alias WKB) time.

See Sec 1.12, 2.8 and the Conclusion for further GR–RPM analogies. The resemblance between RPM's and GR is to a comparable but different extent to the resemblance between GR and the more habitually studied minisuperspace models [462, 463, 464, 546, 311, 642, 193, 129]. The main useful modelling points in this regard are as follows.

- 1) that RPM's linear zero total angular momentum constraint \mathcal{L}_μ is a nontrivial analogue of GR's linear momentum constraint \mathcal{M}_μ . (This is a structure which minisuperspace only possesses in a trivial sense and one which is important as regards the details of a large number of applications, including aspects of the Problem of Time.
- 2) Also, RPM's (unlike minisuperspace) have notions of locality in space and thus of clustering/structure formation; this is useful, among other things, for cosmological modelling.

Note 1) In GR, 1) and 2) are tightly related as both concern the nontriviality of the spatial derivative operator. However, for RPM's, the nontriviality of angular momenta and the notion of structure/inhomogeneity/particle clumping are unrelated. Thus, even in the simpler case of 1- d models, RPM's have nontrivial notions of structure formation/inhomogeneity/localization/correlations between localized quantities.

Note 2) Minisuperspace is, however, closer to GR in having more specific and GR-inherited potentials and indefinite kinetic terms. Thus it and RPM's are to some extent complementary in their similarities to GR, and thus in the ways in which they are useful as toy models of GR.¹⁶

1.10 Classical study of RPM's: outline of Part I

I build up a reasonable set of concrete RPM models. A first key step in understanding these is Sec 2's use of relative Jacobi coordinates (see Fig 32 for a summary of the key moves). Sec 2 also reviews various ways of setting up RPM's and is mostly expected to be of interest to people who have worked with RPM's. A second key step (Sec 3) is in restricting attention for now to 1- d and 2- d models; for N particles, I term these, respectively, N -stop metroland and N -a-gonland. (The first two nontrivial N -a-gonlands I furthermore refer to as *triangleland* and *quadrilateralland*). This is a key step because their configuration spaces are mathematically highly tractable [363, 368, 25]: \mathbb{S}^{N-2} spheres for 1- d and \mathbb{CP}^{N-2} complex projective spaces for 2- d in the pure-shape case. The third key step is that the scaled case's configuration spaces are the cones over these ([35], Sec 3) which include $C(\mathbb{S}^{N-2}) = \mathbb{R}^{N-1}$. The above configuration spaces are all for the choice of plain rather than mirror-image-identified shapes. I argue for plain shapes, at least to start off with, since these giving simpler mathematics and thus are conducive to a wider range of toy model calculations being completeable, comparable and combineable, which is precisely what the study of the Problem of Time needs! While pure-shape models are more straightforward than models with scale for a number of purposes, scaled models shall also be required as regards reasonably realistic semiclassical quantum cosmologies. Also, as a consequence of the cone structure, pure-shape problems occur as subproblems in models with scale [105, 25, 30], so studying these first also makes sense even from this semiclassical quantum cosmological perspective). N.B. that the interesting theoretical parallels between GR and RPM's are unaffected by the choice of plain shapes and of low- d RPM's.

The smallest nontrivial relational examples are scaled 3-stop metroland, pure-shape 4-stop metroland and pure-shape triangleland. Note that while both involve spheres, 4-stop metroland has a simpler physical realization of this than triangleland [35]. Thus study of 4-stop metroland will generally precede that of triangleland in this article. Also note that many 3- and 4-stop metroland results readily extend to the N -stop case. Further noteworthy features of the models are that 4-stop metroland has disjoint nontrivial subsystems and hierarchies of relationally nontrivial subsystems, while triangleland can simultaneously possess scale and nontrivial linear constraints. Quadrilateralland has the useful property of simultaneously possessing all of these features, as well as being more geometrically typical for an N -a-gonland than triangleland (at the cost of this mathematics being somewhat harder than triangleland's. Quadrilateralland is one of the principal frontiers in the first edition of this article. Other possible extensions are 1) to N -stop metroland, N -a-gonland [3- d models rapidly become too hard to handle with increasing N – see Sec 3 for more on this). 2) To mirror-image-identified counterparts and/or counterparts with (partly) indistinguishable particles.

Since Classical Dynamics and QM both benefit considerably from study of the underlying configuration space at the topological and geometrical level, I provide this for pure-shape RPM and then scaled RPM in Sec 3, doing so in great enough generality to anchor mirror-image-identified and indistinguishable particle cases. Since pure-shape N -stop metroland and N -a-gonland have standard and tractable geometries, one can work directly on these spaces/reduce down to them, as well as subsequently having available numerous useful coordinate systems and Methods of Mathematical Physics. For triangleland,

¹⁶Midisuperspace (see e.g. [392, 116, 117, 402]) unites all these desirable features but is unfortunately then computationally too hard for many of the aspects of the Problem of Time.

$\mathbb{CP}^1 = \mathbb{S}^2$, so one has ‘twice as many techniques’. Thus I focus on triangleland in particular in this article’s further sections. (We shall see that this example meets many of the nontrivialities required by Problem of Time strategies, so it is a case I concentrate quite a lot upon). Scaled RPM has as configuration spaces the ‘cones’ corresponding to each of the above pure-shape RPM configuration spaces. This makes clear the *scale–shape* representation of scaled RPM, which plays a big role in the present article. For, pure-shape RPM also occurs as a subproblem within the scale–shape split of scaled RPM and thus remains of direct physical interest. I present a description of shape quantities, including the Cartesian versus Dragt [205] correspondence which distinguishes between the ways the 4-stop metroland and triangleland 2-spheres are realized; this amounts to how one geometrically interprets suitable variables. I also present the useful technique of tessellation of the mass-weighted shape space sphere by its physical interpretation.

Some indirect formulations of relational theories can be reduced to direct ones (SSec 3.14), or independently conceived of in direct terms (SSec 3.13) – the relationalspace approach. For scaled RPM, see [433, 249, 19, 24, 25]. It was also done in 2- d for pure-shape RPM [20, 24, 25]. What one then gets in these examples coincides with what one gets from direct formulation (as I demonstrate in Sec 4). I also provide comparison with relational-absolutist split of Newtonian Mechanics.

I provide a number of less usual and new variational principles of dynamics moves and objects as are appropriate for the relational treatment. Free end-NOS variation, more relational parallels of the total Hamiltonian, the Dirac procedure and of phase space, including also a ‘rigged’ notion of phase space that might suffice for approaches in which \mathfrak{q} is primary.

In Sec 4 I consider the dynamical equations following from the actions of RPM’s in relationalspace/reduced form. This includes useful analogies with more commonly encountered physical systems: rotors, central problems, the Kepler–Coulomb problem. I also provide physical interpretation for RPM’s momenta, isometries and conserved quantities (the middle of these are mathematically $SO(p)$ and $SU(p)$ for 1- and 2- d models, but are realized by quantities which are more general in interpretation in physical space than angular momenta: they have a ‘shape momenta’ interpretation). These are conserved for certain sufficiently shape-independent in loose parallel to the centrality condition in ordinary mechanics implying conservation of angular momentum. As well as the scale–shape split form being widely useful for scaled triangleland, it is useful to consider this in projective-type coordinates that turn out to be akin to parabolic coordinates. I also discuss how RPM’s avoid the absence of configuration space monopole problems.

In Sec 5, I consider power-law potentials and their sums.

In particular, I use zero/constant potentials and HO-like potentials potentials for simpleness.

I then consider potentials chosen via analogy with Cosmology [35]. Here, the heavy slow dynamics of \sqrt{I} or I parallels that of the GR scalefactor a . However, this is now coupled to a simpler light fast finite dynamics of pure shape [finite and tied to well-studied configuration space geometry such as \mathbb{S}^{N-2} or \mathbb{CP}^{N-2}] rather than the rather more complicated light fast infinite dynamics of small GR inhomogeneities. Thus also RPM’s make for a useful qualitative model of the quantum-mechanical origin of structure formation.

I then provide simple exact and approximate solutions for all of these, many of are familiar from elsewhere in physics. I furthermore interpret them in RPM terms (which is new, and overall amounts to using established knowledge to understand RPM models which then have significant further Problem of Time and quantum cosmological properties). I provide qualitative descriptions of solutions, and mathematical forms of some solutions or approximate solutions (mostly in parallel with either ordinary mechanics work or GR cosmology work). These give a reasonable understanding of RPM dynamics. I also include Appendix 5.C as regards whether the observed universe can be directly modelled with RPM’s.

1.11 Quantum study of RPM’s: outline of Part II

This article’s approach to quantization is laid out in Sec 6 in parallel with Quantum Cosmology/quantum geometrodynamics. While it is true that various infinite-dimensional, diffeomorphism related and indefinite configuration space metric signature related aspects are not captured by RPM’s, however, the following major issues are present for RPM’s.

- 1) Kinematical quantization (including an ambiguity paralleling the affine–ordinary geometrodynamics one).
- 2) Time-independent Schrödinger equations, and thus the frozen-formalism aspect of the Problem of Time (see next SSec and Part III for more on this – the principal focus of the present article.)
- 3) Accompanying linear quantum constraints (unless already reduced out, and indeed reduced versus Dirac approach issues [583, 19, 31, 37]).
- 4) Operator-ordering issues [32, 35], including some novel connections between these and absolute versus relative issues, Dirac versus reduced quantization issues, and inner product issues.
- 5) – 9) Finally, c.f. also the Quantum Cosmology issues in SSec 1.13.

I then consider specific examples already familiar from Part I, that go for the simplest possible mathematics to maximize tractability and checkability of subsequent Problem of Time moves, while remaining meaningful as whole-universe models. A further filter on models is that which of the potentials arising from the cosmology-mechanics analogy have the further useful features of being more analytically tractable well-behaved at the quantum level. As N -stop metroland has no Dirac-reduced difference or operator ordering ambiguity, I start in Sec 7 with scaled 3-stop metroland. This gives simple Bessel/Laguerre mathematics which I then reinterpret in terms of the less usual relational physics. I use this to point out and sometimes resolve various foundational issues: closed universe effects lessening the spectrum unless interpreted in a multiverse sense (by energy-interlocking) and non-badness of the semiclassicality. I consider some quantum pure-shape RPM’s in Sec 8. 4-stop metroland and triangleland both have spherical harmonics mathematics, and with potentials included, also have

mathematics that is well-known from Molecular Physics. I use asymptotic as well as exact and perturbative methods of solution. I consider some scaled RPM's in Sec 9. Scaled 4- and N - stop metroland give simple Bessel/Laguerre mathematics, while scaled triangleland can be tackled with well-known spherical and parabolic coordinates mathematics.

This further establishes the useful presence of simple mathematics. This article carefully exports the usefulnesses and limitations of Molecular Physics methods and concepts to Quantum Cosmology.

1.12 RPM's as toy models for Problem of Time: outline of Part III

Part III (especially its Introduction and Conclusion) will be the principal interest in this article for most readers. This notorious problem occurs because the 'time' of GR and the 'time' of ordinary Quantum Theory are mutually incompatible notions. This incompatibility leads to a number of problems with trying to replace these two branches with a single framework in situations in which the premises of both apply, such as in black holes and in the very early universe. There are then yet more analogies between RPM's and GR (see [633, 201, 397, 502, 398, 400, 79, 80, 403, 83, 372, 20, 586, 31, 50, 34, 35, 36, 37, 40] and Part III of this article) at the level of various strategies toward the resolution of the Problem of Time [400, 335].

One facet of the Problem of Time appears in attempting canonical quantization of GR due to its Hamiltonian constraint \mathcal{H} being quadratic but not linear in the momenta, which feature and consequence are shared by \mathcal{E} . Then elevating \mathcal{H} to a quantum equation produces a stationary i.e timeless or frozen wave equation – the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0 , \quad (53)$$

in place of ordinary QM's time-dependent one,

$$i\partial\Psi/\partial t = \hat{\mathbf{H}}\Psi . \quad (54)$$

Here, \mathbf{H} is a Hamiltonian, t is absolute Newtonian time and Ψ is the wavefunction of the universe. (53) is, in more detail,

$$\hat{\mathcal{H}}\Psi := -\hbar^2 \frac{1}{\sqrt{\mathcal{M}}} \frac{\delta}{\delta h^{\mu\nu}} \left\{ \sqrt{\mathcal{M}} \mathcal{N}^{\mu\nu\rho\sigma} \frac{\delta\Psi}{\delta h^{\rho\sigma}} \right\} - \xi \text{Ric}(h; \mathcal{M})' \Psi - \sqrt{h} \text{Ric}(x; h) \Psi + \sqrt{h} 2\Lambda \Psi + \hat{\mathcal{H}}^{\text{matter}} \Psi = 0 , \quad (55)$$

where ' ' implies in general various well-definedness and operator-ordering issues (see Sec 6 for a summary). Correspondingly, for RPM's,

$$\hat{\mathcal{E}}\Psi = \mathbf{E}\Psi . \quad (56)$$

Some of the strategies toward resolving the Problem of Time are as follows (all are presented here in the simpler case of RPM's).

A) Perhaps one is to find a hidden time at the classical level ([400], Secs 11.5, 12 by performing a canonical transformation under which the quadratic constraint is sent to

$$p_{t^{\text{hidden}}} + \mathbf{H}_{\text{true}} = 0 \quad (57)$$

(for $p_{t^{\text{hidden}}}$ the momentum conjugate to some new coordinate t^{hidden} that is a candidate timefunction), which is then promoted to a t^{hidden} -dependent Schrödinger equation

$$i\hbar\partial\Psi/\partial t^{\text{hidden}} = \hat{\mathbf{H}}_{\text{true}}\Psi . \quad (58)$$

The already-mentioned York time is a candidate internal time for GR, an RPM analogue for which is the Euler time. [While JBB time is indeed emergent whilst still at the classical level, it does not provide a linear momentum dependence and so does not serve the above purpose. It is however aligned with the timestandard that is recovered in B) below.] The parabolic form (57) can also be achieved by not rearranging GR but instead appending 'reference fluid' matter [see Secs 11 and 12, which both conceptually and mathematically generalize the way that A) is presented above.]

B) Perhaps (Secs 11.6 and 13) one has slow, heavy 'h' variables that provide an approximate timestandard with respect to which the other fast, light 'l' degrees of freedom evolve [296, 400, 372]. In the Halliwell–Hawking [296] scheme for GR Quantum Cosmology, h is scale (and homogeneous matter modes) and l are small inhomogeneities. In Halliwell–Hawking's scheme [296] the l -part are small inhomogeneities. The Semiclassical Approach involves making

i) the Born–Oppenheimer [136] ansatz

$$\Psi(h, l) = \psi(h) |\chi(h, l)\rangle \quad (59)$$

and the WKB ansatz

$$\psi(h) = \exp(i W(h)/\hbar) \quad (60)$$

(in each case making a number of associated approximations).

ii) One forms the h -equation, which is

$$\langle \chi | \hat{\mathcal{E}} \Psi = 0 \quad (61)$$

for RPM's. Then, under a number of simplifications, this yields a Hamilton–Jacobi equation¹⁷

$$\{\partial W / \partial h\}^2 = 2\{E - V(h)\} \quad (62)$$

where $V(h)$ is the h -part of the potential. One way of solving this is for an approximate emergent semiclassical time $t^{\text{em(WKB)}}(h)$.

iii) One then forms the l -equation

$$\{1 - |\chi\rangle\langle\chi|\}\hat{\mathcal{E}}\Psi = 0. \quad (63)$$

This fluctuation equation can be recast (modulo further approximations) into a $t^{\text{em(JBB)}}$ -dependent Schrödinger equation for the l -degrees of freedom

$$i\hbar\partial|\chi\rangle/\partial t^{\text{em(WKB)}} = \hat{\mathcal{E}}_l|\chi\rangle \quad (64)$$

the emergent time dependent left-hand side arising from the cross-term $\partial_h|\chi\rangle\partial_h\psi$. $\hat{\mathcal{E}}_l$ is the remaining surviving piece of $\hat{\mathcal{E}}$, acting as a Hamiltonian for the l -subsystem.

Note that the working leading to such a time-dependent wave equation ceases to work in the absence of making the WKB ansatz and approximation, which, additionally, in the quantum-cosmological context, is not known to be a particularly strongly supported ansatz and approximation to make. [B) and conceptually-related schemes are further discussed in Secs 11.6 and 13.] $t^{\text{em(WKB)}}$ aligns with $t^{\text{em(JBB)}}$ [whose properties are substantially further covered in Sec 12, and which also ends up playing a role in C) and D) as explained in Secs 15 and 16 as well as providing a ‘middling’ alternative to Relationalism 7) and its Rovellian counterpart].

C) A number of approaches take timelessness at face value (Secs 11.8 and 14). One considers only questions about the universe ‘being’, rather than ‘becoming’, a certain way. This can cause at least some practical limitations, but nevertheless can address at least *some* questions of interest.

C.i) A first example is the *naïve Schrödinger interpretation*. (This is due to Hawking and Page [315, 316], though its name itself was coined by Unruh and Wald [616].) This concerns the ‘being’ probabilities for universe properties such as: what is the probability that the universe is large? Flat? Isotropic? Homogeneous? One obtains these via consideration of the probability that the universe belongs to region R of the configuration space that corresponds to a quantification of a particular such property,

$$\text{Prob}(R) \propto \int_R |\Psi|^2 \mathbb{D}Q, \quad (65)$$

for $\mathbb{D}Q$ the configuration space volume element. This approach is termed ‘naïve’ due to it not using any further features of the constraint equations. This was done for RPM's in [50, 34, 36, 37].

C.ii) The *conditional probabilities interpretation* [502] goes further by addressing conditioned questions of ‘being’. The conditional probability of finding B in the range b , given that A lies in a , and to allot it the value

$$\text{Prob}(B \in b | A \in a; \rho) = \frac{\text{tr}(\mathbf{P}_b^B \mathbf{P}_a^A \rho \mathbf{P}_a^A)}{\text{tr}(\mathbf{P}_a^A \rho)}, \quad (66)$$

where ρ is a density matrix for the state of the system and the \mathbf{P}_a^A denote projectors. Examples of such questions for are ‘what is the probability that the universe is flat given that it is isotropic?’ Or, in the present context, e.g. ‘what is the probability that the triangular model universe is nearly isosceles given that it is nearly collinear?’

C.iii) *Records theory* [502, 248, 80, 83, 292, 28, 29] involves localized subconfigurations of a single instant of time. It concerns issues such as whether these contain useable information, are correlated to each other, and whether a semblance of dynamics or history arises from this. This requires

1) notions of localization in space and in configuration space (these also novelly provide a wider range of configurationally relational objects than the present section's).

2) Notions of information and correlation RPM's are superior to minisuperspace for such a study as they have all of a notion of localization in space, more options for well-characterized localization in configuration space (i.e. of ‘distance between two shapes’, Sec 14.7) through their kinetic terms possessing positive-definite metrics, and exact wavefunctions on which to base notions of information. [This is via Statistical Mechanics (SM) and ‘information = negentropy’].

This article has further novel analysis of Records Theory in Chapter 16: further properties and further proposed approaches with new combinations of these properties.

D) Perhaps instead it is the histories that are primary (*Histories Theory* [248, 306], Secs 11.9, 15).

Combining B) to D) (for which RPM's are well-suited) is a particularly interesting prospect [294], along the following lines (see also [248, 292, 295] and Sec 15.4 for more about this). There is a Records Theory within Histories Theory [248, 292]. Histories decohereing is one possible way of obtaining a semiclassical regime in the first place, i.e. finding an underlying

¹⁷For simplicity, I present this in the case of 1 h degree of freedom and with no linear constraints.

reason for the crucial WKB assumption without which the Semiclassical Approach does not work. What the records are will answer the also-elusive question of which degrees of freedom decohere which others in Quantum Cosmology.

E) Distinct timeless approaches involve *evolving constants of the motion* (‘Heisenberg’ rather than ‘Schrödinger’ style QM), or *partial observables* [545] (see Secs 11.11 and 16.11. These are used in LQG’s *master constraint program* [607], and can also be studied in the RPM arena.

Some approaches to the Problem of Time that do *not* have an RPM analogue include the superspace time approach (which requires indefinite configuration spaces) and ones involving finer details about the diffeomorphisms.

1.13 Analogues in various other aspects of Quantum Cosmology.

To motivate Quantum Cosmology itself, via e.g. inflation, it may then contribute to the understanding and prediction of cosmic microwave background fluctuations and the origin of galaxies [139, 484, 447, 296]. Inflation is currently serving reasonably well [590, 379] at providing an explanation for these. This remains an ‘observationally active area’, with the Planck experiment [515] about to be performed. The underlying Quantum Cosmology, however, remains conceptually troubled (see e.g. [298, 294]). It is via the above contact with testable assertions that hitherto philosophical contentions about QM (in particular as applied to closed systems such as the whole universe) may enter into mainstream physics. Suggestions as to how one might approach such conceptual issues in Quantum Cosmology include [111, 248, 306, 309, 310, 292, 297, 298, 294, 80, 83, 315, 316, 616, 502, 400, 335, 403, 499, 500, 372, 305, 339, 212].

Further foundational issues in Quantum Cosmology that may be addressable at least qualitatively using RPM toy models are as follows (Sec 10).

- 5) Does structure formation in the universe have a quantum-mechanical origin? In GR, this requires midisuperspace or at least inhomogeneous perturbations about minisuperspace, and these are hard to study. (Scaled RPM’s are a tightly analogous, simpler version of Halliwell and Hawking’s [296] model for this; moreover scalefree RPM’s such as this article’s occurs as a subproblem within scaled RPM’s, corresponding to the light fast modes/inhomogeneities.)
- 6) There are also a number of difficulties associated with closed-system physics and observables.
- 7) What are the meaning and form of, the wavefunction of the universe? (E.g. whether a uniform state is to play an important role; in classical and quantum Cosmology, these are held to be conceptually important notions).
- 8) Quantum Cosmology has robustness issues, as regards whether ignoring certain degrees of freedom compromises the outcome of calculations [405].
- 9) Like other branches of physics, Quantum Cosmology has Arrow of Time issues, and, moreover, may have something to say as regards the origin of physics’ various arrows of time [311, 83, 294, 545]. However, I feel that this largely lies outside the scope of the current article.

PART I: CLASSICAL THEORY OF RPM's

2 Examples of relational theories

2.1 Setting up Relational Particle Mechanics (RPM's)

The first task of the relational approach is to provide an unreduced configuration space \mathbf{q} and a group \mathbf{g} of transformations that are held to be physically irrelevant. If one is then to proceed by the indirect means of considering arbitrary \mathbf{g} -frame corrected objects, then by the nature of this correcting procedure, \mathbf{g} should consist entirely of continuous transformations. One can only sometimes proceed directly instead (see Sec 3 for examples).

2.1.1 First choices of a \mathbf{q} for RPM's

1) In the indirect approach, one's incipient notion of space (NOS) is *Absolute space* $\mathbf{a}(d)$ of dimension d . For usual studies of particle mechanics, this is held to be \mathbb{R}^d (equipped with the standard inner product (\cdot, \cdot) with corresponding norm $\|\cdot\|$ built from the d -dimensional identity matrix). Then an incipient *configuration space* $\mathbf{q}(N, d)$ for particle mechanics is “ N labelled possibly superposed material points in \mathbb{R}^d ”, i.e. \mathbb{R}^{Nd} , with coordinates $q^{I\mu}$. This has on it the inner product $(\cdot, \cdot)_{\mathbf{m}}$ with corresponding norm $\|\cdot\|_{\mathbf{m}}$.

2) Because the indirect implementation of configurational relationalism below encodes continuous \mathbf{g} below is held to be continuous, I furthermore need to consider here, at the preliminary level of choosing \mathbf{q} itself, whether physical irrelevance of the non-continuous reflection operation is to be an option or, indeed, obligatory. I.e. should \mathbf{q} be a space of *plain* configurations $\mathbf{q}(N, d)$ or of *mirror-image-identified* configurations $\mathbf{oq}(N, d) = \mathbf{q}(N, d)/\mathbb{Z}_2$ [the O stands for ‘orientation-identified’, meaning that the clockwise and anticlockwise versions are identified, O being a more pronounceable prefix than M]. Mirror-image-identified configurations are always at least a mathematical option at this stage by treating all shapes and their mirror images as one and the same: $\mathbb{R}^{Nd}/\mathbb{Z}_2$. However, one shall see below that in some cases the model has symmetry enough that declaring overall rotations themselves to be irrelevant already includes such an identification. In this case one can not meaningfully opt out of including the reflection, so this fork degenerates to a single prong.

3) In this article, I just consider the case of plain distinguishable particles bar brief mention in this and the next Sec, which serves as a base for a wider range of theories. Distinguishable particles covers the case of generic masses or, via some further unstated mysterious classical labels, the case of equal masses. For indistinguishable particles, consider quotients by the likewise-discrete permutation group on N objects, S_N or the even permutation group A_N so as to continue to exclude the extra reflection generator present in S_N . (Alternatively, one could use the $P < N$ versions of these if only a subset are distinguishable, or a product of such whose P 's sum up to $\leq N$ if the particles bunch up into internally indistinguishable but mutually distinguishable). This would be more Leibnizian (by identifying even more indiscernibles), as well as, more specifically, concordant with the nonexistence of meaningless distinguishing labels at the quantum level. (Though it is fine for particles to be distinguishable by differing in a *physical* property such as mass or charge or spin, which then actively enters the physical equations, and spin only arises at the quantum level.)

4) Some applications require further excision from the configuration space of degenerate configurations (e.g. collinearities or some collisions).

$\dim(\mathbf{q}(N, d)) = Nd$. Note throughout this Section that quotienting out discrete transformations such as reflections or permutations does not affect the configuration space dimension (these amount to taking a same-dimensional portion).

2.1.2 Choice of \mathbf{g} for RPM's

Various possibilities for the continuous group of physically-irrelevant transformations \mathbf{g} are as follows. $\text{Tr}(d)$ are the d -dimensional *translations*. $\text{Tr}(d)$ is the noncompact commutative group, $(\mathbb{R}^d; +)$. $\dim(\text{Tr}(d)) = d$. $\text{Rot}(d)$ are the d -dimensional *rotations*. $\dim(\text{Rot}(d)) = d\{d-1\}/2$; this being 0 in 1- d corresponds to the obvious triviality of continuous rotations in 1- d . $\text{Rot}(d)$ is the special orthogonal group $SO(d)$ of $d \times d$ matrices. $\text{Dil}(d)$ are the d -dimensional *dilations*, alias *homotheties*. $\text{Dil}(d)$ is the noncompact commutative group $(\mathbb{R}^+; \cdot)$ independently of d , so I denote this by Dil from now on. $\dim(\text{Dil}) = 1$.

The *Euclidean group* of d -dimensional continuous isometries is¹⁸ $\text{Eucl}(d) = \text{Tr}(d) \circledcirc \text{Rot}(d)$. $\dim(\text{Eucl}(d)) = d\{d+1\}/2$. The *proper linear group* of $d \times d$ matrices is $\text{Pl}(d) = \text{Rot}(d) \times \text{Dil}$. $\dim(\text{Pl}(d)) = d\{d-1\}/2 + 1$. The d -dimensional ‘non-rotation isometry’ group $\text{Nonrot}(d) = \text{Tr}(d) \times \text{Dil}$. $\dim(\text{Nonrot}(d)) = d + 1$. Finally, the *similarity group* of d -dimensional continuous isometries and homotheties is $\text{Sim}(d) = \text{Tr}(d) \circledcirc \text{Rot}(d) \times \text{Dil}$. $\dim(\text{Sim}(d)) = d\{d+1\}/2 + 1$.

Then if one chooses \mathbf{q} as in the preceding section alongside $\mathbf{g} = \text{Eucl}(d)$, one has scaled RPM (Sec 2.2), or, alongside $\mathbf{g} = \text{Sim}(d)$, one has pure-shape RPM (Sec 2.3), or, alongside $\mathbf{g} = \text{Nonrot}(d)$, one has non-rotational RPM (Sec 2.4).

2.1.3 Further choice of a \mathbf{q} for RPM's

One can also consider the centre of mass motion of the RPM universe to be ab initio meaningless.

¹⁸Here, \circledcirc denotes semidirect product, see e.g. [332]. Strictly speaking, $\text{Eucl}(d)$, $\text{Sim}(d)$ and $\text{Nonrot}(d)$ are the ‘proper’ versions of these groups via not being taken to include the discontinuous reflections.

Then one's incipient configuration space is *relative space* $\mathbf{r}(N, d) = \mathbb{R}^{nd}$ for $n := N - 1$ or *Orelative space* $\mathbf{or}(N, d) = \mathbb{R}^{nd}$. This is equivalent to using $\mathbf{q}(N, d)$ and quotienting out the translations to render absolute position meaningless, but this is devoid of mathematical structure or any extra analogy of GR, so using $\mathbf{r}(N, d)$ makes for a clearer presentation. [Both are presented below because it is not immediately straightforward to see that mechanics on $\mathbf{r}(N, d)$ can be cast in extremely close parallel to mechanics on $\mathbf{q}(N, d)$. For, making this manifest requires passing to relative Jacobi coordinates and noting how all the relevant mechanical formulae then look the same (Sec 2.2.4), which point was missed in the RPM literature prior to my involvement and yet turned out to be the crucial first key in unlocking the detailed understanding of these models.] In this second scheme, then, one is to use $\text{Rot}(d)$, $\text{Pl}(d)$ and Dil to form scaled, pure-shape and non-rotational RPM's respectively.

2.1.4 A deeper relational consideration: which $\mathbf{q}(N, d)$ are actually discernible

It is next necessary to point out a probably non-obvious flaw in the way this Section has hitherto been presenting its mathematical structure. Namely, that N and d have been introduced as if they were on an equal footing as independent free parameters. That does not however comply with relational thinking. For, particle number N is, undeniably, a material property. However, given N particles as the entirety of the contents of one's relational model universe, there only exist $n = N - 1$ independent relative separation vectors, and therefore only the following worlds are discernible.

$$\text{dimension } d = 1 \text{ to } n \text{ with or without mirror image identification .} \quad (67)$$

However, if one tries to use dimension $d > n$, one finds that this is *indiscernible* from the $d = n$ mirror image identified world, and thus *identical* to it as per Leibniz. Thus in RPM's, dimension d is *not* a free parameter; *the different values dimension d can meaningfully take are contingent on the particle number, N .*

Furthermore, this perspective is a very clear way of both anticipating the following next SSec's Note 1)'s subtlety and the ways in which Appendix 3.E's 3 particles in 3- d model is distinct only being at the cost of it being less relational.

A further issue is that some values of the relationalspace (i.e. entirely physical configuration space) dimension k are

i) utterly trivial (0 degrees of freedom).

ii) **relationally trivial** (1 degree of freedom). This is trivial because the relational program concerns expressing one material change in terms of another material change rather than in terms of an meaningless arbitrarily-reparametrizable label-time, so configuration space dimension > 1 is required.

It is however an interesting issue whether this continues to hold at the quantum level (see Sec 10.4). By these criteria, further of the (67) are knocked out as per the next SSec which computes $k(N, d)$.

2.1.5 Outline of the subsequent quotient spaces \mathbf{q}/\mathbf{g}

If absolute orientation (in the rotational sense) is also to have no meaning, then one is left on a configuration space **relational space** $\mathbf{R}(N, d) = \mathbf{q}(N, d)/\text{Eucl}(d)$ or $\mathbf{r}(N, d)/\text{Rot}(d)$ Or, in the case of O-shapes, on *Orelational space* $\mathbf{OR}(N, d) = \mathbf{q}(N, d)/\text{Eucl}(d)$ or $\mathbf{or}(N, d)/\text{Rot}(d)$. These have dimension $k = nd - d\{d-1\}/2 = \{2n + 1 - d\}/2$, i.e. $N - 1$ in 2- d , $2N - 3$ in 2- d and $3N - 6$ in 3- d .

If, instead, absolute scale is also to have no meaning, then one is left on a configuration space termed *preshape space* by Kendall [368], $\mathbf{p}(N, d) = \mathbf{q}(N, d)/\text{Nonrot}(d)$ or $\mathbf{r}(N, d)/\text{Dil}$. Or, one is left on *Opreshape space* $\mathbf{op}(N, d) = \mathbf{or}(N, d)/\text{Nonrot}(d)$ or $\mathbf{or}(N, d)/\text{Dil}$. These have dimension $k = nd - 1$.

If both absolute orientation and absolute scale are to have no meaning, then one is left on what Kendall [368] termed **shape space**, $\mathbf{S}(N, d) = \mathbf{q}(N, d)/\text{Sim}(d)$ or Or, one is left on *Oshape space* $\mathbf{os}(N, d) = \mathbf{or}(N, d)/\text{Sim}(d)$. These have dimension $k = d\{2n + 1 - d\}/2 - 1$, i.e. $N - 2$ in 2- d , $2N - 4$ in 2- d and $3N - 7$ in 3- d . N.B. that shape spaces have no place for a maximal collision. One can regard both preshape space and relational space as intermediaries in reaching shape space. We will really consider the above spaces as augmented to be normed spaces, metric spaces, topological spaces, and, where possible, Riemannian geometries (see Sec 3). Finally note that

$$\mathbf{p}(N, 1) = \mathbf{S}(N, 1) \quad (68)$$

and likewise for all O and indistinguishable-particle cases. This is simply because there are no rotations in 1- d .

Note 1) the above is all within the confines of $d \leq n$ that suffices to span the discernible RPM worlds. Were one to use $d > n$, it could then only be $\text{Rot}(n)$ and not $\text{Rot}(d)$ which acts on the physical configurations. Consequently $\text{Eucl}(n)$ and not $\text{Eucl}(d)$, and $\text{Sim}(n)$ and not $\text{Sim}(d)$ are then also to be used. Without this observation, the formulae for the degrees of freedom indeed manifestly fail! This action by less than $\text{Rot}(d)$ itself has a counterpart within the absolutist setting, most notably as regards the well-known number of degrees of freedom count in linear molecules.

Note 2) Additionally picking off the utterly trivial and relationally trivial cases, the full set of discernible and nontrivial RPM's are as in Fig 1. Note the small but nontrivial k 's among these as the most likely candidates for tractable models. As we shall see, most applications require $d \geq 2$, some require $d \geq 3$ and some require $k \geq 4$.

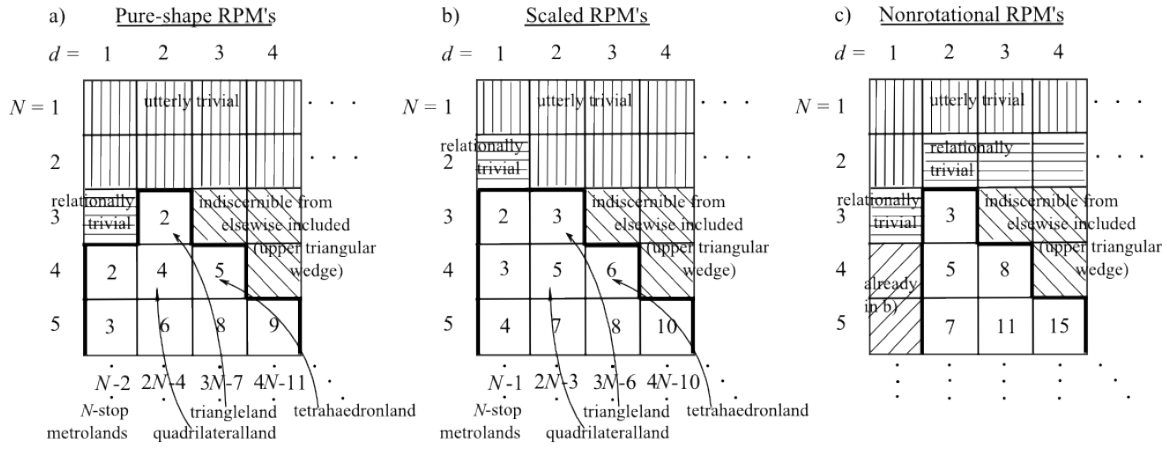


Figure 1: The full set of discernible and nontrivial a) pure-shape RPM's/shape spaces, b) scaled RPM's/relational spaces and c) non-rotational RPM's/preshape spaces. The numbers indicated are the configuration spaces' dimension, k .

I term 1- d RPM's **N-stop metrolands** as each point in them is the depiction of an underground train line. I term 2- d RPM's **N-a-gonlands** as each point in them is a planar N -sided polygon; I refer to the mathematically highly special $N = 3$ case by the neater special name of **triangleland**, and the first mathematically-generic $N = 4$ case as **quadrilateralland**. Finally, I term the 3- d such N -cornerlands as each point in them is a generally nonplanar solid with N corners. The mathematically highly special $N = 4$ case I refer to as **tetrahaedronland**. 3-cornerland is even more mathematically special, but it is also indiscernible from Otriangleland.

N.B. for more than 3 particles, the configurations are really **constellations** (sets of points, which, as the material point-particles, are the relationally-primary content of the theory) rather than what shapes can be made by joining up the dots. For, beyond triangles, the latter becomes ambiguous; while certain coordinate systems and particle labels may imply certain orders of 'joining up the dots', these are practitioner-dependent/particle label dependent rather than intrinsic. This furthermore means that, beyond triangles, some features of laminar/solid shapes are less relevant than others through depending on more than the actual physical entity which is the constellation; this is then reflected in terms of what questions I choose to ask of such shapes. E.g. whether a shape has edges that cross over is a labelling or laminar/solid shape construction dependent property and thus less interesting than configurations with coincident particles. This is an early indication that one will need to think carefully about how to separate out properties of/questions about shapes according to different levels of structure. This paragraph is clearly also why I use the N -cornerland name aside from in the special case of tetrahaedronland, for which there is both a common word and the coincidence that the number of haedra (faces) equals the number of particles.

2.2 Scaled RPM's

In scaled RPM [92, 75, 79, 83], only relative times, relative angles and relative separations are meaningful. E.g. for 3 particles in dimension $d > 1$, scaled RPM is a dynamics of the triangle that the 3 particles form. Scaled RPM was originally [92] conceived for $\mathbf{q} = \mathbf{q}(N, d) = \mathbb{R}^{Nd}$ and $\mathbf{g} = \text{Eucl}(d)$. One can use $\mathbf{oq}(N, d) = \mathbb{R}^{Nd}/\mathbb{Z}_2$ instead in order to accommodate reflections. Then in each case 'best-matching' involves judging a different thing, but is done by the same method the same way [Fig 2].

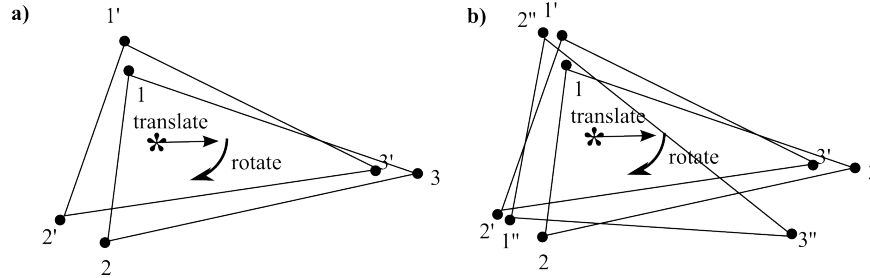


Figure 2: Barbour's demonstration of the concept of best matching. a) In the case of the plain shapes, one keeps one configuration (illustrated here by a triangle, unprimed) fixed, while shuffling the other (primed) around using translations and rotations so as to seek out how to minimize the incongruence between the two. The star is the centre of mass of the primed triangle. b) In the case of the mirror-image-identified shapes, one proceeds likewise, except that one shuffles the second triangle and its mirror image (double-primed) around.

2.2.1 Scaled RPM in terms of particle positions

Let \mathbf{q} be the naïve Nd -dimensional configuration space in d dimensions for N particles, The classical particle mechanics notion of configurational relationalism is implemented by passing to a suitable notion of arbitrary frame. This is achieved by the introduction of a translational auxiliary d-vector \underline{A} and whichever rotational auxiliary corresponds to d . There is none for $d = 1$, a scalar B for $d = 2$ or a 3-vector \underline{B} for $d = 3$. All these cases are encoded by

$$\underline{q}^I - \underline{A} - \underline{B} \times \underline{q}^I . \quad (69)$$

if one allows for $\underline{B} = (0, 0, B)$ in 2- d and $\underline{B} = 0$ in 1- d .

My procedure for constructing actions is that they are to be built as best as possible out of objects that transform well under λ -dependent \mathbf{g} -transformations. I.e. $\partial/\partial\lambda$ acts (and MRI ensures this is λ -independent so equivalently d acts in the MPI form). For scaled RPM, the restriction

$$V = V(\{\underline{q}^I - \underline{q}^J\} \cdot \{\underline{q}^K - \underline{q}^L\} \text{ alone}) , \quad (70)$$

guarantees that the auxiliary corrections straightforwardly cancel each other out within. In examples encountered in practise, these are usually of the form $V(\|\underline{q}^I - \underline{q}^J\|)$ alone. The situation with the kinetic arc element is more complicated in this sense, due to d not being a tensorial operation under d -dependent $\text{Eucl}(d)$ transformations. (This looks more familiar if λ 's are inserted: $\partial/\partial\lambda$ should rather be seen as the Lie derivative $\mathcal{L}_{\partial/\partial\lambda}$ in a particular frame, as per e.g. [593].) This leads to

$$ds_{\text{JBB}}^{\text{ERPM}} = \|d_{\underline{A}, \underline{B}} \underline{q}\|_{\mathbf{m}} \text{ for } d_{\underline{A}, \underline{B}} \underline{q}^I := d\underline{q}^I - d\underline{A} - d\underline{B} \times \underline{q}^I . \quad (71)$$

Finally, the proposed action (a variant of the JBB action) is

$$\mathbf{S}_{\text{JBB}}^{\text{ERPM}} = \sqrt{2} \int \sqrt{W} ds_{\text{JBB}}^{\text{ERPM}} \quad (72)$$

with (70) and (71) substituted into it.

Then the momenta conjugate to the $q_{\alpha A}$ are

$$p_{\alpha I} = m_{I\alpha J\beta} *_{\underline{A}, \underline{B}} \underline{q}^{\beta I} , \text{ for } *_{\underline{A}, \underline{B}} := d/dt^{\text{em}(\text{JBB})}_{\underline{A}, \underline{B}} := \sqrt{2W} d / \|d_{\underline{A}, \underline{B}} \underline{q}\|_{\mathbf{m}} . \quad (73)$$

Then by virtue of the MPI and particular square-root form of the Lagrangian, the momenta obey a primary constraint,

$$\mathcal{E} := \|\mathbf{p}\|_{\mathbf{n}}^2 / 2 + V = E . \quad (74)$$

Note that this is quadratic and not linear in the momenta and that it is physically interpreted as an ‘energy constraint’.

Then variation with respect to \underline{A} gives $\sum_{I=1}^N \underline{p}_I = \underline{C}$, constant, which is 0 at the free end-point (FEP) and thus $\underline{C} = 0$ everywhere. Thus, one obtains

$$\underline{P} := \sum_{I=1}^N \underline{p}_I = 0 \quad (\text{zero total momentum constraint}) . \quad (75)$$

Next, variation with respect to \underline{B} gives $\sum_{I=1}^N \underline{q}^I \times \underline{p}_I = \underline{D}$, constant, which is 0 at the FEP and thus $\underline{D} = 0$ everywhere. Thus

$$\underline{\mathcal{L}} := \sum_{I=1}^N \underline{q}^I \times \underline{p}_I = 0 \quad (\text{zero total angular momentum constraint}) . \quad (76)$$

(Or whichever portion of this that is relevant in the corresponding dimension, i.e. in 1- d there is no $\underline{\mathcal{L}}$ constraint at all, and in 2- d $\underline{\mathcal{L}}$ has just one component that is nontrivially zero: $\mathcal{L} = \sum_{I=1}^N \{q^{I1} p_{I2} - q^{I2} p_{I1}\} = 0$.)

Note 1) These constraints are linear in the momenta.

Note 2) the zero total momentum constraint is interpretable as the centre of mass motion for the dynamics of the whole universe being irrelevant rather than physical. All the tangible physics is in the remaining relative vectors between particles.

Note 3) the zero total angular momentum constraint is interpretable as the absolute angles being irrelevant rather than physical, so that it is in relative angles and relative separations that the physics resides.

The evolution equations are

$$*_{\underline{A}, \underline{B}} p_{I\mu} = -\partial V / \partial q^{I\mu} . \quad (77)$$

The constraints then obey the Poisson bracket algebra [249] whose nonzero brackets are¹⁹

$$\{\mathcal{P}_\mu, \mathcal{L}_\nu\} = \epsilon_{\mu\nu}{}^\gamma \mathcal{P}_\gamma , \quad \{\mathcal{L}_\mu, \mathcal{L}_\nu\} = \epsilon_{\mu\nu}{}^\gamma \mathcal{L}_\gamma . \quad (78)$$

As this is closed, there are no further constraints. These Poisson brackets are just the usual statement that momentum is a vector under rotations and the usual algebra of rotations.

¹⁹1 and 2- d versions are included in the usual sense.

Inverting (73) and applying the configurational relationalism implementing extremization,

$$t^{\text{em(JBB)}} - t^{\text{em(JBB)}}(0) = \underset{\text{of } s_{\text{JBB}}^{\text{ERP}}}{\text{extremum } A, B \text{ of Eucl}(d)} \left(\int \|d_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}} / \sqrt{2W} \right). \quad (79)$$

Note 1) The presence of the $t^{\text{em(JBB)}}(0)$ term here is the familiar freedom of choice of time-origin/‘**calendar year zero**’.

Note 2) One is also free to constantly rescale this time, corresponding to choice of time-unit/‘**tick-duration**’. See Appendix 2.B.3 for how the relational scheme incorporates this feature.

Note 3) Moreover, for the emergent time to be uniquely defined rather than a \mathbf{g} -dependent proto-time, the procedure has to imply some means of freeing itself from dependence on $d\underline{A}$ and $d\underline{B}$. The most obvious such is – illustrated for the subcase of extremization with respect to \underline{B} of the above time-functional – followed by solving the subsequent equation for $d\underline{B}$,

$$W^{-1/2} \partial \|d_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}} / \partial d\underline{B} = 0, \quad (80)$$

and substituting this back into the integral. There is a problem however in general with this choice of extremization, as explained in Fig 3 and backed up by the below GR example.

The alternative is using the extremization of the action; for RPM’s this produces

$$\sqrt{W} \partial \|d_{\underline{A}, \underline{B}} \mathbf{q}\|_{\mathbf{m}} / \partial d\underline{B} = 0, \quad (81)$$

and this is equivalent to (80)[at least away from zeros of W , for which the relational action has problems anyway, c.f. Sec 2.12.5]. This subtlety becomes very necessary in the GR case however, because here the difference between $1/\sqrt{W^{\text{GR}}}$ and $\sqrt{W^{\text{GR}}}$ becomes entangled within the derivative operators that arise ‘by parts’ in the spatial integration, so that the two extremizations produce different equations.

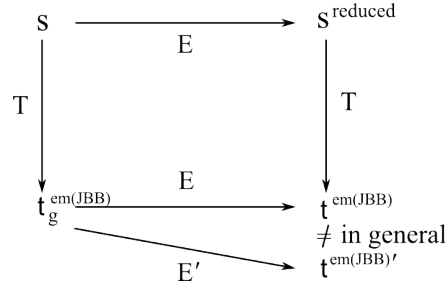


Figure 3: I denote the map from an action to the corresponding emergent (proto)time by T . I denote the map consisting of substituting in the g-extremum of the action by E , and the map consisting of substituting in the g-extremum of the action by E' . Then E and T naturally commute: $TE = ET$, but in general $TE' \neq ET$. This is why I use the g-extremum of S in order to free t of g -dependence.

See Appendix 2.A as regards RPM’s in Hamiltonian form and more relationally meaningful generalizations thereof.

2.2.2 Relative Lagrangian coordinates

C.f. Fig 4 for the $N = 3$ in $2-d$ example of this. N.B. this notion immediately extends to arbitrary N and d .

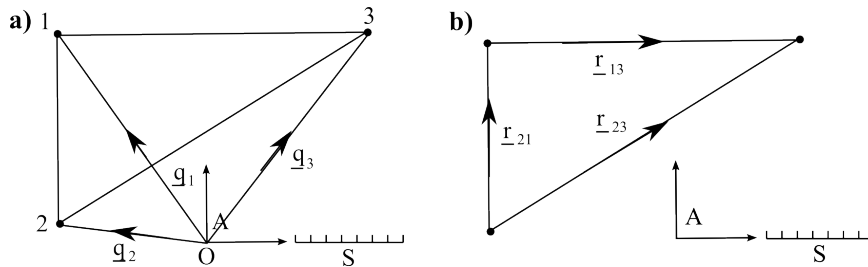


Figure 4: Coordinate systems for 3 particles.

- a) Absolute particle position coordinates ($\underline{q}_1, \underline{q}_2, \underline{q}_3$). These are defined with respect to fixed axes A , a fixed origin O and a fixed scale S .
- b) Relative particle position coordinates $\mathbf{r} = \{\underline{r}^{IJ}, I > J\}$. Their relation to the \underline{q}^I are obvious: $\underline{r}^{IJ} = \underline{q}^J - \underline{q}^I$. For 3 particles, any 2 of these form a basis. No fixed origin enters their definition, but they are still defined with respect to fixed coordinate axes A and scale S .

2.2.3 Scaled RPM in relative Lagrangian coordinates

One renders absolute position irrelevant, by passing to any sort of relative coordinates, leaving one on a configuration space *relative space* = $\mathfrak{r}(n, d) = \mathbb{R}^{nd}$. The most obvious such are the relative Lagrange coordinates. I use that the $d = 3$ working contains everything under the provisos (in Sec 2.2.4), and then comment on each individual case $d = 1, 2, 3$ as these differ significantly. I begin with (72) and eliminate $d\mathbf{A}$ from the Lagrangian form of $\mathcal{P} = 0$

$$d\mathbf{A} = M^{-1} \sum_{I=1}^N m_I \{d\mathbf{q}^I - d\mathbf{B} \times \mathbf{q}^I\} \quad (82)$$

[the Lagrangian counterpart of the Hamiltonian expression (76)]. This results in the (semi-)eliminated ‘Jacobi–Lagrange–Gergely’ [249] action²⁰

$$\mathbf{S}_{\text{J-LG}}^{\text{ERPM}} = \sqrt{2} \int \sqrt{E - V(\mathbf{r}^{IJ} \cdot \mathbf{r}^{KL} \text{ alone})} ds_{\text{J-LG}}^{\text{ERPM}} \quad \text{for} \quad ds_{\text{J-LG}}^{\text{ERPM}2} = \sum \sum_{I < J} \frac{m_I m_J}{M} \|d\mathbf{B} \mathbf{r}^{IJ}\|^2. \quad (83)$$

Note 1) What happens to Barbour’s best matching demonstration in relative coordinates? Given two triangles, instead of leaving one’s vertices fixed and shuffling the other’s around by rigid translations and rotations, one rather 1) lines up the two centres of mass once and for all. 2) One rigidly rotates the relative separation vectors of the second triangle so as to try to minimize the incongruence between the two [Fig 5].

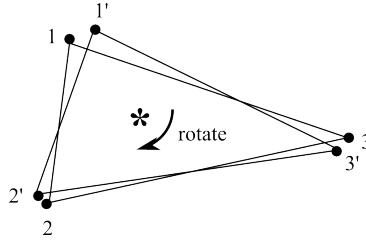


Figure 5: a) The star is now the coincidence of the centre of mass of each system. Shuffling now just involves rotations.

Note 2) As written, these expressions depend on all of the \mathbf{r}^{IJ} ’s rather than on an n -member basis. I first redress this by noting that

$$\mathbf{r}^{IJ}, I = 1 \text{ to } n-1, J = I+1 \text{ form a basis for the relative separation coordinates,} \quad (84)$$

and recasting the previous section’s symmetric but redundant ‘Lagrangian’ expressions asymmetrically yet nonredundantly in terms of these alone [e.g using $\mathbf{r}_{13} = \mathbf{r}_{12} + \mathbf{r}_{23}$].

Note 3) However, once written in terms of a basis subset of the relative Lagrange variables, the kinetic term /arc element is non-diagonal (more widely, the kinetic metric within is not). This hinders progress (e.g. it is more straightforward to envisage classical and quantum separability for some simple potentials with diagonal coordinates). I will get round this by changing to relative Jacobi coordinates in Sec 2.2.4.

Note 4) On the other hand, using relative Lagrange coordinates enables contrast of scaled RPM with the distinct BB77 theory for which these are the original and most natural variables. This is a distinct, non-Newtonian, experimentally refuted and often rediscovered²¹ theory formulated directly in terms of the r^{IJ} and without auxiliaries referring to absolute space. This theory has a kinetic arc element that is exceptionally simple from the Lagrangian perspective,

$$ds^2 = \frac{\delta_{\alpha\beta} \delta_{IK} \delta_{JL}}{|\mathbf{r}^{PQ}|} d\mathbf{r}^{\alpha IJ} d\mathbf{r}^{\beta IK}, \quad (85)$$

which is probably one reason why this theory has been rediscovered so many times. The question then arises whether scaled RPM (and the pure-shape RPM of Sec 2.3) can *also* be cast as a presumably more complicated direct formulation of this kind. Whereupon, not even indirect or unphysical reference to absolute space is required. This is contained in Sec 3.13 in the 1- and 2- d cases. See also Appendix 3.B for further discussion of the BB77 theory.

2.2.4 Relative Jacobi coordinates

Relative Jacobi coordinates are sets of n inter-particle (cluster) separations chosen such that the kinetic term/arc element is diagonal [441, 428].²² Denote the mass matrix alias kinetic metric in the new basis by $\mu_{ij\alpha\beta} = \mu_i \delta_{ij} \delta_{\alpha\beta}$, where μ_i are the Jacobi masses (see below). This has determinant μ and inverse $\nu^{i\alpha j\beta}$. I also use \mathbf{P}_i for the momentum conjugate to \mathbf{R}^i and I for the moment of inertia, $\sum_{i=1}^n \mu_i |\mathbf{R}^i|^2$.

²⁰Here, the total mass $M = \sum_{I=1}^N m_I$.

²¹It was obtained by Reissner (and subsequently rediscovered by Schrödinger and by Barbour–Bertotti [91, 81]). However, this theory was not widely known about, and is incompatible with somewhat more modern mass-anisotropy experiments (e.g. [328, 214]).

²²The more commonly occurring Jacobi coordinate vectors themselves consist of the relative Jacobi coordinate vectors plus one extra centre of mass absolute Jacobi coordinate vector. I use lower-case Latin indices for a basis of relative separation labels 1 to n .

A further useful tidying, particularly convenient at the post-variational level of solving classical and quantum equations, are the mass-scaled relative Jacobi coordinates $\underline{\rho}^i := \sqrt{\mu_i} \underline{R}^i$. I denote the conjugates of these by π_i . The norms of these are then the partial moments of inertia $I^i = \mu_i |\underline{R}^i|^2$. Finally ‘normalized’ mass-scaled relative Jacobi coordinates are obtained by dividing the preceding by $\sqrt{I} = \rho$: $\underline{n}^i = \underline{\rho}^i / \rho$, and ‘normalized’ partial moments of inertia $N^i = I^i / I$. Then $\sum_i \underline{n}^{i2} = \sum_i N^i = 1$. All of these variables are as primary and as natural as the \underline{R}^i . I use \underline{p}_i for the conjugates of the \underline{n}^i .

The first specific nontrivial case for RPM’s has $N = 3$ and so $n = 2$ relative Jacobi coordinates. We take one interparticle separation vector as the first relative Jacobi vector. There are 3 ways of choosing the two particles involved, and an additional $2^2 = 4$ ways to orient the vectors. The second Jacobi vector is from the centre of mass of the first two particles to the third particle; there is no residual freedom at this stage. The total number of ways is thus $3! = 6 =$ number of permutations on 3 objects, while the 3 choices without ascribing an orientation amount to 3 possible different clusterings for 3 particles (i.e. $2 + 1$ partitions into clusters).

Note 1) The two Jacobi vectors thus chosen amount to ‘placing a *Jacobi tree*’ [56] on the constellation of points. N.B. Jacobi trees are *not* like the astronomical constellation figures via having precisely n lines (relative coordinates) and some of the lines emanating from COM’s (the price of diagonality). The name ‘tree’ is well-apt insofar as the Jacobi tree construct does not admit cycles (again unlike constellation figures) whilst always being connected (like constellation figures).

Note 2) The above counting arguments and the form of the tree are both dependent on N but not on d . Moreover, for N particles, the tree has n branches corresponding to the n relative Jacobi vectors.

Note 3) For this simplest case, there is a unique shape of Jacobi tree, which for $d \geq 2$ is a T-shape [Fig 6d)] into which I include the 1- d shape [Fig 6a)] by terming it a ‘squashed T’ (an inclusion and nomenclature I extend to all other types of tree).

Note 4) Each of the T’s is then labelled according to its clustering structure: I use $\{a, bc\}$ read left-to-right in 1- d and anticlockwise in ≥ 2 d , which I abbreviate by (a).²³

Note 5) It is the constellation and not the tree, cluster-label or the shape made by joining the dots that is relational; moreover, we shall see that the choice of tree and of cluster label can have particular *contextual* meaning as regards the regime of study or the propositions being addressed. For now I give as a simple example that if the three particles are the Sun, Earth and Moon, then there is particular significance to having the apex be the Sun (as the Earth and Moon base pair are far more localized) or the Moon (as it is considerably lighter and therefore amenable to treatment as a perturbation).

For $N = 4$, not only are the relative Jacobi coordinates nonunique as regards permutations of the particle labels, but also that there are now two different shapes of Jacobi tree: Jacobi H-coordinates [$2 + 2$ split: Fig 6e)] and Jacobi K-coordinates [$(2 + 1) + 1$ split: Fig 6f)] For the H-coordinates, 1) pick two particles and take the first relative Jacobi vector to be the particle separation vector of these. 2) Take the second relative Jacobi vector to be the particle separation vector of the other two. 3) The third relative Jacobi vector is then the separation vector of the centres of masses of these two clusters. This H-construction can be chosen in 3 ways (number of choices of the $2 + 2$ split), and then there are 8 ways of ascribing the orientations of the 3 vectors (one overall orientation and 2 internal ones). Its labelling is $\{ab, cd\}$, for which I use the shorthand (Hb) For the K-coordinates, step 1) is the same, but then take the second relative Jacobi vector to be between the centre of mass of the first two particles and the third particle. Then the third relative Jacobi vector to be between the centre of mass of this triple cluster and the final particle. This K-construction can be chosen in 4 ways (number of choices of the $3 + 1$ split) and then there are 3 ways to decide how to coordinatize the triple cluster, and then 8 possible ways to orient the 3 vectors. Its labelling is $\{a, bcd\}$ for which I use the shorthand (Ka), or, more precisely picking the clustering within the triple subcluster, $\{a, \{b, cd\}\}$, for which I use the shorthand (Kab).

Note 6) For 4 particles in 2- d , ‘joining the dots’ is no longer unique and which way they are joined is not particularly meaningful. Thus quadrilateralland is in truth rather less about quadrilaterals than triangle-land is about triangles. That is relevant as regards understanding which propositions make particular sense for quadrilateralland.

Note 7) H-coordinates are particularly suited to the study of two pairs of binary stars/diatomics/H atoms, whilst K-coordinates are suited to a binary/diatomic/H atom alongside two single bodies.

The specific forms of the vectors and associated masses are needed for various further applications in this paper, as well as serving as useful examples to ensure the student readers understand the form of the Jacobi construction. For 3 particles, in the (1) cluster [and suppressing the cluster suffixes],

$$\underline{R}_1 = \underline{q}_3 - \underline{q}_2 \quad \text{and} \quad \underline{R}_2 = \underline{q}_1 - \frac{m_2 \underline{q}_2 + m_3 \underline{q}_3}{m_2 + m_3} . \quad (86)$$

²³My general cluster and clustering notation is as follows. I use $\{a...c\}$ to denote a cluster composed of particles a, \dots, c , ordered left to right in 1- d and anticlockwise in 2- d ; I take these to be distinct from their right to left and clockwise counterparts i.e. I consider plain configurations. I insert commas and brackets to indicate a clustering, i.e. a partition into clusters. These notations also cover collisions, in which constituent clusters collapse to a point.

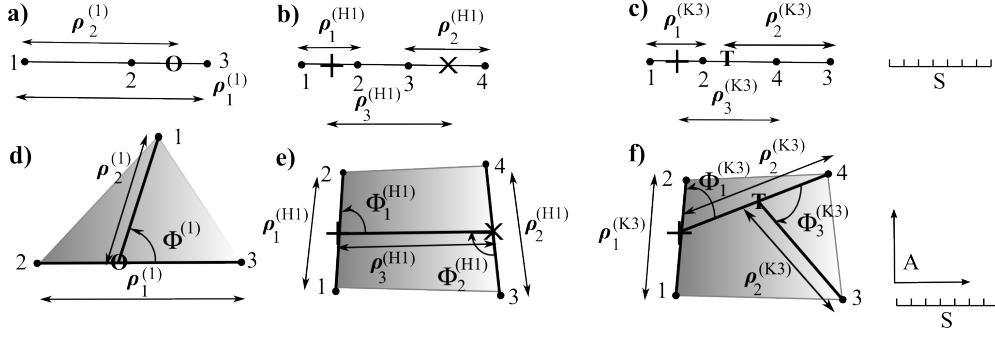


Figure 6: O, +, × and T denote COM(23), COM(12), COM(34) and COM(124) respectively, where COM(ab) is the centre of mass of particles a and b. a) For 3 particles in 1-d, one particular choice of mass-weighted relative Jacobi coordinates are as indicated. For 4 particles in 1-d, b) and c) give, respectively, a particular choice of mass-weighted relative Jacobi H-coordinates and of Jacobi K-coordinates. d) For 3 particles in 2-d, one particular choice of mass-weighted relative Jacobi coordinates are as indicated. I furthermore define $\Phi^{(a)}$ as the ‘Swiss army knife’ angle $\arccos(\rho_1^{(a)} \cdot \rho_3^{(a)} / \rho_1^{(a)} \rho_3^{(a)})$. For 4 particles in 2-d e) and f) give, respectively, a particular choice of mass-weighted relative Jacobi H-coordinates [with $\Phi_1^{(Hb)}$ and $\Phi_2^{(Hb)}$ as the ‘Swiss army knife’ angles $\arccos(\rho_1^{(Hb)} \cdot \rho_3^{(Hb)} / \rho_1^{(Hb)} \rho_3^{(Hb)})$ and $\arccos(\rho_2^{(Hb)} \cdot \rho_3^{(Hb)} / \rho_2^{(Hb)} \rho_3^{(Hb)})$ respectively] and of K-coordinates [with $\Phi_1^{(Ka)}$ and $\Phi_2^{(Ka)}$ as the ‘Swiss army knife’ angles $\arccos(\rho_1^{(Ka)} \cdot \rho_3^{(Ka)} / \rho_1^{(Ka)} \rho_3^{(Ka)})$ and $\arccos(\rho_2^{(Ka)} \cdot \rho_3^{(Ka)} / \rho_2^{(Ka)} \rho_3^{(Ka)})$ respectively]. These are all relative angles: unlike the ρ ’s, they *do not* make reference to axes A or scale S.

The inversion of this (i.e. expressing the \mathbf{r} ’s in terms of the \mathbf{R} ’s) is

$$\mathbf{r}_{12} = -\mathbf{R}_2 - \frac{m_3}{m_2 + m_3} \mathbf{R}_1, \quad \mathbf{r}_{13} = -\mathbf{R}_2 + \frac{m_2}{m_2 + m_3} \mathbf{R}_1 \quad \text{and} \quad \mathbf{r}_{23} = \mathbf{R}_1. \quad (87)$$

The corresponding Jacobi interparticle (cluster) reduced masses μ_i that feature in the diagonal kinetic term are then

$$\mu_1 = \frac{m_2 m_3}{m_2 + m_3} \quad \text{and} \quad \mu_2 = \frac{m_1 \{m_2 + m_3\}}{m_1 + m_2 + m_3}. \quad (88)$$

For the case of equal masses that I commonly use in this article, the above equations reduce to

$$\mathbf{R}_1 = \mathbf{q}_3 - \mathbf{q}_2 \quad \text{and} \quad \mathbf{R}_2 = \mathbf{q}_1 - \{\mathbf{q}_2 + \mathbf{q}_3\}/2. \quad (89)$$

The corresponding Jacobi interparticle (cluster) reduced masses μ_i that feature in the diagonal kinetic term are then

$$\mu_1 = 1/2 \quad \text{and} \quad \mu_2 = 2/3. \quad (90)$$

For the (H2) coordinates,

$$\mathbf{R}_1 = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{R}_2 = \mathbf{q}_4 - \mathbf{q}_3, \quad \mathbf{R}_3 = \frac{m_3 \mathbf{q}_3 + m_4 \mathbf{q}_4}{m_3 + m_4} - \frac{m_1 \mathbf{q}_1 + m_2 \mathbf{q}_2}{m_1 + m_2}, \quad (91)$$

with corresponding cluster masses

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{m_3 m_4}{m_3 + m_4} \quad \text{and} \quad \mu_3 = \frac{\{m_1 + m_2\} \{m_3 + m_4\}}{m_1 + m_2 + m_3 + m_4}. \quad (92)$$

For the case of equal masses that I commonly use in this article, the above equations reduce to

$$\mathbf{R}_1 = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{R}_2 = \mathbf{q}_4 - \mathbf{q}_3, \quad \mathbf{R}_3 = \{\mathbf{q}_3 + \mathbf{q}_4\}/2 - \{\mathbf{q}_1 + \mathbf{q}_2\}/2 \quad (93)$$

with corresponding cluster masses

$$\mu_1 = 1/2, \quad \mu_2 = 1/2 \quad \text{and} \quad \mu_3 = 1. \quad (94)$$

Finally, indeed as announced, T^{ERPM} or ds^{ERPM} are indeed diagonal in these coordinates. Kinetic arc element $ds = \|\dot{\mathbf{q}}\|_{\mathbf{m}}$ maps to $\|\mathbf{dR}\|_{\boldsymbol{\mu}} = \|\mathbf{d\rho}\|$. Thus it looks look very similar to those in particle position coordinates, just involving n instead of N objects. (This amounts to rendering centre of mass motion irrelevant in passing to a relative coordinate description and then the ρ^i are a set of Cartesian coordinates for the relative space.) Thus there is a neat ‘Jacobi map’ $\mathbf{q}^{I\alpha} \rightarrow \mathbf{R}^{i\alpha}$ which will clearly be useful in transcribing the action, Hamiltonian and constraints of RPM’s from particle position expressions to very similar relative Jacobi coordinate expressions. This and the structural uninterestingness of eliminating the translations makes Jacobi coordinates a tidier ‘starting point’. As further substance for this analogy, the total moment of inertia: $\mathbf{I}^{\text{tot}} = \|\mathbf{q}\|_{\mathbf{m}}^2$ maps to the total barycentric moment of inertia $\mathbf{I} = \|\mathbf{R}\|_{\boldsymbol{\mu}}^2 = \|\boldsymbol{\rho}\|^2$. The ‘Euler quantity’ $\mathcal{D} = (\mathbf{q}, \mathbf{p})$ maps likewise to its obvious Jacobi coordinate counterpart $(\mathbf{R}, \mathbf{P}) = (\boldsymbol{\rho}, \boldsymbol{\pi})$ and likewise the angular momentum: $\mathcal{L} = \sum_{I=1}^N \mathbf{q}^I \times \mathbf{p}_I$ to $\sum_{i=1}^n \mathbf{R}^i \times \mathbf{P}_i = \sum_{i=1}^n \boldsymbol{\rho}^i \times \boldsymbol{\pi}_i$. Thus the relative Jacobi coordinates not only diagonalize the kinetic arc element but also have many further properties which are particularly well-suited to the study of RPM’s.²⁴

²⁴I also use $\mathbf{I}^i, \mathcal{D}_i, \mathcal{L}_i$ for the partial counterparts of the above quantities.

2.2.5 Scaled RPM action in relative Jacobi coordinates

As relative Jacobi coordinates are inter-particle (cluster) separations, I take $\mathbf{q} = \mathbf{r}(N, d) = \mathbb{R}^{nd}$, and $\mathbf{g} = \text{Rot}(d)$. Then²⁵

$$\mathbf{S}_{\text{J-JA}}^{\text{ERPM}} = \sqrt{2} \int \sqrt{E - V(\underline{\rho}^j \cdot \underline{\rho}^k \text{ alone})} d\mathbf{s}_{\text{J-JA}}^{\text{ERPM}} \quad (95)$$

$$\text{for } d\mathbf{s}_{\text{J-JA}}^{\text{ERPM}} = ||d\underline{B}\underline{\rho}||, \quad d\underline{B}\underline{\rho}^i := d\underline{\rho}^i - d\underline{B} \times \underline{\rho}^i. \quad (96)$$

Then the conjugate momenta are

$$\pi_{\alpha i} = \delta_{\alpha\beta} \delta_{ij} *_{\underline{B}} \rho^{\beta i}, \quad \text{where } *_{\underline{B}} := d/dt^{\text{em(JBB)}}_{\underline{B}} := \sqrt{2W} d/||d\underline{B}\underline{\rho}|| \quad (97)$$

The surviving constraints are, as a primary constraint,

$$\mathcal{E} := ||\boldsymbol{\pi}||^2/2 + V = E, \quad (98)$$

and, as a secondary constraint

$$\underline{\mathcal{L}} = \sum_{i=1}^n \underline{\rho}^i \times \underline{\pi}_i = 0 \quad (99)$$

from FEP variation with respect to \underline{B} . The evolution equations are

$$*_{\underline{B}} \pi_{i\mu} = -\partial V / \partial \rho^{i\mu}. \quad (100)$$

By applying the Jacobi map, the Poisson brackets are as before other than no longer involving any \mathcal{P}_μ .

The emergent JBB time is now given by

$$t^{\text{em(JBB)}} - t^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{\text{J-JA}}^{\text{ERPM}}}{\text{extremum } \underline{B} \in \text{Rot}(2)} \left(\int ||d\underline{B}\underline{\rho}|| / \sqrt{2W} \right). \quad (101)$$

Note: Barbour's best matching demonstration in terms of relative Jacobi coordinates is presentationally important due to touting Jacobi coordinates as an alternative starting point. This is clearly just a recoordination of Fig 5.

2.3 Pure-shape RPM's

In pure-shape RPM [84, 20, 24], only relative times, relative angles and ratios of relative separations are meaningful. I.e., it is a dynamics of shape excluding size: a *dynamics of pure shape*. E.g. 3 particles in dimension $d > 1$, pure-shape RPM is the dynamics of the shape of the triangle that the 3 particles form. $\mathbf{q} = \mathbf{q}(N, d) = \mathbb{R}^{Nd}$ as before. \mathbf{g} is now $\text{Sim}(d)$.

2.3.1 Pure-shape RPM in particle position coordinates: Barbour's form

[This does not lie within the general format of this article, but can be reformulated as such, c.f. the next SSsec.] Take a MPI action

$$\mathbf{S}_{\text{J-B}}^{\text{SRPM}} = \sqrt{2} \int \sqrt{E - V} d\mathbf{s}_{\text{J-B}}^{\text{SRPM}} \quad (102)$$

$$\text{with } d\mathbf{s}_{\text{J-B}}^{\text{SRPM}} = ||d_{\underline{A}, \underline{B}, c} \underline{\mathbf{q}}||_{\text{m}}/2 \text{ for } d_{\underline{A}, \underline{B}, c} \underline{\mathbf{q}}^A = c\{\underline{\mathbf{q}}^A - \underline{\dot{A}} - \underline{\dot{B}} \times \underline{\mathbf{q}}^A + \{\dot{c}/c\} \underline{\mathbf{q}}^A\}. \quad (103)$$

[I also note that I in $c\mathbf{q}^I$ -coordinates is $c^2\mathbf{I}$ in $\underline{\mathbf{q}}^I$ -coordinates. Moreover, it is also in Barbour's style to not yet make any impositions on the form that the potential is to take.]

The conjugate momenta are then

$$\mathbf{p}_{I\mu} = m_I c \delta_{\mu\nu} \delta_{IJ} *_{\underline{A}, \underline{B}, c} \underline{\mathbf{q}}^{\nu J} \quad \text{where } *_{\underline{A}, \underline{B}, c} := d/dt^{\text{em(JBB)}}_{\underline{A}, \underline{B}, c} := \sqrt{2W} d/||d_{\underline{A}, \underline{B}, c} \underline{\mathbf{q}}||_{\text{m}} \quad (104)$$

Then, as a primary constraint,

$$\mathcal{E} := ||\mathbf{p}||_{\text{n}}^2/2 + V = E. \quad (105)$$

The secondary constraints (75, 76) also hold. Barbour varies with respect to c in a somewhat unusual manner, which he viewed as a toy model of some of the more complicated situation of some of the conformogeometrodynamical approaches in Sec 2.7. However, I disagree [24] with that being a formulation-independent necessity (see the next SSsec). In any case, the outcome is the constraint

$$\mathcal{D} = \sum_{i=1}^N \underline{\mathbf{q}}^i \cdot \underline{\mathbf{p}}_i = 0. \quad (106)$$

Note 1) This is also linear in the momenta and interpreted as zero total dilational momentum constraint.

²⁵Mass-weighting relative Jacobi coordinates does not change the B and C auxiliaries thanks to homogeneous linearity in $\underline{\mathbf{R}}^i, \dot{\underline{\mathbf{R}}}^j$.

Note 2) Barbour then subsequently finds that for the constraints to propagate, $W = E - V$ must be homogeneous of degree -2 . This can be seen to follow from the Poisson bracket

$$\{\mathcal{D}, \mathcal{E}\} = 2\mathcal{E} + \{2W + \sum_{A=1}^N \underline{q}^A \cdot \partial W / \partial \underline{q}^A\} \quad (107)$$

and subsequent lack of propagation unless the right hand side is killed off strongly.

Note 3) The preceding result invalidates actual energy E from occurring in this theory [$E = 0$ in (102) due to otherwise E not being homogeneous of degree -2], by which, alongside the conformogeometro-dynamical counterpart of this [96], Barbour hoped to use to do away with the analogous (and often problematic) notion of cosmological constant.

Note 4) This attitude of Barbour's toward the potential is less restrictive in its assumptions but also renders the treatment less systematic. To date the less restrictive aspect has not given us anything more, while the more systematic scheme is clearer and somewhat faster to do calculations with. The more systematic approach is also based on a more overarching principle that treats kinetic and potential inputs on the same footing as per Sec 1. This ought to be deeper and clearer than just having a rule for correcting velocities/differentials.

Note 5) In pure-shape RPM, the moment of inertia I is a conserved quantity [This follows from (106)].

Note 6) The association of potentials that are homogeneous of degree -2 and of conserved I with scale invariance are well-known elsewhere in the literature (see e.g. [550, 475, 168, 156, 353, 192, 255, 157]). What was new to Barbour's program was the combination of these two well-known elements so as to strike a close analogy with Newtonian physics by using powers of the post-variationally constant I in the potential as some of the powers which have to add up to -2 . See Sec 5 for examples of this. By this means the homogeneity condition on the potential is argued to be not overly restrictive. This is through it permitting e.g. some level of capacity of mimicry of standard potentials such as linear combinations of distinct-power-law potentials, over extensive regimes.

2.3.2 Pure-shape RPM in particle position coordinates: my form

Some additional differences from Barbour's original formulation in my subsequent good $\text{Sim}(d)$ object procedure alias arbitrary $\text{Sim}(d)$ -frame case of arbitrary \mathfrak{g} -frame method are as follows.

1) I employ a different parageodesic principle splitting conformal transformation (PPSCT) representation (explained in Appendix 2.B): the one in which the scale invariance is most natural, through each of \mathbf{T} , $\mathbf{W} = \mathbf{E} - \mathbf{V}$ being homogeneous of degree 0 rather than homogeneous of degree $+2$ and -2 respectively. This also later turns out to be more natural as regards the geometry of the reduced configuration space. Both of the above criteria are clear geometrical naturalities (the former as regards the conformal-geometric meaning of scale invariance itself, while the latter is in terms of obtaining recognizable and natural Riemannian geometry on the reduced configuration space. Barbour's representation, on the other hand, is the mechanically-natural one. In particular, in it the physical dimensions of the various quantities are the usual ones, while in my geometrically natural representation the kinetic term has units of energy/ I while \mathbf{E} and \mathbf{V} have units of energy $\times I$. [However, moving between these and my for of pure-shape RPM's quantities is particularly straightforward as the conversion factor, I , is but a constant in pure-shape RPM]. It is then clear that while E is banished, a new constant \mathbf{E} ('pseudo-energy') of units energy/ I occupies its place, playing a very similar role whereby one does not succeed in banishing cosmological constant type notions by scale-relational thinking, rather just replacing them or transmuting them.

2) The arbitrary $\text{Sim}(d)$ -frame approach takes $\mathbf{W} = \mathbf{E} - \mathbf{V}$ to be homogeneous of degree 0 as part of the starting point rather than a deduction, for it is such that is a good $\text{Sim}(d)$ object.

3) I also use having a new auxiliary C , which is related to [84]'s c by $C = \ln c$. This simplifies the working, which then follows through without any unusual procedures in the variation whatsoever. As such, it largely does away with Barbour's suggestion that pure-shape RPM is a good model of some of the complications found in conformal gravity and the CS + V formulation of GR. On the other hand, Barbour's formulation has useful comparison with the scale-shape split of scaled RPM (c.f. Sec 2.5) and has mechanical/dimensionally natural quantities coincident with scaled RPM's.

Thus my formulation is as follows. The A th particle's position \underline{q}^A is replaced by an arbitrary-frame position

$$\underline{q}^A = \underline{A} - \underline{B} \times \underline{q}^A + C \underline{q}^A. \quad (108)$$

Then the good $\text{Sim}(d)$ object for the potential factor is $W = W(\text{ratios of } \{\underline{q}^A - \underline{q}^B\} \cdot \{\underline{q}^K - \underline{q}^L\} \text{ alone})$. This is free of auxiliary variables by cancellation though the overall homogeneity of degree 0. The nontrivial part of the good $\text{Sim}(d)$ objects construction, paralleling the preceding SSec, is the building of the kinetic term. This is again more straightforward to present in the MRI picture along the lines in [593]: $\partial \underline{q}^{A\mu} / \partial \lambda$ is not in itself a tensorial operation under λ -dependent $\text{Sim}(d)$ transformations. It should rather be seen as the Lie derivative $\mathcal{L}_{\partial \underline{q}^{A\mu} / \partial \lambda}$ in a particular frame. This transforms to the Lie derivative with respect to ' $\partial \underline{q}^{A\alpha} / \partial \lambda$ corrected additively by generators of the translations, rotations and scalings', which gives the combination

$$d_{\underline{A}, \underline{B}, C} \underline{q}^A = d \underline{q}^A - d \underline{A} - d \underline{B} \times \underline{q}^A + d C \underline{q}^A. \quad (109)$$

Then the kinetic arc element is

$$ds_{j-A}^{\text{SRPM}2} = ||d_{\underline{A}, \underline{B}, C} \underline{q}||_{\mathbf{m}}^2 / I. \quad (110)$$

The Jacobi-type action for pure-shape RPM is then

$$\mathbf{S}_{\text{J-A}}^{\text{SRPM}} = \sqrt{2} \int \sqrt{\mathbf{E} - \mathbf{V}(\text{ratios of } \{\underline{q}^I - \underline{q}^J\} \cdot \{\underline{q}^K - \underline{q}^L\} \text{ alone})} d\mathbf{s}_{\text{J-A}}^{\text{SRPM}} . \quad (111)$$

Note: In this formulation, Barbour's triangles demonstration can now be done by using one wooden triangle and an overhead projection image of the compared triangle.

The conjugate momenta are then

$$\underline{p}_A = \delta_{AB} m_A \{\star_{\underline{A}, \underline{B}, C} \underline{q}^B\} / \mathbf{I} , \text{ where } \star_{\underline{A}, \underline{B}, C} := d/dt_{\underline{A}, \underline{B}, C}^{\text{em(JBB)}} = \sqrt{2W} d / \|\underline{d}_{\underline{A}, \underline{B}, C} \mathbf{q}\|_{\mathbf{m}} . \quad (112)$$

[I explain the difference between \star and $*$ in Appendix 2.B.] These obey, as a primary constraint,

$$\mathcal{E} := \mathbf{I} \|\mathbf{p}\|^2 / 2 + \mathbf{V} = \mathbf{E} , \quad (113)$$

which is again quadratic and not linear in the momenta. Again, the momenta additionally also obey as secondary constraints (75), (76), and now also by standard FEP variation of C , a zero dilational momentum constraint (106), the nonzero side of which is aptly nameable as the 'Euler quantity'.

The evolution equations are

$$\star_{\underline{A}, \underline{B}, C} \underline{p}_I = -\partial \mathbf{V} / \partial \underline{q}^I . \quad (114)$$

The scaled RPM Poisson brackets retain the same form in pure-shape RPM, while the new nonzero bracket is

$$\{\mathcal{D}, \mathcal{P}_\mu\} = \mathcal{P}_\mu . \quad (115)$$

I note that \mathcal{L} and \mathcal{D} commute, i.e. the rotations and dilatations do not interfere with each other.

Note 2) The emergent JBB time is now

$$\mathbf{t}^{\text{em(JBB)}} - \mathbf{t}^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{\text{J-A}}^{\text{SRPM}}}{\text{extremum } \underline{A}, \underline{B}, C \in \text{Sim}(2)} \left(\int \|\underline{d}_{\underline{A}, \underline{B}, C} \mathbf{q}\|_{\mathbf{m}} / \sqrt{2W} \right) . \quad (116)$$

2.3.3 Pure-shape RPM in relative Jacobi coordinates

Here $\mathbf{q} = \mathbf{r}(N, d)$ and $\mathbf{g} = \text{Sim}(d)$. Then $d_{\underline{B}, C} \underline{\rho}^i := d\underline{\rho}^i - d\underline{B} \times \underline{\rho}^i + dC \underline{\rho}^i$. The kinetic arc element built from this is

$$d\mathbf{s}_{\text{J-JA}}^{\text{SRPM}2} = \|\underline{d}_{\underline{B}, C} \underline{\rho}\|^2 / \mathbf{I} . \quad (117)$$

$\mathbf{V} = \mathbf{V}(\text{ratios of } \underline{\rho}^i \cdot \underline{\rho}^j \text{ alone})$. The relational action is then

$$\mathbf{S}_{\text{J-JA}}^{\text{SRPM}} = \sqrt{2} \int \sqrt{\mathbf{E} - \mathbf{V}(\text{ratios of } \underline{\rho}^i \cdot \underline{\rho}^j \text{ alone})} d\mathbf{s}_{\text{J-JA}}^{\text{SRPM}} . \quad (118)$$

The conjugate momenta are then

$$\pi_{i\mu} = \delta^{IJ} \delta^{\mu\nu} \star_{\underline{B}, C} \rho_{J\nu} / \mathbf{I} , \text{ for } \star_{\underline{B}, C} := d/dt_{\underline{B}, C}^{\text{em(JBB)}} = \sqrt{2W} d / \|\underline{d}_{\underline{B}, C} \underline{\rho}\| . \quad (119)$$

These obey as a primary constraint

$$\mathbf{I} \|\boldsymbol{\pi}\|^2 / 2 + \mathbf{V} = \mathbf{E} . \quad (120)$$

There is also a secondary zero angular momentum constraint (99) and a secondary zero dilational momentum constraint,

$$\mathcal{D} := \sum_{i=1}^n \underline{\rho}^i \cdot \underline{\pi}_i = 0 . \quad (121)$$

The evolution equations are

$$\star_{\underline{B}, C} \pi_{\mu i} = -\partial \mathbf{V} / \partial \rho^{\mu i} . \quad (122)$$

Exploiting the Jacobi map, the Poisson brackets are as before but with no \mathcal{P}_μ involved anymore. The emergent time is

$$\mathbf{t}^{\text{em(JBB)}} - \mathbf{t}^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{\text{J-JA}}^{\text{SRPM}}}{\text{extremum } \underline{B}, C \in \text{Pl}(2)} \left(\int \|\underline{d}_{\underline{B}, C} \mathbf{q}\|_{\mathbf{m}} / \sqrt{2W} \right) . \quad (123)$$

2.4 Nonrotational RPM

I give this one straight off in my preferred conceptualization of relative Jacobi coordinates and geometrically natural objects. One needs $d > 1$ (i.e. rotational nontriviality) for this to be distinct from SRPM. This theory is a more tractable arena for investigation of implications of scale invariance than Barbour's theory itself.

Here,

$$ds_{J-JA}^{NRPM\ 2} = ||d_C \rho||^2 / I \text{ and } d_C \rho = d\rho_i + dC\rho^i \quad (124)$$

alongside $V = V(\text{a function of ratios of the } \rho^i \text{ components alone})$. Note that there is now no 'dot product' rotational restriction! Then the action is

$$\mathbf{S}_{J-JA}^{NRPM} = \sqrt{2} \int \sqrt{E - V(\text{a function of ratios of the } \rho^i \text{ components alone})} ds_{J-JA}^{SRPM} . \quad (125)$$

The conjugate momenta are

$$\pi_{i\alpha} = \delta_{ij} \delta_{\alpha\beta} \{ * \rho^{j\beta} + * C \rho^{j\beta} \} / I \text{ for } \star_C := d/dt_C^{\text{em(JBB)}} := \sqrt{2W} d / ||d_C \rho|| . \quad (126)$$

These momenta obey energy and dilational momentum constraints as above: (120), (121).

The evolution equations are

$$\star_C \pi_{\mu i} = -\partial V / \partial \rho^{\mu i} . \quad (127)$$

The emergent time is

$$\mathbf{t}^{\text{em(JBB)}} - \mathbf{t}^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{J-JA}^{NRPM}}{\text{extremum } \mathcal{C} \in \text{Dil}} \left(\int ||d_{\underline{A}, \underline{B}, C} \mathbf{q}||_{\mathbf{M}} / \sqrt{2W} \right) . \quad (128)$$

Note 1) Nonrotational RPM is much easier to handle than SRPM in 3- d .

Note 2) Nonrotational RPM suffices to investigate theoretical changes due to considering length ratios and angles alone (which is a theoretically desirable thing to investigate since only ratios of lengths are in practise measured, even if absolute angles still remain in the model).

Note 3) By Note 1), this theory might have some value toward whether scale-invariant physics can reproduce astrophysics and cosmology (to the same kind of extent that Newtonian cosmology can reproduce actual cosmology, see Appendix 5C).

2.5 Scale-shape split of scaled RPM

This is a useful further way of looking at scaled RPM (and is a working original to this article, which may be viewed as a 'missing link' between various of the preceding workings. I begin in q 's because it is valuable to compare this with Barbour's formulation of SRPM. For the mass-normalized coordinates $u^{I\alpha} = \sqrt{m_I} q^{I\alpha}$, under the homothetic ansatz

$$u^{I\alpha} = \sigma s^{I\alpha} \quad (129)$$

such that

$$\mathcal{C} := \sum_{I=1}^N \underline{s}_I \underline{s}^I - 1 = 0 , \quad (130)$$

one has the usual relational action

$$\mathbf{S}_{J-SSS}^{\text{ERPM}} = \sqrt{2} \int \sqrt{W} ds_{J-SSS}^{\text{ERPM}} \text{ with } ds_{J-SSS}^{\text{ERPM}} = ||\sigma d_{\underline{A}, \underline{B}} \mathbf{s} + d\sigma \mathbf{s}||_{\mathbf{M}} . \quad (131)$$

[One could write this as $ds = |d_{\underline{A}, \underline{B}, \sigma}|_{\mathbf{M}}$ with this symbol taking the same meaning as in Barbour's formulation of pure-shape RPM, however the key difference is that σ is *not* now an auxiliary, so the FEP condition does *not* now apply to it. The cross-term is also zero, by the constraint (130) implying $\sum_{I=1}^N \underline{s}_I \underline{s}^I = 0$, by which the kinetic term cleanly splits into shape and scale blocks,

$$ds_{J-SSS}^{\text{ERPM} 2} = d\sigma^2 + \sigma^2 ||d_{\underline{A}, \underline{B}} \mathbf{s}||^2 . \quad (132)$$

Then the conjugate momenta are

$$p_\sigma = * \sigma \text{ and } p_{I\mu}^s = \sigma^2 *_{\underline{A}, \underline{B}} s_{I\mu} . \quad (133)$$

These obey the constraint

$$\sigma p_\sigma = p_{I\mu}^s s^{I\mu} \quad (134)$$

either as a primary constraint from the first form of the action or as an enforcer of (130). They also obey the primary constraint

$$\mathcal{E} = ||p_s||_{\mathbf{M}}^2 / 2\sigma^2 + V = E \quad (135)$$

and the secondary constraints

$$0 = \underline{\mathcal{P}} = \sum_{I=1}^N \underline{p}_I = \sum_{I=1}^N \{ \sigma ds_I + d\sigma s_I \} , \quad 0 = \underline{\mathcal{L}} = \sum_{I=1}^N \underline{s}^I \times \underline{p}_I . \quad (136)$$

Thus the zero total angular momentum constraint is identified as being a pure-shape constraint. Another difference between uneliminated scaled RPM's scale-shape split and pure-shape RPM is that the Hamiltonian now has a multiplier (or cyclic velocity if one applies the almost-Dirac procedure, see Appendix A) that will enter Hamilton's equations. In the Barbour-style approach, (134) with q for σ is replaced by (including FEP variation where now appropriate)

$$p_\sigma = 0, \text{ and so } 0 = \sum_{i=1}^n \underline{q}^i \cdot \underline{p}_i = \mathcal{D} \quad (137)$$

follows. This makes pure-shape RPM and the scale-shape split of scaled RPM quite different at the uneliminated level, with Barbour's formulation of pure-shape RPM being closer than my own one. Moreover, pure-shape RPM and scale-shape split of scaled RPM will be shown to be much more closely linked in the reduced = direct relationalspace formulation. Also, pure-shape RPM can then be seen as a σ -nondynamical version of this, in which (130) is now unnecessary and the $p_\sigma = 0$ form of (134) arises in a new way. I.e. as a secondary constraint from variation with respect to pure-shape RPM's new C -auxiliary. However, the way in which pure-shape RPM is the shape portion of scaled RPM comes out more cleanly in the direct formulation of the next Chapter.

One could alternatively work at the level of the Jacobi coordinates with a scale-shape split

$$\rho^{i\alpha} = \Sigma S^{i\alpha} \text{ such that } \mathcal{C}' := \sum_{i=1}^n \underline{S}_i \cdot \underline{S}^i - 1 = 0, \quad (138)$$

which works out exactly as above except that there is now no momentum constraint (that being automatically incorporated).

Note 1) By the definition of moment of inertia and (130) or (138), the scale Σ here is the square root of moment of inertia $\rho = \sqrt{I}$, which is the dimensionally natural quantity for it to be.

Note 2) If one uses T, V, E instead, then the σ^2 factor in (135)'s kinetic term gets cancelled out, but reappears in the total 'pseudoenergy' and the potential. This is the usual PPSC with $\Omega^2 = I$ (see Appendix 2.B) and permits alignment with the pure-shape RPM convention.

2.6 Geometrodynamics and its relational formulation

2.6.1 Many routes to relativity

General relativity (GR) is often thought of in the spacetime formulation of Einstein's original conceptualization and derivation [220, 221] (supplemented by Vermeil–Weyl–Cartan–Lovelock mathematical simplicity theorems [620, 630, 161, 431]). The corresponding action principle is that of Hilbert and Einstein,

$$\mathbf{S}_{\text{EH}}^{\text{GR}} = \int d^4x \sqrt{|g|} \text{Ric}(g). \quad (139)$$

However, GR can be arrived at along many different routes²⁶

Route 1) is the above.

Route 2) is Cartan's route [162].

Routes 3) and 4) are the 2-way passage between the spacetime and Arnowitt–Deser–Misner (ADM) [60] split space-time formulations in which GR is a dynamics of 3-geometries (see below).

Route 5) is Sakharov's [552], in which GR is the elasticity of space due to particle physics.

Route 6) is Fierz–Pauli's [229], in which GR emerges from consideration of a spin-2 field on Minkowski spacetime.

Note 1) note that there are a number of other (mostly later) programs that could well be considered as routes. See e.g. the below commentary on the 7th and 8th routes. One could also view in this way various programs involving 'beins' or spinors [593, 622, 510] (including Ashtekar variables [62, 607]).

Note 2) Additionally, various (partial or total) unifications lead to aspects of GR (such as the closed relativistic string necessarily containing a spin-2 mode [272]).

The analogies between RPM's and GR are via GR as geometrodynamics and conformogeometrodynamics, so I concentrate on these in this SSec and the next respectively.

²⁶That is the perspective of the famous physicist John Archibald Wheeler [633, 465], who, among other things, was a co-founder of geometrodynamics. This perspective bears some similarities to Poincaré's conventionalism as subsequently championed by Reichenbach [525] and Grünbaum [282]. On the other hand, Kant had long previously argued in denial the diversity of possible scientific points of view. I here comment that physically equivalent formulations can still be *conceptually* different (whether in terms of which philosophy they implement or via which mathematical axioms are needed for such results to exist). As such, two routes turning out to be (partly) equivalent formulations of a single theory constitutes an interesting *theorem*, whether rigorous or heuristic. This is how I view recovery of Newtonian mechanics from BB82 and also how Chapter 3 arrives at the same structures as Chapter 2.

2.6.2 ADM's form of geometrodynamics

For a geometrodynamical formulation of GR [in terms of 3-metrics $h_{\mu\nu}(x^\omega)$ on a fixed topology Σ , for simplicity taken to be compact without boundary; x^ω are spatial coordinates], proceed as follows. Apply the following Arnowitt–Deser–Misner (ADM) 3 + 1 split to the spacetime metric,

$$g_{\Gamma\Delta} = \begin{pmatrix} \beta_\mu\beta^\mu - \alpha^2 & \beta_\gamma \\ \beta_\delta & h_{\gamma\delta} \end{pmatrix}. \quad (140)$$

Use the definition of the extrinsic curvature²⁷

$$K_{\mu\nu} = -\{1/2\alpha\}\{\partial h_{\mu\nu}/\partial t - \mathcal{L}_\beta h_{\mu\nu}\}. \quad (141)$$

Then the above and various geometrical identities, the Einstein–Hilbert action (139) can be rearranged to the ADM action

$$\mathbf{S}_{\text{ADM}}^{\text{GR}} = \int dt \int d^3x \sqrt{h} \alpha \{K_{\mu\nu} K^{\mu\nu} - K^2 + \text{Ric}(h) - 2\Lambda\}. \quad (142)$$

The corresponding manifestly-Lagrangian form of the action is then

$$\mathbf{S}_{\text{ADM-L}}^{\text{GR}} = \int dt \int d^3x \sqrt{h} \alpha \{T_{\text{ADM-L}}^{\text{GR}}/\alpha^2 + \text{Ric}(h) - 2\Lambda\}, \quad (143)$$

for

$$T_{\text{ADM-L}}^{\text{GR}} = \{h^{\mu\rho}h^{\nu\sigma} - h^{\mu\nu}h^{\rho\sigma}\}\{\dot{h}_{\mu\nu} - \mathcal{L}_\beta h_{\mu\nu}\}\{\dot{h}_{\rho\sigma} - \mathcal{L}_\beta h_{\rho\sigma}\}/4. \quad (144)$$

Varying this with respect to α and β^μ respectively gives the GR Hamiltonian constraint (47) and the GR momentum constraint (48), which is interpreted as the physical content residing not in the choice of coordinatization but in the remaining scaled shape information in the 3-metric.

2.6.3 Various interpretations of geometrodynamics

Geometrodynamics has been subjected to a number of interpretations, in great part due to Wheeler.

Interpretation 1) His original attempt as an interpretation, widely regarded as failed [631], was to consider vacuum GR in these dynamical terms as a theory of everything. In later interpretations, matter source terms have been considered a necessity; fortunately, these are manageable.

Interpretation 2) The most standard interpretation, however, is the superspace one as underlying the ADM action. Then $T_{\text{ADM}}^{\text{GR}}$ can be rewritten in the Lagrangian–DeWitt form

$$T_{\text{ADM-L-D}}^{\text{GR}} = ||\partial \mathbf{h}/\partial t - \mathcal{L}_F \mathbf{h}||_{\mathcal{M}}^2/4 \quad (145)$$

Note also that this is in terms of the undensitized GR kinetic metric.

The resulting GR momentum constraint (48) is interpretable geometrically as follows. GR is not just a dynamics of 3-metrics (‘metrodynamics’) but also that moving the spatial points around with diffeomorphisms does not affect the physical content. The remaining information in the metric concerns the ‘underlying geometrical scaled shape’. Thus GR is, more closely, a dynamics of 3-geometries in this sense (a geometrodynamics) on the quotient configuration space superspace(Σ). This description is less (rather than completely non-) redundant (the partial redundancy being due to the Hamiltonian constraint not yet being addressed). Interpreting the Hamiltonian constraint is more problematic.

Interpretation 3) A historically intermediate attempt at interpreting geometrodynamics was the *thin sandwich approach*, in which Baierlein, Sharp and Wheeler [71] regarded $h_{\mu\nu}$ and $\dot{h}_{\mu\nu}$ as knowns. [I.e. the ‘thin’ limit of taking the bounding ‘slices of bread’ $h_{\mu\nu}(\lambda = 1)$ and $h_{\mu\nu}(\lambda = 2)$ as knowns.] This is solved for the spacetime ‘filling’ in between, in analogy with the QM set-up of transition amplitudes between states at two different times [632]. In more detail,

BSW 0) They take as primary a conventional difference-type action for geometrodynamics.

BSW 1) They vary this with respect to the ADM lapse α .

BSW 2) They solve resulting equation for α .

BSW 3) They substitute this back in the ADM action to obtain a new action: the BSW action,

$$\mathbf{S}_{\text{BSW}}^{\text{GR}} = 2 \int dt \int d^3x \sqrt{h} \sqrt{T_{\text{ADM-L-D}}^{\text{GR}} \{\text{Ric}(h) - 2\Lambda\}}. \quad (146)$$

BSW 4) They then take this action as one’s primary starting point.

²⁷The shift vector β^μ is the more common form of representing the $\text{Diff}(\Sigma)$ auxiliary variables. α is the lapse as arising from presupposing the 3 + 1 split of spacetime. See Appendix 2.A.5 for comparison of these with various mathematically aligned but conceptually different notions.

BSW 5) They vary with respect to the shift β^μ to obtain the GR linear momentum constraint $\mathcal{M}_\mu = 0$.
 BSW 6) They posit to solve the Lagrangian form of $\mathcal{M}_\mu = 0$ for the shift β^μ .
 BSW 7) They posit to then substitute this into the computational formula for the lapse α .
 BSW 8) Finally, they substitute everything into the formula for the extrinsic curvature to construct the region of spacetime contiguous to the original slice datum.

Note 1) Unfortunately the p.d.e. involved in this attempted elimination is the *thin sandwich equation*, which is difficult and not much is known about it (obstructions based on this are termed the **Thin Sandwich Problem**). Counterexamples to its solubility have been found [110, 656], while good behaviour *in a restricted sense* took many years to establish (see Bartnik and Fodor [100]).

Note 2) I also give myself the liberty of adding a cosmological constant to the potential factor of this action. This does not significantly change the above procedure or the thin sandwich problem.

Note 3) BSW 0) to 3) are a multiplier elimination; they can be reformulated as a cyclic velocity elimination i.e. a passage to the Routhian (c.f. Appendix 2.A) directly equivalent to the Euler–Lagrange action to Jacobi action move mentioned in the Introduction, or a cyclic differential elimination.

Note 4) BSW 4) to 6) directly parallel Best Matching 1) to 3) modulo the upgrade from multiplier shift to cyclic velocity/differential of the instant. Note that this means that in the geometrodynamics case, Best Matching is hampered in practise by the Thin Sandwich Problem. A general phrasing for this is the obstruction, whether practical-computational or provable-analytical, to

$$\text{solving } \mathcal{L}_{\text{inz}} = 0 \text{ for whichever form of } \mathbf{g}^Z \text{ auxiliaries it contains .} \quad (147)$$

Moreover, ‘sandwich’ is clearly a geometrodynamical term whereas best matching is a far more universal one, so in fact I choose the name **Best Matching Problem** for the general theoretical problem (147) of which the Thin Sandwich Problem is but geometrodynamics’ particular case.

Note 5) They do not consider the culmination move Best Matching 4) of this which (merely formally) produces an action on superspace.

Note 6) BSW 7) is, however, a primitive version of the $\mathbf{t}^{\text{em}(\text{JBB})}$ computation (via the \mathbf{N} to $\dot{\mathbf{I}}$ to $d\mathbf{I} = d\mathbf{t}^{\text{em}(\text{JBB})}$ and hence $\mathbf{t}^{\text{em}(\text{JBB})}$ progression). There is a presupposed (BSW) versus emergent (relational approach) difference in the status of the computed object too.

Note 7) As far as I know, Christodoulou [171, 172] was the first person to cast BSW 7) as an emergent time [differential form of (32), which he termed the ‘chronos principle’]. Like Wheeler, he did not base this on relational first principles. [Though Christodoulou is certainly aware of some of what Barbour and I have held to be relational: “*They contain the statement that time is not a separate physical entity in which the changing of the physical system takes place. It is the measure of the changing of the physical system itself that is time*”]. He deduced the generalization of (147) to include gauge fields minimally-coupled to GR involving shift and the Yang–Mills generalizations of the electric potential [173]. I credit Aleveizos for pointing out this connection between Barbour’s and Christodoulou’s work.

Note 8) BSW 8) is subsequent step concerning dynamical evolution which for geometrodynamics has nontrivial geometrical content.

Note 9) I have thus clearly laid out how the technical content of Barbour’s Best Matching notion resides within a very immediate generalization of the original BSW paper. Wheeler did not, however, have the relational significance that underpins this procedure, or the cyclic auxiliaries formulation of the next SSSec.

Interpretation 4) Next, Wheeler asked [633] why \mathcal{H} takes the form it does and whether this could follow from first principles (‘7th route’ to GR) rather than from mere rearrangement of the Einstein equations. To date, there are two different ‘7th routes’ which tighten wide classes of ansätze down to the GR form (see [21] for a comparison of these).

Route 7A) Hojman–Kuchař–Teitelboim’s from deformation algebra first principles [322, 602] (this assumes embeddability into spacetime). In this approach $\mathcal{O} - \mathcal{L}_\beta$ arises as the *hypersurface derivative* δ_β , so $\mathbf{T}_{\text{ADM-L-D}}^{\text{GR}}$ becomes

$$\mathbf{T}_{\text{HKT}}^{\text{GR}} = ||\delta_\beta \mathbf{h}||_{\mathcal{M}}^2 / 4 . \quad (148)$$

Route 7B) The ‘relativity without relativity’ approach [94, 47, 15, 17, 16, 21], which lies within the relational program, arrives at actions similar to the BSW one, and is the subject of the next two SSSecs.

2.6.4 GR on relational foundations

In good part due to the compellingness of Mach’s arguments, Einstein reconceptualized the nature of space and time, albeit there is considerable confusion as to how GR is and is not Machian. SR transmutes rather than eliminates absoluteness issues – one passes from Newton’s set of unexplainedly privileged notions to another set, the ‘SR inertial frames’. Doing so amounts to a destruction of simultaneity, which is replaced by the universal significance of lightcones. In SR it is often said that space and time can also be regarded as fused into the spacetime of Minkowski. (However, SR breaks only isolation of space and time, not their distinction [142].) Whitrow [635] and Barbour [90] are also dismissive of Minkowski’s considering

the individual notions of time and space to be “*doomed to fade away.*”) Moreover, this is itself a fixed background structure, possessing a timelike Killing vector and privileged spatial frames that precede the introduction of actual material physics into one’s worldview. However, next, in GR matter now exerts an influence on the form of space and time, by which it is in general curved rather than flat as Minkowski spacetime is. While in some senses that has a Machian character, Einstein’s inception of the theory did not concretely build up on Mach’s ideas [632, 76], so whether the theory actually implements these has been a source of quite some argument. (This is not helped by Mach’s ideas being all of multiple, poorly understood and vague as regards concrete mathematical implementations). We shall see below, however, that directly building up on some of Mach’s ideas happens to lead one to (a portion of) GR, arriving to it in its (geometro)dynamical form [81, 76, 94, 21], so that Einstein’s GR does, in any case, happen to contain this philosophically desirable kernel. It should also be mentioned here that GR is, in any case, additionally a theory of gravity, which is both confirmedly more precise than Newton’s and succeeds in fitting in with the rest of physics insofar as it is a relativistic field theory. And, while the role of spacetime in GR has often been touted, one should not forget that GR also admits a dynamical interpretation in terms of evolving spatial 3-geometries, i.e. geometrodynamics.

In this approach to GR, one has to preliminarily choose a residual NOS in the sense of a fixed spatial topology Σ . (See Appendix 14.A for discussion of the relational undesirability of this.) This, I take to be a compact without boundary one for simplicity, and 3- d as this suffices to match current observations. Then the redundant configuration space \mathfrak{q} is chosen to be $\text{Riem}(\Sigma)$ – the space of positive-definite 3-metrics on Σ . This choice of 3-metric objects is a simple one (one geometrical object, rather than more than one), moreover one that is

A) useful for physics given its significance in terms of lengths.

B) Mathematically rich – one then gets a connection for free – the metric connection – as well as the notions of curvature and of Hodge star.

Nevertheless, one is entitled to view this choice as a possibly inessential simplicity postulate.

Next, $\text{Diff}(\Sigma)$ the group of 3-diffeomorphisms on Σ , is a suitable \mathfrak{g} , i.e. the physically irrelevant transformations between the indiscernible 3-metrics that correspond to the same 3-geometries.

The quotient $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$ is called superspace(Σ) [see Sec 3.12 for more]. The arbitrary $\text{Diff}(\Sigma)$ -frame expressions are most easily discussed in the form $\mathcal{O}_{\dot{F}}h_{\mu\nu} := \dot{h}_{\mu\nu} - \mathcal{L}_{\dot{F}}h_{\mu\nu}$ rather than ‘bare’ $\dot{h}_{\mu\nu}$. From these base objects, one can construct the frame-corrected objects of the spatial metric geometry. As these transform well under the 3-diffeomorphisms of the spatial 3-metric geometry, the $\text{Diff}(\Sigma)$ corrections are manifest only as corrections to the metric velocities.

Here, one uses the cyclic velocity of the grid, \dot{F}^μ (the type of frame auxiliary relevant to geometrodynamics) instead of the Lagrange multiplier shift β^μ .

Then $\mathcal{L}_{\dot{F}}$ is the Lie derivative with respect to F^μ , $\mathcal{O}_F := \mathcal{O} - \mathcal{L}_{\dot{F}}$ is the geometrodynamical ‘best-matching’ derivative. This is the same mathematical quantity as the hypersurface derivative δ_β , albeit now conceived of from within the foundational assumption of space alone rather than spacetime, and using a cyclic velocity of the grid rather than a shift multiplier at the level of the Principles of Dynamics. The MPI counterpart is $d_F = d - \mathcal{L}_{dF}$.

The action that one builds so as to implement temporal relationalism is [94, 16, 48, 21]

$$\mathbf{S}_{\text{BFO-A}}^{\text{GR}} = 2 \int d\lambda \int_{\Sigma} d^3x \sqrt{h} \sqrt{T_{\text{BFO-A}}^{\text{GR}} \{\text{Ric}(h) - 2\Lambda\}} = \sqrt{2} \int d\lambda \int_{\Sigma} d^3x \sqrt{h} \sqrt{\text{Ric}(h) - 2\Lambda} ds_{\text{BFO-A}}^{\text{GR}}, \quad (149)$$

$$\text{where } T_{\text{BFO-A}}^{\text{GR}} = \|\mathcal{O}_F h_{\mu\nu}\|_{\mathcal{M}}^2/4 \text{ or } ds_{\text{BFO-A}}^{\text{GR}}{}^2 = \|d_F h_{\mu\nu}\|_{\mathcal{M}}^2/2, \quad (150)$$

$$\text{and } \mathcal{O}_F h_{\mu\nu} = \mathcal{O}h_{\mu\nu} - \mathcal{L}_{\dot{F}}h_{\mu\nu} \text{ or } d_F h_{\mu\nu} = dh_{\mu\nu} - \mathcal{L}_{dF}h_{\mu\nu}. \quad (151)$$

One can also obtain [27] the BFO–A action from another unfamiliar action. This is via

1) using the instant-frame version of the 3 + 1 ADM split of the the spacetime metric,

$$g_{\Gamma\Delta} = \begin{pmatrix} \dot{F}_\mu \dot{F}^\mu - \dot{\gamma}^2 & \dot{F}_\gamma \\ \dot{F}_\delta & h_{\gamma\delta} \end{pmatrix}. \quad (152)$$

See Appendix 2.A.5 for the interpretation of $\dot{\gamma}$ and explanation of its similarities and differences with α , $\dot{\mathbf{I}}$ and N .

2) Using the corresponding ‘instant–frame’ formulation of the extrinsic curvature,

$$K_{\alpha\beta} = -\{1/2\dot{\mathbf{I}}\}\{\dot{h}_{\alpha\beta} - \mathcal{L}_{\dot{F}}h_{\alpha\beta}\}, \quad (153)$$

to obtain, from the Einstein–Hilbert action,

$$\mathbf{S}_A^{\text{GR}} = \int dt \int d^3x \sqrt{h} \dot{\mathbf{I}} \{K_{\alpha\beta} K^{\alpha\beta} - K^2 + \text{Ric}(h) - 2\Lambda\}. \quad (154)$$

Then the corresponding manifestly-Lagrangian form is

$$\mathbf{S}_{L-A}^{\text{GR}} = \int dt \int d^3x \sqrt{h} \dot{\mathbf{I}} \{ \|\mathcal{O}_F \mathbf{h}\|_{\mathcal{M}}^2/4 \dot{\mathbf{I}}^2 + \text{Ric}(h) - 2\Lambda \}. \quad (155)$$

3) The BFO-A action now follows from the A action by using Routhian reduction to eliminate $\dot{\mathbf{I}}$ (an even closer parallel of Jacobi's deparametrization than BSW's lapse multiplier elimination procedure).

Note 1) The BFO-A action bears many similarities to the better-known BSW one, but supercedes it in attaining manifest temporal relationalism.

Note 2) The BFO-A and BSW actions become indistinguishable for minisuperspace, likewise the pair of actions (143, 155). Therein, $\mathcal{M}^{\mu\nu\rho\sigma}(h_{\gamma\delta}(x^\omega))$ collapses to an ordinary 6×6 matrix $\mathcal{M}_{\Gamma\Delta}$ or further in the diagonal case (a 3×3 matrix \mathcal{M}_{AB}) – the ‘minisupermetric’.

Of these actions, $\mathbf{S}_{\text{L-BFO-A}}^{\text{GR}}$ is the one of most interest to the relationalist, so I perform the variation for it below. The conjugate momenta are

$$\pi^{\mu\nu} = \sqrt{h} \mathcal{M}^{\mu\nu\rho\sigma} *_F h_{\rho\sigma} \quad \text{where} \quad *_F := d/dt_F^{\text{em(JBB)}} := \sqrt{2\{\text{Ric}(h) - 2\Lambda\}} d/\|d_F \mathbf{h}\|_{\mathcal{M}}. \quad (156)$$

(46) being MRI, there must likewise be at least one primary constraint, which is in this case the GR Hamiltonian constraint (47) to which the momenta contribute quadratically but not linearly. Variation with respect to F_μ yields as a secondary constraint the GR momentum constraint (48), which is linear in the gravitational momenta. Variation with respect to $h_{\mu\nu}$ provides the usual GR evolution equations (modulo functions of the constraints), which very straightforwardly propagate the above constraints via the Bianchi identities.

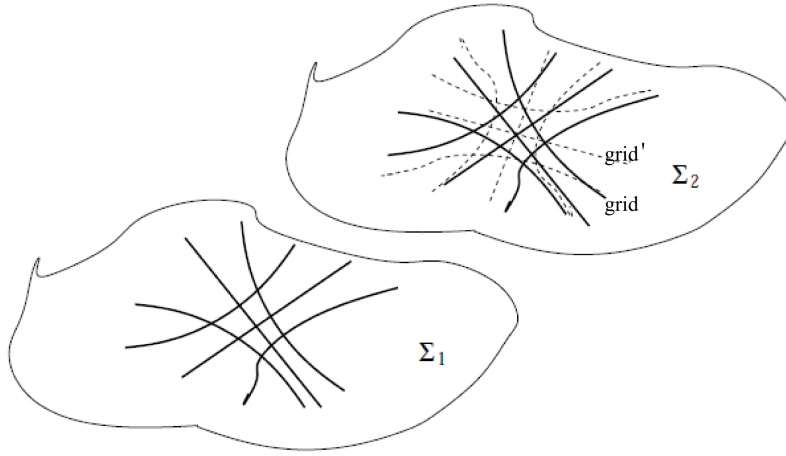


Figure 7: Barbour's construction for 3-metrics: one now shuffles grids on Σ_2 so as to minimize incongruence with Σ_1 's grid.

As for standard ADM, the momentum constraint is interpretable geometrically as GR not being just a dynamics of 3-metrics (‘metro-dynamics’) but furthermore that moving the spatial points around with 3-diffeomorphisms does not affect the physical content of the theory. The remaining information in the metric concerns the ‘underlying geometrical shape’. This is how GR is, more closely, a dynamics of 3-geometries in this sense (a geometrodynamics [633, 201] on the quotient configuration space $\text{superspace}(\Sigma) = \text{Riem}(\Sigma)/\text{Diff}(\Sigma)$). This description is less (rather than completely non-) redundant (the partial redundancy being due to the Hamiltonian constraint not yet being addressed). \mathcal{H} arises in this scheme as a primary constraint and \mathcal{M}_μ as a secondary constraint. In terms of $*$, the evolution equations are the standard ADM ones (modulo functions of the constraints). The constraint algebra closes as per (168). Finally, in this context crediting also Christodoulou [172],

$$t^{\text{em(JBB)}} - t^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{\text{BFO-A}}^{\text{GR}}}{\text{extremum } F \in \text{Diff}(\Sigma)} \left(\int \|d_F \mathbf{h}\|_{\mathcal{M}} / 2\sqrt{W} \right). \quad (157)$$

2.6.5 Relativity without relativity

Furthermore, adopting the relational first principles, *without* assumption of additional features derived in ADM's approach, leads to the recovery of the BFO-A action of GR as one of very few consistent choices within a large class of such actions. E.g. one does not need to assume the GR form of the kinetic metric or of the potential. Relational postulates plus a few simplicities already give this since the Dirac procedure [209] prevents most other likewise-simple choices of kinetic term T^{trial} from working [94, 14, 17, 16, 21, 491, 492]. This involves an action $\text{JBB}[P(\langle \text{Riem}(\Sigma), \mathcal{M}^W \rangle, \text{Diff}(\Sigma))]$ for \mathcal{M}^W the general ultralocal supermetric $\mathcal{M}_{\mu\nu\rho\sigma}^W = h_{\mu\rho}h_{\nu\sigma} - Wh_{\mu\nu}h_{\rho\sigma}$ and a potential ansatz such as $W^{\text{trial}} = A \text{Ric}(h) + B$ for A, B constant.

$$\mathbf{S}_{\text{trial}}^{\text{RWR}} = \sqrt{2} \int \sqrt{W^{\text{trial}}} dS_{\text{quad}}^{\text{trial}}. \quad (158)$$

This relational formulation of GR is furthermore robust to the inclusion of a sufficiently full set of fundamental matter sources so as to describe nature. Thus electromagnetism is included in [94], Yang–Mills theory in [47], spin-1/2 fermions in [15] and

scalar and spin-1/2 fermion gauge theories [15, 17]. Moreover, it does *not* do so in a way by which relational postulates alone *pick out these matter theories*. ([21, 16, 26] demonstrate how earlier claims to the contrary [94] in fact contain additional tacit simplicity assumptions. A further gap, not yet addressed, is that the possibility of second class constraints was not being properly catered for in the existing exhaustion proof.)

One way some of these things are accommodated is through lifting the homogeneous quadraticity restriction [415, 15] (which is not per se relational). In particular this allows for mechanics with linear ‘gyroscopic’ terms, as well as moving charges, spin-1/2 fermions and the usual theories coupling these to gauge fields, scalars and GR. The form of action here is

$$\mathbf{S}_{\text{trial-A}}^{\text{RWR}} = \int \int_{\Sigma} d\Sigma \{ \sqrt{2} \sqrt{\mathbf{W}^{\text{trial}}} ds_{\text{quad}}^{\text{trial}} + ds_{\text{lin}}^{\text{trial}} \} , \quad (159)$$

with $\mathbf{T}_{\text{lin}}^{\text{trial}}$ then not contributing to the quadratic constraint or to the emergent JBB time (thus this is not entirely universal, though the potential associated with such fields does sit inside the square root and thus itself does contribute to these things). $ds_{\text{lin}}^{\text{trial}}$ does however contribute to the linear constraints.

Note 1) This action is clearly MPI and so temporal-relationalism-implementing. Modulo degeneracy issues, it is one of the Finsler-type geometries (in fact a *Randers*-type geometry [523]; he considered the spacetime version of such a geometry as an alternative to GR, whereas I get it in the context of dynamics). And this clearly generalizes to a much wider range of combinations of roots and sums.

Note 2) This also renders clear how to extend Christodoulou’s Chronos Principle working to include fermions.

Other simplicities involve neglecting the possibility of metric–matter cross-terms and other terms that one knows appear in the canonical formulation of more general fields, as well as the restriction on allowed orders of derivatives that is much more ubiquitous in theoretical physics. By such means, some more general/complicated possibilities can be included among the relationally-formulable theories:

- 1) Brans–Dicke theory [94],
- 2) Proca theory [16], and
- 3) local-SR-cone-violating and equivalence principle violating vector–tensor theories [26].

Thus, contrary to the claims/speculations in [94], relationalism does *not* derive or imply the equivalence principle or a universal light cone. (The former effectively requires some form of geometrodynamical equivalence principle postulate [26, 322], while the latter requires a number of nonrelational simplicity assumptions.)

2.7 Relational formulations of conformogeometrodynamics

In the geometrodynamical formulation, GR’s true dynamical degrees of freedom are Wheeler’s “2/3 of superspace”, which his subsequent postdoc Jimmy York worked out the nature of [651, 652, 654, 655]. Conformal superspace: the space of all conformal 3-geometries on a fixed spatial topology, i.e. the result of quotienting $\text{Riem}(\Sigma)$ by the semidirect product of $\text{Diff}(\Sigma)$ and $\text{Conf}(\Sigma)$, the conformal transformations, gives the right count for this whilst being geometrically natural. However do bear in mind that the 2/3 of Superspace that conformal superspace picks out might, however, not be directly related with the 2/3 of superspace picked out by the Hamiltonian constraint itself.

2.7.1 Lichnerowicz’s and York’s conformogeometrodynamics

I refer to routes to and from conformogeometrodynamics as the “8th” routes in accord with earlier discussion of Wheeler’s six routes and the various versions of 7th route. Traditionally, conformogeometrodynamics is viewed as a convenient decoupling leading to substantial mathematical and numerical tractability. Lichnerowicz’s work in this direction [424] is based on maximal slicing (49), which York [652, 654] generalized to CMC slicing (50) $\text{const} = K$ (the trace of the extrinsic curvature, which is also equal to $-\pi/2\sqrt{h}$). This is an important generalization as regards spatially-compact spacetimes, for which maximal slicing cannot be propagated (and important as regards numerical GR too [106, 269]). We will subsequently see that the CMC is proportional to a hidden internal time (York time).

The constraints are decoupled in this formulation because the momentum constraint happens to be conformally covariant so one can solve it irrespective of the subsequent solution of the conformally-transformed Hamiltonian constraint for the physical scale. In the maximal slicing case this equation takes the form

$$8\Delta\phi = \text{Ric}(h)\phi - \pi_{\mu\nu}\pi^{\mu\nu}/\sqrt{h}\phi^7 , \quad (160)$$

which is known as the *Lichnerowicz equation*. In the CMC case, this generalizes to the *Lichnerowicz–York equation*,

$$8\Delta\phi = \text{Ric}(h)\phi - \pi_{\alpha\beta}\pi^{\alpha\beta}/\sqrt{h}\phi^7 + \pi^2/6\sqrt{h}\phi^5 . \quad (161)$$

Maximal slicing is maintained if the lapse solves the maximal lapse-fixing equation (LFE)

$$\Delta N = N \text{Ric}(h) \quad (162)$$

albeit this is frozen for compact spatial topology. CMC slicing is maintained if the lapse solves the CMC LFE

$$2\{N \text{Ric}(h) - \Delta N\} + N\pi^2/2h = \{\pi/\sqrt{h}\} . \quad (163)$$

2.7.2 Foundations for conformogeometrodynamics

This is Route 8B, in parallel to the above route 7B.

Question* Is there any conformogeometrodynamical parallel ‘route 8A’ of the HKT route?

2.7.3 Conformal relationalism: the frozen theory

Consider $\mathfrak{q} = \text{Riem}(\Sigma)$ and $\mathfrak{g} = \text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{Conf}(\Sigma)$ where $\text{Conf}(\Sigma)$ is the group of conformal transformations on Σ . The configuration spaces had been previously studied e.g. in [652, 655, 232, 233]. Then one has somewhat more of a problem than usual in getting ‘good’ $\text{Conf}(\Sigma)$ -objects. Then conformal superspace $\text{CS}(\Sigma) = \text{Riem}(\Sigma)/\text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{Conf}(\Sigma)$. One needs a ϕ such that the metric and it form a simple, internally-conformally-invariant pair

$$\phi \longrightarrow \omega^{-1}\phi, \quad h_{\mu\nu} \longrightarrow \omega^4 h_{\mu\nu} \quad (164)$$

so $h_{\mu\nu}\phi^4$ is $\text{Conf}(\Sigma)$ -invariant and it is out of this that the action is to be built. Writing this is a *local* scale–shape split (since ϕ depends on position). Then such an action is

$$\mathbf{S}_{\text{proto-ABFO}}^{\text{GR}} = \int d \int_{\Sigma} d^3x \sqrt{h} \phi^6 \sqrt{\phi^{-4} \{\text{Ric}(h) - 8\Delta\phi/\phi\}} d\mathbf{s}_{\text{proto-ABFO}}^{\text{GR}} \quad (165)$$

where²⁸

$$\begin{aligned} d\mathbf{s}_{\text{BO}}^{\text{GR}2} = & \{ \phi^{-4} h^{\mu\rho} \phi^{-4} h^{\nu\sigma} - \phi^{-4} h^{\mu\nu} \phi^{-4} h^{\rho\sigma} \} \circ_F \{ \phi^4 h_{\mu\nu} \} \circ_F \{ \phi^4 h_{\rho\sigma} \} = \\ & \{ h^{\mu\rho} h^{\nu\sigma} - h^{\mu\nu} h^{\rho\sigma} \} \{ d_F h_{\mu\nu} + 4h_{\mu\nu} d\phi/\phi \} \{ d_F h_{\rho\sigma} + 4h_{\rho\sigma} d\phi/\phi \}. \end{aligned} \quad (166)$$

(Note that the combination $\{\text{Ric}(h) - 8\Delta\}\phi$ is conformally invariant.)

This action then gives as a primary constraint the Lichnerowicz equation (160), \mathcal{M}_μ as a secondary constraint from F_μ -variation, and the maximal slicing condition (49) from a part of the FESH ϕ -variation. However the other part of this last variation entails frozenness for the CWB Σ ’s of interest (this parallels the shortcoming with Lichnerowicz’s work that York overcame, though I now recast this as a frozen VOTIFE – i.e. a fixing equation for the velocity of the instant $\dot{\mathbf{I}}$ – rather than for the lapse N), and ultimately as a frozen DOTIFE.

2.7.4 Conformal relationalism: the alternative theory

Barbour and Ó Murchadha [96] and ABFO [48] got round this frozenness by considering a new action obtained by dividing the above one by $\text{Vol}^{2/3}$ (a manoeuvre reminiscent of the Yamabe Conjecture [649, 563] familiar from the GR IVP literature). This amounts to again fully using $\mathfrak{g} = \text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{Conf}(\Sigma)$. However the subsequent variational principle no longer gives GR and is furthermore questionable as an alternative theory [48, 16, 17]). ABFKO [49] got round the frozenness in a different way (see the next SSSec).

2.7.5 Conformal relationalism: the recovery of York’s formulation

Instead by using $\mathfrak{g} = \text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{VPCConf}(\Sigma)$, where $\text{VPCConf}(\Sigma)$ are the volume-preserving conformal transformations on Σ as implemented by using

$$\hat{\phi} = \phi \left/ \left\{ \int d^3x \sqrt{h} \phi^6 \right\}^{1/6} \right. . \quad (167)$$

These are associated with the constant mean curvature condition (50). Then $\{\text{CS} + \text{V}\}(\Sigma) = \text{Riem}(\Sigma)/\text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{VPCConf}(\Sigma)$ [49] (which previously featured in e.g. [652]).

Question. $\{\text{CS} + \text{V}\}(\Sigma)$ not to my knowledge yet been studied in detail from a geometrical perspective.

[The $\text{CRiem}(\Sigma) = \text{Riem}(\Sigma)/\text{Conf}(\Sigma)$ analogue of preshape space has been studied geometrically in e.g. [201, 232, 233], in the last of which it is termed ‘pointwise conformal superspace’].

In this scheme the primary constraint (now denoted by $\mathcal{H}_{\hat{\phi}}$) becomes the Lichnerowicz–York equation (161) and FESH ϕ -variation now gives the CMC condition (50) alongside a VOTIFE that successfully maintains this. This scheme involves $\text{JBB}(\langle \text{Riem}(\Sigma), \mathcal{M} \rangle, \text{Diff}(\Sigma) \mathbin{\text{\textcircled{S}}} \text{VPCConf}(\Sigma))$.

Question. The addition of matter to the ABFKO scheme has not to date been extensively studied.

²⁸For simplicity, I present this with no Λ term; see [48] for inclusion of that. Also, they themselves considered this using two separate multipliers instead of a single more general auxiliary whose velocity also features in the action and has to be free end hypersurface varied. The way this is presented here is that of [49] and [43]. See also [27, 43] for justification of the type of variation in use.

[No significant hindrances are known to date [438]. Other at least partially-relevant results are the addition of matter to the Barbour–Ó Murdhadha/ABFO scheme [48] and in the ‘shape dynamics’ program [263, 265].

Question. There are hints that there is *not* a ‘relativity without relativity’ result for conformogeometrodynamics [26]. Can this be verified by working within a fully CS + V picture with general trial actions?

The ‘shape dynamics’ program [97, 264, 266, 263, 150, 386, 265] is somewhat related to this, though it is as yet not clear to me whether this continues to be as well-motivated from a relational perspective.

2.8 RPM’s and GR: 106 analogies and 39 differences

These analogies are my main motivation for studying RPM’s. Take the below to be a summary of those encountered so far, a few new ones that are appropriate at the present position of development in this article. On the other hand, I also refer forward to other Secs for many further ones, due to it being best for the article to lay out some more context before these appear.

2.8.1 Analogies and Differences at the level of GR’s most standard features

Difference 1) RPM’s are not explicitly theories of gravitation. E.g. they are not rigidly restricted to having Newtonian-gravitation-like potentials or a GR cosmology-like dynamical structure.

Analogy 1) RPM’s are, however, explicitly theories of background-independence, which is a feature of GR (and theories beyond GR) emphasized by Einstein, in LQG and in the build-up to M-theory, and which I am arguing to be conceptually important enough that GR is a gestalt theory of gravity and background independence.

Thus RPM’s are valuable in isolating this aspect so as to see what comes of its study in the absence of some of the other complicating features.

Analogy 2) RPM’s are appropriate as toy models for closed system/whole universe physics.

Difference 2) RPM’s do not possess any elements of SR.

This is less debilitating than might be expected, since the NM to SR passage involves trading one set of absolute structures for another; it is GR which frees us of some such (see Sec 11).

Difference 3) RPM’s do not possess a nontrivial notion of spacetime.

However, there is a case that all physics prior to SR was based on dynamics, and this current has continued to run in all physical theories since and might ring truer, pace Minkowski (c.f. Sec 2.6.4). RPM’s do moreover possess a notion of strutting (point identification map and lapse), it is just that this is not furtherly a slicing invariant higher- d geometrical structure like GR spacetime.

Analogy 3) RPM’s are a dynamical formulation paralleling in particular GR’s geometrodynamical formulation.

Analogy 4) RPM’s and GR are both relational theories in this article’s sense that is tied to dynamical formulations.

Difference 4) RPM’s are finite rather than infinite/field-theoretic.

This is in part a difference with benefits due to it rendering RPM’s more tractable than GR, e.g. no regularization issues and less equation well-definedness issues (see Sec 6.6).

On the other hand, it does preclude some interesting aspects; an example for now is that it precludes analogies of the manifestly field-theoretic fifth and sixth routes to GR.

Finally, I comment that in some cases the presumption that field theory is required for a number of further issues to manifest themselves is in fact mythical (see SSSec 11.3.4).

Difference 5) RPM’s do not make contact with the inherently nonlinear nature of GR.

This is not unexpected, since the nonlinearity signifies that gravity gravitates and is thus a feature of this part of the gestalt rather than the background-independent part which RPM’s model.

Analogy 5) Analogies 1), 2) and 3) render RPM’s useful as qualitative models for classical and quantum cosmology.

Analogy 6) RPM’s are furthermore useful thus through having an analogous notion of clumping/inhomogeneity/structure and thus of structure formation.

Difference 6) There is no nontrivial analogue of which I am aware of black holes in RPM’s. [I.e. of the principal arena other than cosmology in which QM and GR can be simultaneously relevant.

2.8.2 Analogies at level of configuration spaces

Analogy 7) In GR-as-geometrodynamics, the incipient straightforward configuration space role of $\mathfrak{t}(N, d)$ ’s is played by the space $\text{Riem}(\Sigma)$ of Riemannian 3-metrics on a fixed spatial topology Σ .

Analogy 8) Both RPM’s and GR are constrained dynamical systems.

Analogy 9) Both can be cast to obey configurational relationalism implemented indirectly by arbitrary \mathfrak{g} –frame corrections.

Analogy 10) The role of $\text{Rot}(N, d)$ ’s as group of irrelevant transformations \mathfrak{g} is played by the 3-diffeomorphism group, $\text{Diff}(\Sigma)$. This analogy goes further through both groups being nonabelian and having nontrivial orbit structure (though, as that requires $d > 2$, this further aspects are beyond the scope of this article’s specific examples).

Moreover, I am not aware of any consequences specifically of nonabelianness of \mathfrak{g} for the Problem of Time. N.B. this is not one of the groups that plays a global role in [332]’s account.

Analogy 11) The role of Dil as a further contribution to the group of irrelevant transformations is played in a certain sense by the conformal transformations $\text{Conf}(\Sigma)$ and in a certain sense by $\text{VPCConf}(\Sigma)$ [49]. This analogy goes further in the sense that these are all scales, though Dil is global. (Here this is global in the sense of pertaining to the whole system rather than to any particular cluster therein.) $\text{Conf}(\Sigma)$ is local and $\text{VPCConf}(\Sigma)$ is ‘local excluding one global degree of freedom’ (the global volume).

Analogy 12) Relational space = $\mathfrak{r}(N, d)/\text{Rot}(d)$ corresponds to superspace(Σ) = $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$ [201, 203, 230].

Analogy 13) Preshape space is the analogue of “ $\text{CRiem}(\Sigma)$ ” [30] alias “pointwise version of $\text{CS}(\Sigma)$ ” [233].

Analogy 14) Shape space = $\mathfrak{r}(N, d)/\text{Rot} \times \text{Dil}$ for analogous to conformal superspace [655, 48]

$\text{CS}(\Sigma) = \text{Riem}(\Sigma)/\text{Diff}(\Sigma) \times \text{Conf}(\Sigma)$ [652, 654, 653, 655, 48, 233].

Analogy 15) Both RPM’s and GR admit scale–shape splits. See Sec 3.8.1 for details about RPM and GR scale variables, and below and Secs 3.6, 3.7, 3.10, 3.12 for details about the shape variables in each case.

Analogy 16) $\text{CS} + \text{V}$ is analogous also to relationalspace (now in scale–shape split form, see Sec 3 for more on this).

2.8.3 Analogies between RPM’s and GR at the level of actions

Analogy 17) Both can be cast to obey temporal relationalism implemented by MRI/MPI in each case.

Analogy 18) There is a relational action (149) for GR-as-geometrodynamics that is the direct counterpart of the indirectly-formulated scaled RPM action in particular.

Analogy 19) There is also a relational $\text{CS} + \text{V}$ action [49] that has further parallels with scale–shape split form of scaled RPM action.

Partial Analogy 20) Also, the conformal gravity action has further parallels with pure-shape RPM; here, homogeneity requirements are taken care of in both by incorporation of powers of some dimensionful quantity. Namely, I in pure-shape RPM and the volume of the universe in an alternative relational theory of conformal gravity [48]. This is not GR, but this action is argued to have a GR application in Sec 14.

Analogy 21) Reading off (3) and (46), energy E and cosmological constant Λ (up to factor of -2) play an analogous role at the level of the relational actions. Thus, to some extent E is a toy model for the mysteries surrounding Λ (c.f. discussion in Sec 2.3, but also contrast Analogy 47).

The way in which that the physical equations follow from the relational action for GR-as-geometrodynamics and from the indirectly formulated RPM actions then have many parallels.

Analogy 22) Dirac’s argument from MRI/MPI and the square root form of the action gives each a primary constraint quadratic in the momenta.

Analogy 23) The ensuing quadratic constraints – the Hamiltonian constraint(47) and the RPM ‘energy constraint’ (42i) – are analogous. This is then the basis of a number of further RPM-GR analogies. N.B. this is temporal relationalism.

Analogy 24) Variation with respect to the auxiliary \mathfrak{g} -variables gives each relational theory constraints linear in the momenta.

Analogy 25) Forms of these: for GR, the momentum constraint $\mathcal{M}_\mu := -2D_\nu \pi^\nu_\mu = 0$ from variation with respect to F^μ , and, for RPM’s, the zero total angular momentum constraint $\underline{\mathcal{L}} = \sum_{i=1}^n \underline{\rho}^i \times \underline{\pi}_i$.

Analogy 26) pure-shape RPM’s zero total dilational momentum constraint, $\mathcal{D} := \sum_{i=1}^n \underline{\rho}^i \cdot \underline{\pi}_i = 0$ closely parallels the well-known GR maximal slicing condition [424, 652, 654], $h_{\mu\nu} \pi^{\mu\nu} = 0$.

Analogy 27) The best matching originally conceived of for RPM’s is essentially the same as a portion of BSW’s approach to GR.

Analogy 28) Both subsequent workings involve their own version of an emergent JBB time, each aligned with a number of other well-known notions of time.

Analogy 29) ‘Dilational’ conjugates to scale quantities are e.g. \mathcal{D} itself (now taken to be just an object – the *Euler quantity* [20] – rather than being a constraint equal to zero) that is conjugate to $\ln \rho$. And e.g. the York quantity [653] $Y = \frac{2}{3} h_{\mu\nu} \pi^{\mu\nu} / \sqrt{h}$ (which is a rescaling and reweighting of the object involved in the maximal slicing condition likewise regarded as a quantity rather than being equal to zero by the slicing condition) that is conjugate to \sqrt{h} . The York quantity is 4/3 times the mean curvature K ; it indexes constant mean curvature (CMC) slices of (regions of) spacetime. This additional geometrical interpretation would appear to have no counterpart in RPM’s.

2.8.4 Some differences between RPM’s and geometrodynamics

Helpful Difference 7) In GR, nontrivial structure and nontrivial linear constraint are tightly related as both concern the non-triviality of the spatial derivative operator. In particular, minisuperspace has neither. However, for RPM’s, the nontriviality of angular momenta and the notion of structure/inhomogeneity/particle clumping are unrelated. Thus, even in the simpler case of 1- d models, RPM’s have nontrivial notions of structure formation/inhomogeneity/localization/correlations between localized quantities. This isolates one of these 2 midisuperspace aspects in the scaled 1- d models, whilst having both in the pure-shape and 2- d models.

Next, the rotations cannot emulate a number of further features that are more specifically of the 3-diffeomorphisms themselves.

Difference 8) The constraint algebras for RPM's (78, 115) does not 'criss-cross' (i.e. the constraints are not integrability conditions for each other), unlike in the Dirac Algebra of geometrodynamics,

$$\begin{aligned}\{\mathcal{M}_\mu(x^\gamma), \mathcal{M}_\nu(y^\gamma)\} &= \mathcal{M}_\mu(y^\gamma)\delta_{,\nu}(x^\gamma, y^\gamma) + \mathcal{M}_\nu(x)\delta_{,\mu}(x, y) , \\ \{\mathcal{M}_\mu(x^\gamma), \mathcal{H}(y^\gamma)\} &= \mathcal{H}(x)\delta_{,\mu}(x^\gamma, y^\gamma) , \\ \{\mathcal{H}(x^\gamma), \mathcal{H}(y^\gamma)\} &= h^{\mu\nu}(x^\gamma)\mathcal{H}_\nu(x^\gamma)\delta_{,\mu}(x^\gamma, y^\gamma) + h^{\mu\nu}(y^\gamma)\mathcal{H}_\nu(y^\gamma)\delta_{,\mu}(x^\gamma, y^\gamma) .\end{aligned}\tag{168}$$

Thus, for GR but not for RPM's, if one has only some of the constraints, one discovers the missing ones as integrabilities [209, 600, 491, 14]. It follows from this that in RPM's temporal and configurational relationalism are separate issues, but in GR they have to be taken together as a package.

Difference 9) The presence of the $h^{\mu\nu}$ factor in the right-hand-side of the Poisson bracket of two Hamiltonian constraints for diffeomorphisms causes these to substantially differ from rotations in the following ways (paraphrasing [600, 601]). One may speak of rotations without saying what it is that is being rotated 'whether it is a black cube, a green cube or a yellow cat'. Furthermore, one may commute two rotations and the result is well-defined in terms of the original rotations and nothing else. However in geometrodynamics, in order to evaluate the commutator of 2 normal deformations, one requires the original deformations and one also has to say that they were acting on a spatial surface with metric $h_{\mu\nu}$.

Difference 10) Also, RPM's also have nothing like the embeddability/hypersurface deformation interpretation of the Dirac Algebra (which is very diffeomorphism-specific). Consequently, RPM's have nothing like the Hojman–Kuchař–Teitelboim [322] first-principles route to geometrodynamics (see also the next SSSec).

Difference 11) There is also no RPM counterpart of the 'relativity without relativity' [94, 21, 26] first-principles route to geometrodynamics.

Analogy 30) One can however view the scale–shape split approach to scaled RPM as paralleling the first-principles route to conformogeometrodynamics ('8th route to relativity') in [49], as per Analogies 26 and 29.

Difference 12) However, this RPM model has no analogue of the velocity-of-the-instant fixing equation (VOTIFE) for RPM's, which limits their usefulness as toy models as regards some of the as yet little-understood features of the scheme.

Further discussion of configuration spaces, RPM's whose scale parts mimic the cosmological scale dynamics, QM, and both facets of and strategies for the Problem of Time each bring out a number of further analogies and differences. Finally, I present a figure in the conclusion that gathers these together into a flowchart of dependency

2.8.5 Many routes to Newtonian Mechanics

Newtonian Mechanics itself can also be considered as lying at the end of many routes. Moreover there are analogies between some of these routes and some of the routes to GR.

NM Route 0) Newton's own route is sui generis rather than pairable with any GR route.

NM Route 1) Cartan's route to Newtonian mechanics bears close parallels to Einstein's traditional route to GR.

NM Route 2) The variational work of Euler and Lagrange bears parallels with the Einstein–Hilbert action and, even more closely, with the ADM–Lagrangian action.

Note 1) Various of the above differences then apply, while scaled RPM parallels BFO-A's route 7B) to GR. Despite the above resemblances, I argue that there is *not* a close parallel between the BB82 route to Newtonian Mechanics and the *tightening* aspects of the relativity without relativity (RWR) and Hojman–Kuchar–Teitelboim (HKT) deformation algebra 'seventh routes' to GR. For, that tightening arises from the restrictiveness of the emergent or assumed constraint criss-cross of GR, while the RPM's have far less constraint interweaving.

Note 2) Also, for scaled RPM, the variant of RWR that Ó Murchadha and I investigated [491, 14] (see also Sec 2.9) fails here. This is because it relies on linear constraints first appearing via being integrabilities of \mathcal{H} , but the analogue \mathcal{E} analogue of this does not criss-cross with the linear constraints.

Note 3) There is some criss-cross in pure-shape RPM but it is not of the right sort to work in this way either. In scaled RPM, the standard RWR technique of using consistency under constraint propagation leads to the result that if one sets \mathbf{V} arbitrary, then one is led to $\mathbf{V}(|\underline{x}^{ab}| \text{ alone})$. In fact, that's how the original BB82 paper proceeds. In pure-shape RPM, the RWR technique furthermore gives the homogeneity property of \mathbf{V} . No more than this can be gleaned. Done the HKT way [regarding in each case a particular algebra, now $\text{Eucl}(d)$ and $\text{Sim}(d)$ respectively, as fundamental and demanding that constraint ansätze close in precisely that way], for the BB82 formulation I find that

$$\sum_{\mathbf{A}=0}^{\infty} f(|\underline{x}^{IJ}|) \mathbf{l}^{I_1 \dots I_A} \mathbf{p}_{I_1} \dots \mathbf{p}_{I_A} \tag{169}$$

survives, while in pure-shape RPM furthermore $f^{I_1 \dots I_A} = \mathbf{l}^{I_1 \dots I_A}(\underline{x}^{IJ}) k(\underline{x}^{IJ})$ for $\mathbf{l}^{I_1 \dots I_A}$ homogeneous functions of degree \mathbf{A} and k a homogeneous function of degree -2 . These represent considerable generalizations of scaled and pure-shape RPM, corresponding to more complicated MRI actions than Jacobi-type actions.

Note 4) One can view the 'Newtonian Mechanics for the island universe subsystems within RPM' account of Appendix 5.C as a relational route to Newtonian Mechanics.

2.9 Further compatibility restrictions between \mathfrak{q} , \mathfrak{g} , actions thereupon and Nature

Relationalism 9) [E.A.]

A) **Nontriviality** \mathfrak{g} cannot be too big [if $\dim(\mathfrak{g}) \geq \dim(\mathfrak{q}) - 1$ then the relational procedure will yield a theory with not enough degrees of freedom to be relational, or even an inconsistent theory].

B) Further **structural compatibility** is required.

Example 1) If one uses such as $\text{Sim}(d_1)$ or $\text{Eucl}(d_1)$ with $\mathfrak{q}(N, d_2)$, it is usually for $d_1 = d_2$ (or at least $d_1 \leq d_2$).

Example 2) If one uses $\text{Diff}(\Sigma_1)$ to match to $\text{Riem}(\Sigma_2)$, it is usually for $\Sigma_1 = \Sigma_2$, or at least for the two to be obviously related (e.g. taking out $\text{Diff}(\mathbb{S}_1)$ from $\text{Riem}(\mathbb{S}^1 \times \mathbb{S}^1)$)

C) More concretely, \mathfrak{g} has to have a (preferably natural) group action on \mathfrak{q} .

Relationalism 10) [Barbour and I] In looking to do fundamental modelling, one may well have a strong taste for **lack of extraneity** (c.f. Sec 1), e.g. always taking out all rotations rather than leaving a preferred axis.

I) $\text{Aut}(\mathfrak{a})$ is then a very obvious designate for this, though some subgroup of $\text{Aut}(\mathfrak{a})$ might also be desirable, and there's the issue of Aut up to which level of structure. Then $\text{Aut}(\mathfrak{a})$ and its subgroups definitely comply with 1) and are wont to comply with the concretization of 2).

II) One may also have a strong taste for eliminating as many extraneous properties as possible [Barbour's view], though under some circumstances there may be strong practical reasons to not eliminate some [my counter-view].

Example 1) One may wish to retain the scale, thus using $\text{Eucl}(d)$ rather than $\text{Sim}(d)$ and $\text{superspace}(\Sigma)$ or $\{\text{CS} + \text{V}\}(\Sigma)$ instead of $\text{CS}(\Sigma)$, so as to have a viable cosmological theory and possibly for reasons of time provision [see Sec 11].

Example 2) Quotienting out $\text{Diff}(\Sigma)$ and $\text{Conf}(\Sigma)$ does not interfere with the existence of spinors, but for further/other choices one has a lack of such guarantees [510], likely compromising one's ability to model spin-1/2 fermion matter. Moreover, it is well-known that some choices of Σ by themselves preclude the existence of fermions (see Sec 3), and that orientability of Σ is also often desired.

In more detail, One has a background NOS \mathfrak{a} , which has properties deemed redundant, all or some of which can be modelled as such. There is then some level of structure $\langle \mathfrak{a}, \mathcal{P}_{\text{modelled}} \rangle$ within the $\langle \mathfrak{a}, \mathcal{P}_{\text{Total}} \rangle = \langle \mathfrak{a}, \mathcal{P}_{\text{modelled}}, \mathcal{P}_{\text{unmodelled}} \rangle$. One then builds the configuration space of actual tangible entities to lean on \mathfrak{a} , so I denote it $\mathfrak{q}(\mathfrak{a})$. Then finally one passes to

$$\mathfrak{q}(\mathfrak{a})/\text{Aut}(\langle \mathfrak{a}, \mathcal{P}_{\text{modelled}} \rangle) . \quad (170)$$

BB82 has a means of modelling continuous transformations at the level of Riemannian geometry. As per Sec 2.1, one can also conceive of \mathfrak{a} in such a way that discrete isometries can be considered as well as continuous ones. If one fails to be able to model $\mathcal{P}_{\text{Total}}$, then the automorphisms in question will not be freeing one of all aspects of \mathfrak{a} . This is clearly the case for $\text{superspace}(\Sigma)$: quotienting out $\text{Diff}(\Sigma)$ from $\text{Riem}(\Sigma)$ clearly does not free one from what choice of Σ one made in setting up $\text{Riem}(\Sigma)$. This is because Σ pertains to the level of topological structure, and rendering the diffeomorphisms irrelevant is only a freeing from background *metric* structure.

The consummate relationalist would then aim to remove all sources of indiscernibility and actors that cannot be acted upon. Taking out $\text{Rot}(2)$ but not $\text{Rot}(3)$ in a 3- d world would make no sense. But if a feature is held to play a role, it is kept. See Sec 16.6.1 as regards the status of scale, and Sec 14 as regards whether spatial topology in GR is discernible and can be acted upon. The objective of relationalism is thus

$$\mathfrak{q}(\mathfrak{a})/\text{Aut}(\langle \mathfrak{a}, \mathcal{P}_{\text{held to be unphysical}} \rangle), \quad (171)$$

or possibly even

$$\mathfrak{q}(\mathfrak{a})/\text{Aut}(\langle \mathfrak{a}, \mathcal{P}_{\text{total}} \rangle) \quad (172)$$

Relationalism 11) [BFO–A]. Note that \mathfrak{q} (the entity taken to have some tangible physical content) has the **a posteriori right to reject** [94, 16, 21, 26] a proposed \mathfrak{g} by triviality or inconsistency that go beyond 1), arising instead via, for the moment, the Dirac procedure yielding further constraints.

Example. If one attempts to use $\mathfrak{g} = \text{id}$ with $\text{Riem}(\Sigma)$ and a BSW-like trial action (like Ó Murchadha [491] or I [14]), one finds that enlarging \mathfrak{g} to $\text{Diff}(\Sigma)$ is *enforced* by the \mathcal{M}_μ arising as an integrability of \mathcal{H} . This also exemplifies how not all subgroups of Aut are guaranteed to serve.

Note 2) Additionally, a given \mathfrak{q} , \mathfrak{g} pair still constitutes a substantial ambiguity as to the form of the action principle. This can sometimes be truncated with simplicity postulates like that the action is not to contain higher than first (or occasionally second) derivatives, but ambiguities beyond that still often remain.

Example 1) The potential is completely free in scaled RPM and free up to homogeneity in pure-shape RPM.

Example 2) One can have GR or Euclidean GR or strong gravity for the $\text{Riem}(\Sigma)$, $\text{Diff}(\Sigma)$ pair with action restricted to contain at most first label-time derivatives and quadratically and at most second spatial derivatives.

I do not for now have anything particularly *relational* to say about Note 2); by the presence of such adjunct conditions, it would appear that relationalism does *not* exert a highly unique control over the form that theoretical physics is to take [though Relationalism 10 and 11) can sometimes help with this].

2.10 Relationalism in other theoretical settings

2.10.1 Relationalism and the Ashtekar variables formulation of GR

Ashtekar variables approaches are also relational in this sense of Barbour’s, modulo that the pure gravity case is not castable in temporally relational form. For, from a nonrelational elimination-of-the-lapse perspective (c.f. the equivalence of difference and product-type actions in Sec 1.4 and the specific working in Sec 2.6.3) this glitch is due to the following. The lapse-uneliminated action is purely linear in the lapse. This is due to the well-known ‘pure-T’ character of Ashtekar’s canonical action for pure GR, in contrast to the more usual ‘ $T - V$ ’ form of the geometrodynamical action. Thus the variation with respect to the lapse produces an equation independent of the lapse. Thus the lapse cannot now be eliminated from its own variational equation. However, it is clear that addition of matter fields breaks this pure linearity in the lapse, so this unusual accident blocking the elimination of the lapse from its own variational equation goes away and a temporally relational action can be obtained. Thus the Barbour-relational literature’s fixation with GR in geometrodynamical form is down to a matter of taste/of each paper’s applications, rather than being in any way being indicative of Barbour-relationalism only applying to the older geometrodynamics and not the newer Ashtekar variables work that underlies LQG. Configurational relationalism for the Ashtekar variables case clearly has its \mathfrak{g} enlarged by an $SU(2)$ as compared to the geometrodynamical case discussed above; variation with respect to the $SU(2)$ auxiliaries produces the $SU(2)$ Yang–Mills–Gauss constraint that is ubiquitous in the Ashtekar variables literature. Sec 2.6.5’s matter configuration and ‘relativity without relativity’ results are, moreover, specific to geometrodynamical-type formulations.

Question* Thus, if one is interested in relational first principles with less structure assumed in Ashtekar variables formulations also, one would need to consider afresh whether analogous results hold in the Ashtekar variables context.

The present article’s main interest however is in RPM’s, for which there are particularly many parallels with the geometrodynamical and conformogeometrodynamical formulations of GR. I note that these are fine for conceptual consideration of the Problem of Time (and are indeed the arena for which such has been most highly developed [400, 335]).

Question** Conduct a full survey of whether Ashtekar variables and the subsequent LQG program substantially changes any of the approaches to the Problem of Time.

2.10.2 Question*: is there a supersymmetric extension of RPM and does it toy-model supergravity?

Supersymmetric particle mechanics exists (see e.g. [239]). Presumably then supersymmetric RPM also exists. Here the indirect formulation’s incipient NOS $\mathfrak{a} = \mathbb{R}^d$ is replaced by a Grassmann space. What are then the sequence of configuration spaces? The appropriate \mathfrak{g} ’s? The isometry groups of the reduced configuration spaces? (See Sec 3 for these for the usual RPM’s.) Does this toy-model the canonical geometrodynamics form of supergravity to any significant extent (e.g. do this article’s Analogies and Differences carry over; and do supersymmetric RPM’s and supergravity theories share more?)

Note 1) This circumvents Sec 2.9 lack of spinors impasse.

Note 2) Also, supersymmetry is conceptually strange in how the product of two supersymmetry operations produces a *spatial* transformation, and possibly also via how the supersymmetry constraint in supergravity is the square root of the Hamiltonian constraint. As such, there is some chance that supersymmetry is one of not conceptually sound, or conceptually deeper than, other transformations, and the relational perspective could be one that had more to say about this matter. As such, *the relational program might be able to provide its own prediction* as to whether to expect supersymmetry in nature.

Note 3) Were supersymmetry to exist in nature it could well impose limitations on applicability of Relationalism 3) and its less-than-phase-space development [45].

2.10.3 RPM’s versus string theory

Aside from the Introduction’s use of the preclusion of perturbative strings to sharpen up criteria for relationalism, there is some conceptual parallel between passing from studying point particles to studying strings and passing from point particles to studying relational quantities only. From a conceptual perspective, these are of comparable in value; whether each of these is rich, tractable and has anything to say about Quantum Gravity is another matter. The latter is moreover more conservative – the relational quantities come from careful thought about the original problem rather than replacing it with a distinct problem as in string theory. This gives less new structure leading perhaps to less mathematical richness, but it it would appear to be a safer bet as regards not making hypotheses (“*hypotheses non fingo*”).

Exporting the configurational relationalism/best matching ideas to other parts of physics could conceivably be of interest even if not accompanied by the further demands of no background spatial structures and of temporal relationalism. E.g. one could consider what form these ideas take for target space theories of which strings on a given background are one example. Finally, one would expect nonperturbative M-theory to have *both* of these features (relationalism *and* extended objects).

Question** I conjecture that the notions of relationalism covered in this article (modulo variants, since some are forks) are

necessary ingredients for any philosophically-convincing and truly GR-embracing notion of background independence,²⁹ and thus that M-theory would, among other things, need to be a relational theory in this sense. Establish or refute this.

2.11 Appendix A. Auxiliary variables: multipliers versus cyclic velocities/differentials

2.11.1 A.1 FENOS variation renders these representations equivalent

Consider for the moment that the auxiliary g^Z variables take their usual guise as multipliers in the generalized sense (coordinates whose velocities do not occur in the action). E.g. one conventionally views in this way the electric potential $\Phi(x^\gamma)$, and the shift $\beta^\mu(x^\gamma)$ and lapse $\alpha(x^\gamma)$ of GR split with respect to spatial hypersurfaces. This is the way that translation and rotation correction variables were viewed in the original BB82 paper [92]. Then variation with respect to the g^Z produces multiplier equations,

$$\nabla L / \nabla g^Z = 0, \quad \text{or} \quad \nabla d\tilde{s} / \nabla g^Z = 0 \quad (173)$$

which are the constraints that one is expecting if one is indeed to obtain thus a gauge theory with gauge group \mathfrak{g} . These constraints then use up both the auxiliary variables' degrees of freedom and an equal number of degrees of freedom from the \mathfrak{q} . So indeed one is left with a theory in which the dynamical variables pertain to the quotient space $\mathfrak{q}/\mathfrak{g}$. (I.e. variables among the original \mathfrak{q} which are entirely unaffected by which choice of \mathfrak{g} -frame is made).

However, this multiplier viewpoint is not the only viable interpretation. One can also use an interpretation as velocities associated with an auxiliary cyclic coordinate. This is possible because of the entirely unphysical nature of auxiliary variables. As it is physically meaningless for these to take a fixed value anywhere, in particular, they must take arbitrary rather than fixed values at the 'end points' of the variation. More precisely, the appropriate type of variation is **free end notion of space (FENOS)** variation (alias *variation with natural boundary conditions*) [185, 236, 140, 415]. This is in particular a portmanteau of *free end point (FEP)* variation for finite theories, and *free end spatial hypersurface (FESH)* variation for field theories. This argument can be viewed as non-entities not acting or as the various end-NOS values producing indiscernible and thus identical physics.

Next, note that FENOS variation involves more freedom than standard fixed end variation, i.e. it involves a space of varied curves of notions of space (which are just varied curves in the FEP subcase) that is larger. It is by this means that it is tolerable for it to impose more conditions than the more usual fixed-end variation does. Then consider configurations $\{Q^A, g^Z\}$ for g^Z entirely arbitrary through being unphysical. With this in mind, I start from scratch, with a more general Lagrangian portmanteau or arclength parameter in which both the auxiliaries g^Z and their velocities \dot{g}^Z in general occur: $L = L[\dot{Q}^A, \dot{g}^Z, Q^A, g^Z]$ or $d\tilde{s}[dQ^A, dg^Z, Q^A, g^Z]$.

Then, because g^Z is auxiliary, variation with respect to g^Z is free, so $\delta g^Z|_{\text{end-NOS}}$ and are not controllable. Thus one obtains 3 conditions per variation,

$$\nabla L / \nabla g^Z = \dot{P}_Z, \quad \text{or} \quad \nabla d\tilde{s} / \nabla g^Z = dP_Z, \quad \text{each alongside} \quad P_Z|_{\text{end-NOS}} = 0. \quad (174)$$

If the auxiliaries g^Z are multipliers m^Z , (174) reduces to

$$P_Z = 0, \quad \nabla L / \nabla m^Z = 0 \quad (175)$$

and redundant equations. I.e. the end-NOS terms automatically vanish in this case by applying the multiplier equation to the first factor of each. This is the case regardless of whether the multiplier is not auxiliary and thus standardly varied, or auxiliary and thus FENOS varied, as this difference translates to whether or not the cofactors of the above zero factors are themselves zero or not. Thus the FENOS subtlety in no way affects the outcome in the multiplier coordinate case.

If the auxiliaries g^Z are cyclic coordinates c^Z , the above reduces to

$$P_Z|_{\text{end-NOS}} = 0 \quad (176)$$

and

$$\dot{P}_Z = 0 \quad \text{or equivalently} \quad dP_Z = 0 \quad (177)$$

which implies that

$$P_Z = C(x^{\mu_p}), \quad \text{invariant along the curve of notions of space.} \quad (178)$$

Then $C(x^{\mu_p})$ is identified as 0 at either of the two end NOS's (176), and, being invariant along the curve of notions of space, is therefore zero everywhere. Thus (177) and the definition of momentum give

$$\nabla L / \nabla \dot{c}^Z := P_Z \quad \text{or equivalently} \quad \nabla d\tilde{s} / \nabla dc^Z = 0. \quad (179)$$

So, after all, one does get equations that are equivalent to the multiplier equations (173) at the classical level. Albeit, there are conceptual and foundational reasons to favour the latter – now that fits in with temporal relationalism as laid out in the Introduction. For previous discussion of this case in the literature, see [84, 48, 16, 49, 27, 43].

Note: see [43] for more on the variational procedure used in conformal gravity and the CS + V formulation, which lies beyond the scope of the present article.

²⁹To avoid confusion, I take care to note here that some other uses of the term 'background (in)dependence' in the string theory literature have a different meaning, namely vacuum-choice independence of string perturbations.

2.11.2 A.2 Multiplier elimination is then equivalent to passage to the Routhian

We also need to establish that the a priori distinct procedures of cyclic velocity elimination (known as Routhian reduction [415]) and multiplier elimination are equivalent. Consider $L[Q^A, \dot{Q}^A, g^Z]$ or $d\tilde{s}[Q^A, \dot{Q}^A, g^Z]$. If these are taken to be multiplier coordinates m^Z , then variation yields $0 = \nabla L / \nabla m^Z$ or $\nabla d\tilde{s} / \nabla m^Z$. If this is soluble for m^Z , one can replace it by $m^Z = m^Z(Q^A, \dot{Q}^A)$, and then substitute that into L . If the g^Z are taken to be velocities corresponding to cyclic coordinates c^Z , then FENOS variation yields

$$C^Z = \nabla L / \nabla \dot{c}^Z \text{ or } \nabla ds / \nabla dc^Z = P^Z = 0 . \quad (180)$$

This is soluble for \dot{c}^Z or dc^Z iff the above is soluble for m^Z . However one now requires passage to the Routhian portmanteau R in eliminating \dot{c}^Z from the action: $\int d\lambda \int_{\Sigma_p} d\Sigma_p R = \int_{\Sigma_p} d\Sigma_p \{L - \dot{c}^Z P_Z\}$ or $\int \int_{\Sigma_p} d\Sigma_p dR = \int_{\Sigma_p} d\Sigma_p \{d\tilde{s} - dc^Z P_Z\}$ But the last term is zero, by (180), so Routhian reduction in the cyclic velocity interpretation is equivalent to multiplier elimination in the multiplier coordinate interpretation.

2.11.3 A.3 (Differential-)Almost-phase spaces, -Hamiltonians and -Dirac procedures

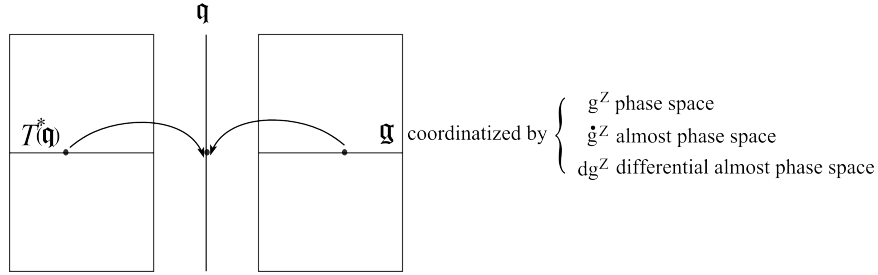


Figure 8: Phase space, almost phase space and differential almost phase space are types of ‘bi-bundle’ as indicated.

The phase space has a Hamiltonian $H(Q^A, P_A, m^Z)$, the almost phase space has an almost-Hamiltonian $A(Q^A, P_A, \dot{c}^Z)$, and the differential almost phase space has a differential almost-Hamiltonian $dA(Q^A, P_A, dc^Z)$. Clearly if one has no \mathfrak{g} the first two reduce to the same, whilst the second and third are MRI and MPI and thus interchangeable by (de)parametrization.

The Dirac procedure involves appending constraints by multipliers in passing to the total Hamiltonian; this is in line with the above \mathfrak{g} -extension of phase space. However, if one wishes to remain temporally relational by MRI, one would append constraints by cyclic velocities in passing to a total almost-Hamiltonian, which I term the almost-Dirac procedure. And, if one wishes to remain temporally relational by MPI, one would append constraints by cyclic differentials in passing to a total (differential-)almost-Hamiltonian, which I term the differential-almost-Dirac procedure.

Likewise, one has now not Hamilton’s equations

$$\nabla H / \nabla p_A = \dot{q}^A , \quad \nabla H / \nabla q^A = -\dot{p}_A , \quad \text{augmented by } \nabla H / \nabla m^Z = 0 , \quad (181)$$

but rather almost-Hamilton’s equations

$$\nabla A / \nabla p_A = \dot{q}^A , \quad \nabla A / \nabla q^A = -\dot{p}_A , \quad \text{augmented by } \nabla A / \nabla \dot{c}^Z = 0 , \quad (182)$$

or the manifestly relational differential-almost-Hamilton’s equations

$$\nabla dA / \nabla p_A = dq^A , \quad \nabla dA / \nabla q^A = -dp_A , \quad \text{augmented by } \nabla dA / \nabla dc^Z = 0 . \quad (183)$$

Example 1) The ERPM total almost-Hamiltonian and differential almost-Hamiltonian are

$$A_{\text{Total}}^{\text{RPM}} = \circ I H + \circ \underline{A} \cdot \underline{P} + \circ \underline{B} \cdot \underline{L} , \quad dA_{\text{Total}}^{\text{RPM}} = dI H + d\underline{A} \cdot \underline{P} + d\underline{B} \cdot \underline{L} . \quad (184)$$

Example 2) In the case of geometrodynamics, the well-known form of the (total) Hamiltonian shared between the ADM and BSW approaches is [209, 60]

$$H_{\text{Total}} = \int d\lambda \int_{\Sigma} d^3x \sqrt{h} \{ \alpha \mathcal{H} + \beta^\mu \mathcal{M}_\mu \} . \quad (185)$$

On the other hand, the A and BFO-A approaches share instead the total (differential-)almost-Hamiltonian

$$A_{\text{Total}} = \int d\lambda \int_{\Sigma} d^3x \sqrt{h} \{ \dot{I} \mathcal{H} + \dot{F}^\mu \mathcal{M}_\mu \} , \quad \text{or } dA_{\text{Total}} = \int \int d^3x_{\Sigma} \sqrt{h} \{ dI \mathcal{H} + dF^\mu \mathcal{M}_\mu \} . \quad (186)$$

The Hamiltonian collapse problem. This article’s relational perspective at the classical level suffers from how, despite the different appearance from the usual of its Lagrangians, both its Lagrangians and the usual approach to physics’ lead to post-variationally equivalent (differential-almost-) Hamiltonians. Hamiltonians being crucial in many ways for the passage

to quantum theory, the classical advances in the understanding of relationalism have a large tendency to *not* directly provide any insights as to how to think relationally at the quantum level.

Note: the *physical* part of these ‘(differential) almosts’ remains unchanged. One is driven to this subtle difference in representation of the unphysical parts by the wish to work with manifestly relational formulations of the Principles of Dynamics.

2.11.4 A.4 One further family of forms for the action

Starting off with a Legendre transformation to the almost-Hamiltonian variables,

$$\mathbf{S} = \int d\Sigma \int \{\dot{\mathbf{Q}}^{\mathbf{A}} \mathbf{P}_{\mathbf{A}} - \mathbf{A}_{\text{Total}}\} d\lambda = \int d\Sigma \int \{\dot{\mathbf{Q}}^{\mathbf{A}} \mathbf{P}_{\mathbf{A}} - \dot{\mathbf{I}} \mathbf{H} - \dot{\mathbf{c}}^{\mathbf{Z}} \mathcal{L}_{\text{InZ}}\} d\lambda = \int d\Sigma \int \{\mathbf{Q}^{\mathbf{A}} \mathbf{P}_{\mathbf{A}} - \mathbf{H} - \mathbf{c}^{\mathbf{Z}} \mathcal{L}_{\text{InZ}}\} d\mathbf{t}^{\text{em(JBB)}} \quad , \quad (187)$$

where the second form is MRI thanks to using the almost-Hamiltonian, the upstairs form is MPI, and the downstairs form is obtained by setting the label time to being the emergent JBB time. The downstairs form (or the same as the downstairs form but with different interpretations pinned on the time-variable) is commonly found in the literature; the above establishes equivalence between this and MRI forms.

As they will be needed in Part III’s path integral type constructions, and a number of the expressions within are well-known, I now specialize the above equation to the GR and ERPM cases.

$$\mathbf{S}_{\text{GR}} = \int d^3x \int \{\dot{h}_{\mu\nu} \pi^{\mu\nu} - \mathbf{A}_{\text{Total}}\} d\mathbf{t} = \int d^3x \int \{\dot{h}_{\mu\nu} \pi^{\mu\nu} - \dot{\mathbf{I}} \mathcal{H} - \dot{\mathbf{F}}^{\mu} \mathcal{M}_{\mu}\} d\mathbf{t} = \int d^3x \int \{h_{\mu\nu} \pi^{\mu\nu} - \mathcal{H} - \mathbf{F}^{\mu} \mathcal{M}_{\mu}\} d\mathbf{t}^{\text{em(JBB)}} \quad , \quad (188)$$

for which the second form’s multiplier counterpart is the commonest (ADM) form in the literature, and the downstairs form coincides mathematically with the proper time formulation.

$$\mathbf{S}_{\text{ERPM}} = \int \{\dot{\underline{\rho}}^i \cdot \underline{\pi}_i - \mathbf{A}_{\text{Total}}\} d\lambda = \int \{\dot{\underline{\rho}}^i \cdot \underline{\pi}_i - \dot{\mathbf{I}} \mathcal{E} - \dot{\underline{\mathbf{B}}} \cdot \underline{\mathcal{L}}\} d\lambda = \int \{\rho^i \underline{\pi}_i - \mathcal{E} - \mathbf{B} \cdot \underline{\mathcal{L}}\} d\mathbf{t}^{\text{em(JBB)}} \quad , \quad (189)$$

for which the downstairs form reinterprets and extends the usual mechanics form for this action. In the specific case of scaled N -stop metroland (needed for Sec 15.1.4), (189) becomes

$$\mathbf{S}_{\text{N-stop ERPM}} = \int \{\dot{\rho}^i \underline{\mathbf{p}}_i - \mathbf{A}_{\text{Total}}\} d\lambda = \int \{\dot{\rho}^i \underline{\mathbf{p}}_i - \dot{\mathbf{I}} \mathcal{E}\} d\lambda = \int \{\rho^i \underline{\mathbf{p}}_i - \mathcal{E}\} d\mathbf{t}^{\text{em(JBB)}} \quad . \quad (190)$$

In the specific case of the indirect presentation of the scaled triangle (needed for Sec 15.1.6), this becomes

$$\mathbf{S}_{\Delta\text{-SRPM}} = \int \{\dot{\underline{\rho}}^i \cdot \underline{\pi}_i - \mathbf{A}_{\text{Total}}\} d\lambda = \int \{\dot{\underline{\rho}}^i \cdot \underline{\pi}_i - \dot{\mathbf{I}} \mathcal{E} - \dot{\mathbf{B}} \mathcal{L}\} d\lambda = \int \{\rho^i \underline{\pi}_i - \mathcal{E} - \mathbf{B} \mathcal{L}\} d\mathbf{t}^{\text{em(JBB)}} \quad . \quad (191)$$

2.11.5 A.5 On \mathbf{N} , $\dot{\mathbf{I}}$, α and $\dot{\gamma}$

These are four symbols for the ‘same’ quantity albeit each rests on different foundations. The last two expressions come from presupposing spacetime and performing on it the ADM split and my parallel split (20) respectively. These are called the lapse and the velocity of the instant (VOTI). The next two come from relational approaches: the non-manifestly temporally relational BSW formulation and the manifestly temporally relational BFO-A one. They are, respectively, the emergent lapse and the emergent VOTI; moreover, in these formulations, these quantities come with a common computational formula that is not present for the other two (unless one performs manoeuvres that transcend to the BSW or BFO-A action). Finally, the labeller of the instants is identifiable by comparison with the standard ADM formulation and its interpretation as being the proper time of the constituent elements (taken to include an idealization of observers where relevant) on the instants in question. This makes good physical sense. $\gamma = \tau$ is particularly identifiable thus, with $\dot{\gamma}$ then being the derivative of proper time with respect to coordinate time, whereas $\dot{\mathbf{I}}$ is then to be viewed as the emergent version of this derivative.

The lapse has long been known to be of this nature, the α to $\dot{\gamma}$ difference reflecting rather a difference in what is to be varied: the lapse itself or the instant that sits within the definition of the lapse. This was not considered prior to the relational

program probably because of the need for subtlety in the variational procedure in the second case representing a major jump-up from the first case, in which that subtlety is absent. On the other hand, whether to use \dot{I} or N is but a difference in naming this combination that is widespread throughout the results of the variational procedure. The \dot{I} name is then the more consistent one given the relational approach's obligation to use \dot{F}^μ rather than the manifest temporal-relationalism-breaking β , and also to reflect the homogeneous factor of $1/d\lambda$ in the emergent object.

In parallel, for RPM's I is identifiable as Newtonian time on an emergent footing, with \dot{I} then being the derivative of this by label time. This fits in that GR is already-MRI so its coordinate time has label status. The general case has

$$\dot{I} = d(\text{instant-labelling time})/d(\text{meaningless label parameter time}) \quad (192)$$

The instant-labelling time has special status by being a property of the instants rather than being just a mere parametrization of the 'time-line' itself. However, it is not unique in that sense, e.g. scale time is a property of instants, so there is more to it than that. Proper time is very observer-associated, though more broadly it is actual of fictitious material blobs that it is attached to. I consider it interesting that Newtonian and proper times have a common origin from the current perspective; I would not claim to fully understand why, though proper time is, locally an SR concept, and I have already commented on the conceptual closeness of SR and Newtonian Mechanics from the absolute versus relational perspective (Sec 2.6.4).

In conventionally-formulated GR, the lapse is defined as a function $\alpha(t, x^\mu)$ that arises as part of the ADM split.

$$\alpha(x^\mu, \text{coordinate time}) := d(\text{proper time})/d(\text{coordinate time}) . \quad (193)$$

Note that in field theories this is not like a coordinate time due to its spatial dependence. That makes lapse a partly conceptually undesirable quantity, unlike the DOTI, which ends up being the conceptually desirable proper time notion. Only then does consideration of the Lagrangian form of the equation for its variation produce a computational formula for it, so that the functional form of that formula has no bearing on the definition of the quantity. This is also all true of the $\dot{\gamma}$ in my parallel approach.

On the other hand, in the relational approach, the computational formula for \dot{I} comes first. For this to coincide with the existing concepts, one then also needs to stipulate (or have some other primary reason) why to supplement this with the demand that this quantity be treated as a function of GR label time and spatial position alone, rather than as the functional which the computational formula would seem to imply. One possibility here, though it remains tentative in my opinion, is that this demand come from the expectation that the t^{em} quantity being simultaneously introduced be able to serve as a privileged label via having the same functional dependence as the general label-time. Only then at the end is the spacetime metric and subsequent spacetime Riemannian geometry reconstructed including identification of \dot{I} among the components of this reconstructed spacetime metric.

Finally, in the MPI relational presentation, one uses the DOTI dI (differential of the instant) in place of the VOTI \dot{I} . This can be identified as an emergent version of the differential of the proper time or of the Newtonian time in the GR and mechanics contexts respectively.

2.12 Appendix B: Parageodesic principle split conformal transformations (PPSCT's)

2.12.1 B.1 Jacobi's Principle: geodesic principle versus parageodesic principle

The physical quantity is $d\tilde{s}$. In terms of this, Jacobi's principle is a **geodesic principle** for some geometry (a Riemannian one in Jacobi's own setting, though readily extendible to arbitrary-signature Riemannian geometry, and to, following Synge, Finslerian geometry; I additionally mean each of these in a finite-infinite portmanteau sense). In the (arbitrary-signature) Riemannian case, the equations of motion are indeed here of geodesic form,

$$\tilde{D}_{\text{abs}}^2 q^A = \tilde{*} \tilde{*} q^A + \tilde{\Gamma}^A_{BC} \tilde{*} q^B \tilde{*} q^C = 0 . \quad (194)$$

Two difficulties with this perspective are as follows.

- 1) This is entirely on a case-by-case basis: one requires a different geometrization for each system.
- 2) Specific models come in families. Restricting for now to the (arbitrary-signature) Riemannian case, all the possible potential factors W 's for a given configuration space's material contents; the configuration space may well then have a natural W -independent kinetic line-element ds such that

$$d\tilde{s} = w ds \quad (195)$$

for weight function $w = \sqrt{2W}$.

It would then often be desirable to use this geometrization. In terms of such a geometrically-natural ds , Jacobi's Principle is a **parageodesic principle**, the equations of motion now being (23). However, geometrical naturality (and, even more widely) convenience, need not be uniquely defined. Thus if one splits $d\tilde{s}$, how one splits it is nonunique.

$$d\tilde{s} = \{w/\Omega\} \{\Omega ds\} \quad (196)$$

will also do. I term this a **parageodesic principle splitting conformal transformation (PPSCT)**, the first factor being an ordinary conformal transformation of the metric arc element, and the second factor being the corresponding *compensatory transformation* of the potential factor.³⁰ This occurring for GR and RPM's constitutes Analogy 31)

2.12.2 B.2 PPSCT-tensorialities of the objects of the Principles of Dynamics

1) Squaring and assigning the transformation to the (arbitrary-signature) Riemannian kinetic metric within the ds^2 , the transformation is

$$M_{AB} \longrightarrow \overline{M}_{AB} = \Omega^2 M_{AB} \quad , \quad W \longrightarrow \overline{W} = W/\Omega^2 \quad . \quad (197)$$

2) I adopt the (for metric geometry, very usual) convention that the basic transformation involves 2 powers of the conformal factor Ω . Then the kinetic metric is a PPSCT-vector and the potential factor is a PPSCT-covector (and so, overall, the product-type parageodesic principle action is PPSCT-invariant).

3) Moreover, $*$ is then a PPSCT-covector [immediately from its formula (20)]; this means it is in fact highly nonuniquely defined, at least at this stage in the argument.³¹

4) Clearly from the invariance of the action, performing such a transformation should not (and does not) affect the physical content of one's classical equations of motion.

5) If one then considers the emergent timefunction's PPSCT-vectoriality to carry over to the difference-type action formulations' timefunction (including to the Newtonian/proper time *interpretation* usually pinned upon these timefunctions therein), then one has unravelled a more complicated manifestation of the conformal invariance for difference-type actions too.

$$\begin{aligned} \overline{\mathbf{S}} &= \int \int_{\Sigma} d\Sigma \{ \overline{T} - \overline{V} \} d\overline{\mathbf{t}} = \int \int_{\Sigma} d\Sigma \{ \overline{M}_{AB} \overline{*}_g Q^A \overline{*}_g Q^B / 2 - \overline{V} \} d\overline{\mathbf{t}} = \int \int_{\Sigma} d\Sigma \{ \Omega^2 M_{AB} \Omega^{-2} *_g Q^A \Omega^{-2} *_g Q^B / 2 - \Omega^{-2} V \} \Omega^2 d\mathbf{t} \\ &= \int \int_{\Sigma} d\Sigma \{ M_{AB} *_g Q^A *_g Q^B / 2 - V \} d\mathbf{t} = \int \int_{\Sigma} d\Sigma \{ T_t - V \} d\mathbf{t} = \mathbf{S} \quad . \end{aligned} \quad (198)$$

I.e., one has a *3-part conformal transformation* $(\mathbf{M}, W, *) \longrightarrow (\overline{\mathbf{M}}, \overline{W}, \overline{*}) = (\Omega^2 \mathbf{M}, \Omega^{-2} W, \Omega^{-2} *)$. [Note that in the case of the relational formulation, the PPSCT of \mathbf{M} , W directly implies the 3-part conformal transformation, so that these are not then distinct entities, but the 3-part conformal transformation is a distinct entity if one *does not* presuppose relational foundations.] Working through how the scaling of \mathbf{M} , W and the timefunction \mathbf{t} conspire to cancel out at the level of the classical equations of motion reveals interesting connections (Sec 12) between the simplifying effects of using the emergent timefunction on the equations of motion and those of the rather better-known affine parametrization [622, 593].

6) The evolution equations of motion following from a product-type action (29 in 3) will clearly be invariant under the PPSCT, as the action that they follow from is.

7) The conjugate momenta are PPSCT-invariant:

$$\overline{P}_B = \overline{M}_{AB} \overline{*}_g Q^A = M_{AB} *_g Q^A = P_B \quad . \quad (199)$$

Thus PPSCT concerns, a fortiori, configuration space rather than phase space.

8) As N^{AB} is the inverse of M_{AB} , it scales as a PPSCT-covector,

$$N^{AB} \longrightarrow \overline{N}^{AB} = \Omega^{-2} N^{AB} \quad . \quad (200)$$

Then (197ii), (199) and (200), $Quad$ is a PPSCT-covector.

9) In cases with nontrivial configurational relationalism (for which the above arc elements and stars, and thus momentum-emergent-time-velocity relations and evolution equations, are \mathfrak{g} -corrected), there are also linear constraints from variation with respect to \mathfrak{g} -auxiliaries. Then by the pure linearity in the momenta of these as manifested in (31) and by 6), these are conformally invariant.

10) The total Hamiltonian is PPSCT-invariant by N and $Quad$ vector-covector cancellation and neither \mathfrak{m}^Z nor \mathcal{L}_{inZ} scaling in the first place. The same shape of argument immediately apply also to the total (differential) almost-Hamiltonian.

11) Finally, the Liouville operator $P_A \dot{Q}^A$ is PPSCT-invariant, so that the last SSsec of the previous Appendix's form for the action is also indeed PPSCT-invariant.

2.12.3 B.3 Further applications

1) The constant conformal factor case gives the tick-duration freedom: \mathbf{t}^{em} to $k^2 \mathbf{t}^{em} = \mathbf{t}_k^{em}$, so

$$\mathbf{t}_k^{em} - \mathbf{t}_k^{em}(0) = k^2 \int ds / \sqrt{2W} \quad (201)$$

³⁰I used to term this a banal conformal transformation [32], but now have reasons to believe that the most fundamental setting for it is the parageodesic principle splitting, so I now name it after that.

³¹To not confuse ' $\mathbf{t}^{em(JBB)}$ ' as present in the previous literature' [79, 22] and the PPSCT-covector discovered in [32] and explained in this Appendix, I denote the latter by $\overline{\mathbf{t}}$. One can also think of this as the emergent lapse, VOTI and DOTI scaling as PPSCT-vectors.

for $\mathbf{t}_k^{\text{em}}(0) = k^2 \mathbf{t}^{\text{em}}(0)$.

2) The evolution that follow have two simplifications for or particular choices of parameter that are generally different (the two coincide if the potential is constant).

Simplification A) use of emergent time.

Simplification B) use of geodesic rather than parageodesic form, the former corresponding to ‘the dynamical curve being an affinely-parametrized geodesic on configuration space’. In this case I denote the timefunction by $\mathbf{t}^{\text{aff-geo}} = \mathbf{t}^{\text{geom}}$, the latter name coming from the shape-space geometrical naturality of this time.

3) There is a link to the affine transformations of geodesic equations expositied in Sec 12.

4) PPST’s serve to set up the foundations for one of the article’s

Note: considering 3) and 4) together amounts to following Misner’s Hamiltonian treatment [464], now taking it further in tracing the variational principle origin of his conformal transformation of the Hamiltonian constraint back to the relational product-type Jacobi parageodesic principle.

2.12.4 B.4 Further notation for this article’s PPST representations

For scaled RPM, the PPST representations are centred about the mechanical representation $\Omega = 1$ for $M_{\text{AB}}, E - V$; the corresponding time is t_{mech} , usually abbreviated to t . The scaled triangleland flat representation has $\Omega^2 = 4I$ and objects in this representation are denoted by checks. Occasionally I also use $\Omega^2 = I$, which I denote by bars.

For pure-shape RPM, I choose to centre the PPST representations about the geometrically-natural representation, so that $\Omega = 1$ for M_{AB} and $E - V$; the corresponding emergent time is \mathbf{t}^{geom} , usually abbreviated to \mathbf{t} . I then need to use the five-pointed star \star for its derivative to avoid confusion with ERPM’s six-pointed star $*$ which is distinct due to the difference in PPST representation-centring conventions. In SRPM, the mechanical representation (‘mech’ subscript) then has $\Omega = \sqrt{I}$. Also, the flat representation of pure-shape 4-stop metroland has $\Omega = \{1 + \mathcal{R}^2\}/2$ and objects in this representation are then denoted by bars. The pure-shape triangleland flat representation has $\Omega = 1 + \mathcal{R}^2$ and objects in this representation are also denoted by bars. The pure-shape triangleland breved representation has $\Omega = 2$.

2.12.5 B.5 Problems with (para)geodesic principles

A principal hope/application of configuration space studies is representing motions by geodesics on configuration space.

Problem of edges/singularities of configuration space Geodesics can however run into the these, (e.g. the $N = 3, d = 1$ straight (half-) line geodesics that go into the triple collision). Thus it is interesting to investigate the nature of these edges and singularities (c.f. Sec 3). It is then helpful that the study of singularities for particle mechanics is well-developed [206], particularly for the classical $1/|\mathbf{x}^{IJ}|$ potentials though also for similar potentials such as $1/|\mathbf{x}^{IJ}|^l, l > 0$. These can include curvature singularities on configuration space, c.f. Sec 3.6.5 and 3.8.7. While studying this at the level of the relationalspace is less common, at least some such studies have been done [478, 249, 250]. Ideally, one would like to know how typical it is for the motion to hit such a boundary and what happens to the motion after hitting the boundary. One possibility is that boundaries are singular, a simple subcase of this being when the boundaries represent curvature singularities of the configuration space. E.g. the $N = 3, d = 2$ or 3 relational space metric blows up at the triple collision [250], while the shape metric is better behaved in being of constant positive curvature [478]. Also, the $N = 4, d = 1$ shape metric is of finite curvature.

Problem of zeros, poles and nonsmoothness. Additionally note that one requires not the ‘bare’ metrics M_{AB} but the physical $\tilde{M}_{\text{AB}} = W M_{\text{AB}}$ for each W in order to encode motions as geodesics. Now, clearly, performing such a conformal transformation generally alters geodesics and curvature. Additionally, this transformation cannot necessarily be performed on extended regions since it requires a smooth nonzero conformal factor while W in general has zeros or unbounded/rough behaviour. This limits such use of Jacobi-type actions as geodesics. In the case of Newtonian gravitational potentials, this is a condition of having to stay within the so-called Hill’s regions (see e.g. [441]). Here, such zeros are always ‘halting points’ in the sense that $\mathbf{P}_i = 0$ there by $0 = W = T(\mathbf{P}_i)$ (conservation of energy and positive-definiteness of the mechanical T).

Analogy 32) Both GR and RPM’s possess this problem of zeros, poles and nonsmoothness.

Difference 13) Moreover, the nature of such zeros is different in GR. This is due to definiteness versus indefiniteness of the kinetic term [Difference 16)], which is relevant by zeros of the potential corresponding to zeros of the kinetic term via the quadratic constraint. Definiteness causes a barrier which indefiniteness causes merely spurious zeros [464] since the kinetic term now need not get stuck at zero momentum, but, rather, the motion may continue through the zero on the Superspace null cone, which is made up of perfectly reasonable Kasner universes.

Note 1) To illustrate that the presence of zeros in the potential term is an important occurrence in GR, note that Bianchi IX spacetimes have an infinity of such zeros as one approaches the cosmological singularity. Furthermore, these spacetimes are important due to the BKL conjecture. The above argument was put forward in [151] to argue against the validity of the use of the ‘Jacobi principle’ to characterize chaos in GR [597].

Note 2) more generally,

$$0 < \Omega(\mathbf{q}^A) < \infty, \quad \Omega(\mathbf{q}^A) \text{ smooth} \quad (202)$$

is needed throughout the region of physical interest for the two-way conformal transformation in question to be defined.

2.13 Appendix C: Barbour relationalism versus Rovelli relationalism

Secs 1 and 2 involved Leibniz–Mach–Barbour’s use of the term ‘relationalism’ [7, 120, 436, 75, 76, 517, 82, 152, 584, 89, 92, 79, 83, 94, 88, 6]; see [545] for Rovelli’s distinct use of the same word. In outline [30], Rovelli’s classical relationalism involves objects not being located in spacetime (or space) but being located with respect to each other [c.f. Relationalism 0]: this looks like common ground albeit statements of this kind by themselves lack concreteness]. Rovelli also has a relationalist postulate that states for a particular subsystem only make sense with respect to another subsystem (this is held to carry over to QM, and Crane was part of this conceptualization and has gone on to use his own form [188] of it in his own program). I comment on this further in Secs 14 and 16 after having introduced a number of other concepts and machinery. Rovelli then speculates that these two relational postulates might be linked (p 157 of the online version of [545]): “*Is there a connection... This is of course very vague, and might lead nowhere, but I find the idea intriguing.*” See Sec 16.3 for what links I have found in this regard. Barbour on the other hand, has specific spatial and temporal relationalism postulates along the abovementioned lines. These embody particular ideas of Mach (that time is to be abstracted from change) and Leibniz (the identity of indiscernibles), each of which is sharply implemented by particular mathematics at the classical level, as outlined above.

Rovelli also endorses the Leibnizian timelessness postulate Relationalism 5), however he does not use the MRI/MPI implementation (unlike Barbour). Indeed, in Sec 11 I explain how Rovelli’s program and Barbour’s belong to *separate* families of timeless conceptions of physics, Barbour’s being more heavily populated by other theoreticians (at least traditionally and also regards internal diversity). They both agree on the Machian ‘time is abstracted from change’ postulate Relationalism 6), but they differ as regards what change to involve. For Rovelli, *any* individual change will do, whilst Barbour insists on abstracting time from the *totality* of changes. N.B. that Rovelli is not insisting on keeping time *well* though, (but part of the issue is *can* he with such a democratic rather than cumulative perspective on which change he is doing the abstraction from). On the other hand, Barbour’s approach picks out a totality, but does not say how to interpret it (I argue as to how it should be interpreted in Sec 12).

The starting point of consideration of the configuration spaces alone, on which Jacobi-type variational principles are defined is considered only by Barbour and not by Rovelli. This then takes Barbour straight to the situation in which no variable is distinguished as time at the kinematic level. On the other hand, Rovelli characterizes this as an essential difference between non-relativistic and relativistic mechanics in his approach, in Barbour’s approach this distinction has dissolved. The recovery of Newtonian dynamics from relational particle models also has features by which such models are closer in objective structure to GR than Rovelli generally holds nonrelativistic mechanics models to be in [545]. Thus relational particle models provide tractable models outside the scheme in [545].

Barbour’s relationalism has the virtue of being more in line with Leibniz and Mach’s thinking. (Thus alongside having attained a concrete mathematical implementation of these ideas it is definitely of interest to the foundations of physics and theoretical physics and so deserves full investigation). However, I would not dismiss the possibility that Rovelli’s relationalism is *also* in line with other (interpretations of) Leibniz, Mach or other such historical luminaries, and, in any case, has original value and is useful in a major Quantum Gravity program (LQG [539, 607]).

Note 1) Sec 6, 11, 14 and 16) for further differences between Rovelli and barbour relationalism.

Note 2) My own relationalism is much more Barbour-like (but not identical to Barbour’s). As well as my more detailed analysis of which groups and group– \mathbf{q} pairings, I go further than Barbour in Secs 4, 6, 11 and 14; some of this may bridge to Rovelli–Crane types of relationalism, or be compatible with that (e.g. giving Rovelli relationalism more structure: see Sec 16).

Note 3) I do not take some of the relational postulates too seriously or with too much hope of rigorous implementation . See also the Conclusion Sec 16).

3 Geometry of relationalspace and RPM's thereupon

Understanding the configuration space \mathbf{q} is very important for, firstly, classical dynamics and quantum mechanics in general. Secondly this article's Relationalism 3) postulate is that \mathbf{q} is primary (though I *discuss whether* to adhere to this postulate rather than always taking it for granted). Thirdly, \mathbf{q} is important as regards a number of Problem of Time approaches (the timeless case in particular very much concerns the study of configurations and configuration spaces).

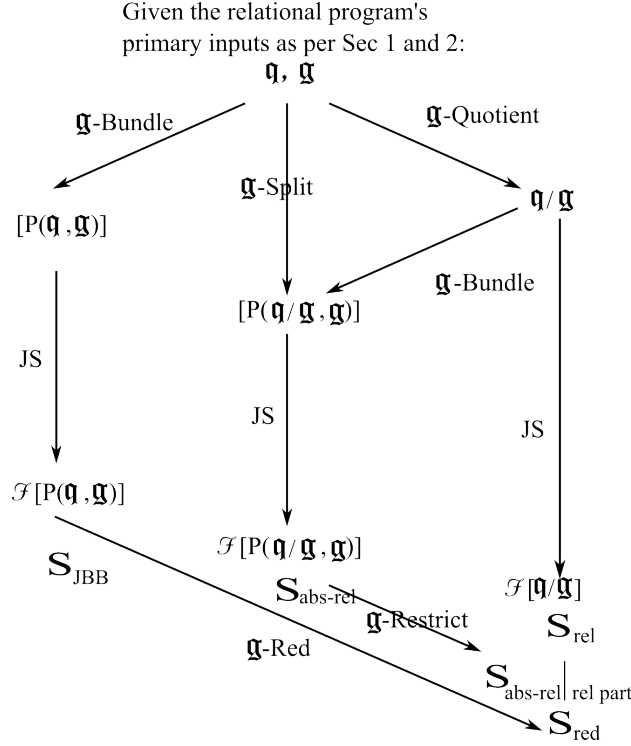


Figure 9: A, B, C.I) and D can be identified as paths of moves on this diagram. The \mathcal{F} 's are just the spaces the actions themselves live on. A priori, it is not clear whether the cluster of three actions in the bottom-right corner will coincide.

In this chapter, I approach \mathbf{q} for RPM's from first principles. I do so due to the availability of Kendall's study [363, 368] (along with his book's coauthors Barden, Carne and Le and his former student Small, who wrote an earlier book on this subject [579]), which, despite not being done in the context of mechanics, turns out to be the right mathematics for the configuration space study of pure-shape RPM. On the other hand, the cone construction gives the scale-shape split scaled RPM configuration spaces from pure-shape RPM's configuration spaces (shape spaces). Then I term constructing the natural mechanics action $\mathbf{S}^{\text{relationalspace}}$ associated with this by the Jacobi-Synge parageodesic principle

A) the **relationalspace approach**. I identify this as a map $\mathbf{g}\text{-Rel}: (\mathbf{q}, \mathbf{g}) \longrightarrow \mathcal{F}(\mathbf{q}/\mathbf{g})$, with, via Fig 9,

$$\mathbf{g}\text{-Rel} = \text{JS} \circ \mathbf{g}\text{-Quotient} . \quad (203)$$

This is a priori disjoint from the work so far in this article, i.e.

B) the **indirect approach** that constitutes Chapter 2, which I identify via Fig 9 as the map

$$(\text{Best Matching with respect to } \mathbf{g}) := \mathbf{g}\text{-BM} = \mathbf{g}\text{-Red} \circ \mathbf{g}\text{-JBB} := \mathbf{g}\text{-Red} \circ \text{JS} \circ \mathbf{g}\text{-Bundle} , \quad (204)$$

which is, a priori, another map $\mathbf{g}\text{-Rel}: (\mathbf{q}, \mathbf{g}) \longrightarrow \mathcal{F}(\mathbf{q}/\mathbf{g})$ and eventually leads to an \mathbf{S}^{red} (the best matching has not been performed yet in this article other than in the very trivial case of 1- d scaled RPM).

Thus one will for the moment need to take it on trust that this investigation will indeed join up with Sec 2's, representing therefore a second, independent foundation (a 'second route' to RPM's in Wheeler's parlance). I demonstrate in Secs 3.13, 3.14, 3.15 that the two match up, at the level of

C.I) **configuration space reduction**, which is in this context one and the same as the best matching procedure and the first 5/6ths of the relational-mechanical counterpart of BSW.

I note that reduction can also be done at other levels:

C.II) (almost-)phase space reduction, and

C.III) Reduction at the level of the QM equations (e.g. in Sec 6).

[There is also

C.IV) reduction *after* quantization alias the *Dirac quantization scheme* which is at the level of the QM solutions themselves.] Note also the additional scheme

D) rearranging Newtonian Mechanics by **passing to absolute–relational split generalized coordinates**, so as to look at the relational portion sitting within this.

Note 1) [A generalization of D)] = **g**-Split in Fig 9 can be viewed as part of a third map from (\mathbf{q}, \mathbf{g}) to $\mathcal{F}(\mathbf{q}/\mathbf{g})$. However, I only make use of the first two steps of D) in the present article.

Note 2) These are useful through D) sharing kinematical structure with A) to C), and this kinematical structure then being well-known in D) as part and parcel of D) being an elsewhere considerably studied subject! In other words, D) is a source of coordinate systems that continue to be useful in setting up A) and in the reduction C) of B). Examples of such are Jacobi coordinates (Sec 2.2.4), spherical-type and inhomogeneous coordinates (Sec 3.6), parabolic-type coordinates, Dragt coordinates [205], and democratic invariants [662, 427] (all in 3.10), and no doubt further such in the study of 3- d cases beyond the scope of this article. Some overall-useful references for D) are e.g. Iwai [348], Montgomery [475, 476], Hsiang [326], and in work reviewed by Littlejohn and Reinsch [428]. The Molecular Physics–RPM connection was first envisaged by Gergely, though he tapped this connection to a far smaller extent than my own research program has done. The lack of awareness of N -body physics among Theoretical Physicists is a somewhat worrisome trend, e.g. Theoretical Physics textbook QM tends to be very misleading as regards it being configuration space rather than space that is central (by overuse of 1-particle examples at the expense of N -body problem examples. The present article looks to partly reverse that trend, both by making use of Molecular Physics techniques and analogies and by showing that N -body physics has its own versions of background-dependence versus background-independence and nontrivially exhibit many of Theoretical Physics’s conceptual problems as regards time and closed universes.

Note 3) 4 particles in 2- d is *not* habitually covered in the Molecular Physics literature, however I provide contact with distinct pre-existing sources of useful coordinate systems that are readily applicable to this problem in Sec 3.10.

Note 4) The absolute–relational split action $\mathbf{S}_{\text{abs–rel}}$ is useful in Sec 6 in pointing out quantum discrepancies between the particular mathematical forms of absolute and relational QM.

Note 5) This Sec’s success with the reduction approach for 1- and 2- d RPM’s amounts to the following.

Difference 14) There is no Thin Sandwich/Best Matching Problem for 1- and 2- d RPM’s, unlike in GR.

Analogy 33) If one wishes to continue to have such an obstruction, however, 3- d RPM’s (for which the entirely physically reasonable collinearities have singular inertia tensors in 3- d , obstructing the elimination of the rotational auxiliary) remain impassible here.

Note 6) This Sec’s scheme A) has *no* Barbour demonstration: in it, one is held to already have the right sequence of shapes.

Note 7) Butterfield [153] terms C.I) (that I term configuration space reduction) ‘relational’ and C.II) (phase space reduction) ‘reductive’ (Belot [112, 113] also compares these two approaches). However, I consider these both to be reductions in different formalisms and coincident for many purposes.

Note 8) I will subsequently find that (Sec 6) for this article’s RPM examples, all of A to C look to be coincident, bar C.IV, which can differ due to interference of operator-ordering issues. I also find that D) (the relational–absolute split of Newtonian Mechanics) can likewise be distinct.

Note 9) The indirect formulation A) retains the virtue of being the closest analogue to the geometrodynamical formulation of GR alongside all of its uses and developments in Quantum Cosmology and the Problem of Time.

3.1 Preliminary definitions

I use the following additional index types. \mathcal{A} for preshape space coordinates, \mathbf{a} for shape space coordinates, with $\bar{\mathbf{a}}$ for radial such in 2- d and $\tilde{\mathbf{a}}$ for angular such in 2- d .

I denote by $Q_{N,d}$ the nontrivial *quotient map*: $\mathfrak{S}(N, d) \longrightarrow \mathfrak{p}(N, d)$. I usually abbreviate this to Q .

Lemma 1 (real representations) The configuration space $\mathbf{q}(N, d)$ can be represented by $\mathbf{q} = \{\mathbf{q}^I\}$, the particle position coordinates. The relative configuration space $\mathbf{r}(N, d)$ can be represented by $\mathbf{r} = \{\mathbf{r}^i\}$, a basis set of relative coordinates. Examples of such are 1) a basis subset of the relative particle positions $\mathbf{r}^{IJ} = \mathbf{q}^I - \mathbf{q}^J$. 2) the relative Jacobi coordinates \mathbf{R}^i . Preshape space $\mathfrak{p}(N, d)$ can be represented by $\bar{\mathbf{r}} = \{\bar{\mathbf{r}}^{\mathcal{A}}\}$ obtained from the preceding by normalization. N.B. that preshape space is all that one needs for NRPM.

3.2 Topological structure of (pre)shape space

Note that at the topological level, $\mathbf{a}(d) = \mathbb{R}^d$, $\mathbf{q}(N, d) = \mathbb{R}^{Nd}$ and $\mathbf{r}(N, d) = \mathbb{R}^{nd}$ are all simple and well-known. Next, it is natural to ask what topology the various reduced configuration spaces have. Figs 10 and 11 set about a first-principles investigation of this for simple cases (along the lines of [20]).

I subsequently found that systematized investigations into the question of shape space topology had already been done in 1- and 2- d by Kendall and co-workers [368, 362, 363]. (Whilst these investigations did not consider such as the configuration

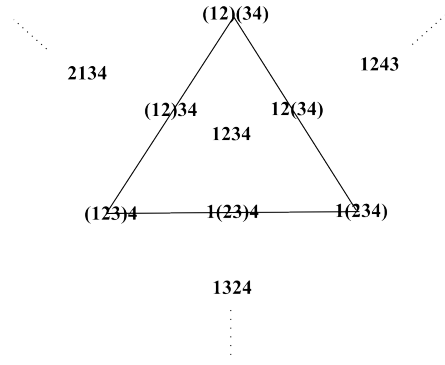


Figure 10: A sketch of the construction for the example of 4-stop metroland, regarding the particles 1, 2, 3, 4 as distinguishable and the order 1234 as distinct from 4321. Starting in the order 1234, one has a 2- d region of shape space bounded by three double-collision line segments, and simultaneous double and triple collision points. Next, deduce which regions lie adjacent to this by continuous deformation into permutations one swap away from the preceding 2- d region. Keep on going until all permutations of the labels have been covered and thus the whole configuration space has been covered (permit no identically-labelled region to occur more than once, by use of identification as necessary).

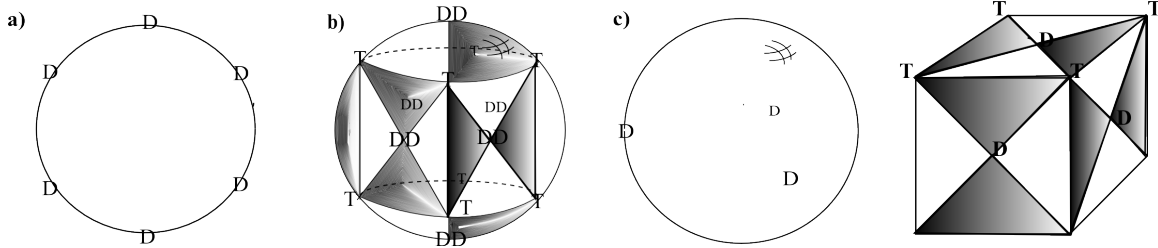


Figure 11: Applying Figure 10's procedure to various cases results in the following.

- a) $\mathfrak{S}(3, 1) = \mathbb{S}^1$. The six D's are double collisions, and they cut up the circle into 6 arcs.
- b) $\mathfrak{S}(4, 1) = \mathbb{S}^2$. The 8 T's are triple collisions and the 6 DD's are double-double collisions. Each DD is attached to 4 T's, and each T to 3 T's and 3 DD's, in each case by single double collision lines. There are also 36 line segments that are double collisions (D). These tessellate the sphere into 24 identical disjoint regions.
- c) $\mathfrak{S}(3, 2) = \mathbb{S}^2$. There are three D points, so, if these points are excised, this is the 'pair of pants' [477, 478]. Note that these 3 points are the half/mirror-image-identified version of a) [i.e. Fig 13a)].
- d) N.B. quadrilateralland is harder to visualise than the previous subsections' shape spaces, due to greater dimensionality as well as greater geometrical complexity and a larger hierarchy of special regions of the various possible codimensions. The topologically-defined decorations on \mathbb{CP}^2 are 6 pairs of spheres (each of which is a triangleland) which share 4 T points and 3 DD points. As these are 2- d , how they fit together is likewise visualizable. They are arrayed as the corners and face centres of the identified cube. Thus each shaded triangle in fact represents a pair of spheres. Removing this set is like forming the connected pants rather than 4-stop metroland's slice-up into 24 disjoint regions, since the codimension is 2 and not 1. There is also a net of 4-stop metrolands with one being a D point on an \mathbb{RP}^2 mirror image identified by rotation in 2- d ; note the similarity between these and the half/mirror-image-identified part of b) [i.e. Fig13b)].

space of a mechanics, I bridged the gap via the Jacobi–Synge procedure ([25], Sec 3.13–3.15). Kendall's own interests in these spaces have been pure-mathematical geometry and statistical applications. (Given a set of monoliths, or quasars, say, how many approximate alignments are needed therein for further explanations beyond mere chance to be necessary?). Kendall also worked on the 3- d case, but found that to be much harder [368, 365, 366, 367]. In contrast, 1- and 2- d are straightforward, leading to their immediate exploitability via RPM's as whole-universe models being toy models for a wide range of Problem of Time strategies. Kendall's work proceeds via the simpler preliminary treatment of preshape space, to which I next turn.

Lemma 1 ('Preshape space is always easy'). $\mathfrak{p}(N, d) = \mathbb{S}^{nd-1}$ homeomorphically.

Proof Immediate from the definition of $\mathfrak{p}(N, d)$.

Corollary 1 $\mathfrak{S}(N, 1) = \mathfrak{p}(N, 1) = \mathbb{S}^{n-1} = \mathbb{S}^{N-2}$.

Lemma 3 (Beltrami coordinates). Straightforwardly, \mathbb{S}^{nd-1} can be coordinatized by $\mathcal{X}^A = \rho^{a\alpha}/\rho^{11}$ for $a, \alpha \neq 1, 1$.

Lemma 4 (complex representations). i) In 2- d , relative configurations can be represented by n complex numbers z^a – the *homogeneous coordinates*.

ii) Assume that not all of these coordinates are simultaneously 0 (i.e. exclude the *maximal collision*). Then 2- d shapes can be represented by $N - 1$ independent complex ratios, the *inhomogeneous coordinates* Z^a .

[This is straightforwardly established by dividing the complex numbers in i) by a particular z^a and ignoring the 1 among the new string of complex numbers.]

I denote the corresponding polar forms by $z^a = \rho^a \exp(i\phi^a)$ and $Z^a = \mathcal{R}^a \exp(i\Phi^a)$. These are independent ratios of the z^a , and so, physically their magnitudes \mathcal{R}^a are ratios of magnitudes of Jacobi vectors, and their arguments Φ^a are now angles

between Jacobi vectors, which are entirely relational quantities. Also, I use $|Z|_c^2 := \sum_a |Z^a|^2$, $(\ , \)_c$ for the corresponding inner product, overline to denote complex conjugate and $|\ |$ to denote complex modulus.

3.2.1 The simpler shape spaces as topological surfaces

‘Partial Periodic Table’ Theorem. At the topological level, the discernible shape spaces include 3 simple series [Fig 12]:

- i) $\mathfrak{S}(N, 1) = \mathbb{S}^{n-1} = \mathbb{S}^{N-2}$,
- ii) $\mathfrak{S}(N, 2) = \mathbb{CP}^{N-2}$ and
- iii) $\mathfrak{S}(d+1, d) = \mathbb{S}^{\{d-1\}\{d+2\}/2}$.

Proof i) Corollary 1. ii) Begin with Lemma 3.ii)’s standard coordinatization of \mathbb{CP}^{N-2} . Then quotient $\exp(i\alpha)$, i.e. $U(1)$, i.e. the $SO(2)$ rotations: in the form $\mathbb{CP}^{n-1} = \mathbb{S}^{2n-1}/U(1)$, this is a well-known result of Hopf [which generalizes the most basic Hopf fibration, $\mathbb{CP}^1 = \mathbb{S}^2 = \mathbb{S}^3/U(1)$]. iii) is Casson’s theorem (proven e.g. on p20-22 of [368]).

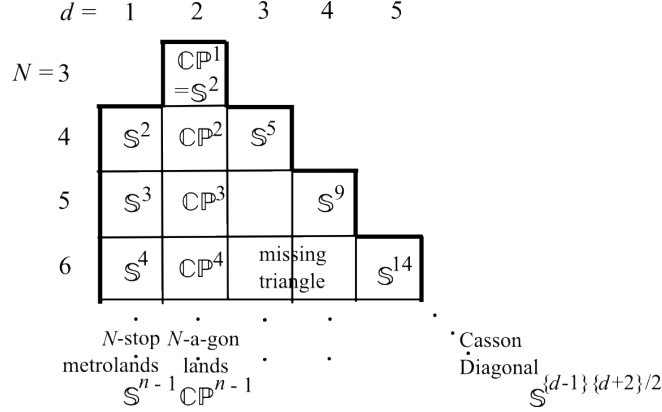


Figure 12: ‘Partial Periodic Table’ of the discernible shape spaces at the topological level. This and its metric counterpart (Fig 17) are the second key to unlocking RPM’s.

- Note 1) there is indeed agreement on the overlap of i) and iii) by the well-known homeomorphic equivalence of \mathbb{CP}^1 and \mathbb{S}^2 . Thus $\mathfrak{S}(3, 2) = \mathbb{S}^2$ homeomorphically.
- Note 2) There are further shape spaces for $N \leq d$ but these are all indiscernible from each other and from $\mathfrak{OS}(d+1, d)$, so I treat them in the OShapespace SSec.
- Note 3) i), ii) and iii) are series with particular simple patterns, at least at the topological level (Fig 12); this is the first half of the second key observation toward understanding simple concrete examples of RPM’s.
- Note 4) Once one also takes into account only the discernible shape spaces as per Sec 2.1, this is all the shape spaces bar the missing triangle in Fig 12. In here, the spaces are likely new in this context rather than known from elsewhere in mathematics, even at the topological level. See Ch. 2-5 of [368] for a partial characterization of the new spaces at the topological level.³²
- Note 5) \mathbb{CP}^{n-1} involves n lines, whilst n lines can be used to form whichever Jacobi tree for an N -a-gon. This is a lucid insight as to why \mathbb{CP}^{n-1} is naturally representable as the space of all N -a-gons.

3.2.2 Some detailed topological properties of (pre)shape space

Use this as a topological characterization, parts of which have further physical significance. In the 3- d case, this topological characterization is as good a description of what the topology is for the spaces in question [368].

Now that $\mathfrak{p}(N, d)$, $\mathfrak{S}(N, 1)$ and $\mathfrak{S}(N, 2)$ have been identified as spheres and complex projective spaces, the following classically and quantum-mechanically useful topological information about paths on, and obstructions in, these configuration spaces becomes available. $\mathfrak{p}(N, d) = \mathbb{S}^{nd-1}$, $\mathfrak{S}(N, 1) = \mathbb{S}^{n-1}$ and $\mathfrak{S}(N, 2) = \mathbb{CP}^{n-1}$ are compact without boundary and Hausdorff. Following from [313], the homotopy groups (useful for the classification of classical paths) of $\mathfrak{p}(N, d)$ exhibit the simple

$$\text{pattern } \pi_p(\mathfrak{p}(N, d)) = \pi_p(\mathbb{S}^{nd-1}) = \begin{cases} \mathbb{Z} & p = nd - 1 > 1 \\ 0 & p < nd - 1 \end{cases}.$$

From [611], the first few homotopy groups in the remaining wedge are

	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	
$\mathfrak{p}(4, 1) = \mathfrak{s}(3, 2) =$	\mathbb{S}^2	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2
$\mathfrak{p}(5, 1) = \mathfrak{p}(3, 2) =$	\mathbb{S}^3		\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2
$\mathfrak{p}(6, 1) =$	\mathbb{S}^4			\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}
$\mathfrak{p}(7, 1) = \mathfrak{p}(4, 2) = \mathfrak{s}(4, 3)$	\mathbb{S}^5				\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
$\mathfrak{p}(8, 1) =$	\mathbb{S}^6					\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}
$\mathfrak{p}(9, 1) = \mathfrak{p}(5, 2) =$	\mathbb{S}^7						\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0

³²Also see e.g. [368, 365, 366, 367] for 4 particles in 3- d .

By $\mathfrak{S}(N, 1) = \mathfrak{p}(N, 1)$, the corresponding results for the N -stop metrolands $\mathfrak{S}(N, 1)$ are readily obtained from the above by setting $d = 1$. The pattern is $\pi_p(\mathfrak{p}(N, 1)) = \pi_p(\mathbb{S}^{N-1}) = \begin{cases} \mathbb{Z} & p = n - 1 > 1 \\ 0 & p < n - 1 \end{cases}$, while the remaining cases' table above includes the other $\mathfrak{S}(N, 1)$ results. Likewise, the Casson diagonal is included by using $\{d - 1\}\{d + 2\}/2$ in place of nd .

From [313], it follows that the homology and cohomology groups of preshape space are

$$H_p(\mathfrak{p}(N, d)) = H_p(\mathbb{S}^{nd-1}) = \begin{cases} \mathbb{Z} & p = 0 \text{ or } nd - 1 \\ 0 & \text{otherwise} \end{cases} = H^p(\mathbb{S}^{nd-1}) = H^p(\mathfrak{p}(N, d)).$$

Thus, for N -stop metroland,

$$H_p(\mathfrak{S}(N, 1)) = \begin{cases} \mathbb{Z} & p = 0 \text{ or } n - 1 \\ 0 & \text{otherwise} \end{cases} = H^p(\mathfrak{S}(N, 1)).$$

The first and second Stiefel–Whitney classes are trivial for all spheres [486]. The main use of this is that it implies that $\mathfrak{p}(N, d)$ are all orientable and admit a nontrivial spin structure, and likewise for $\mathfrak{S}(N, 1)$.

From [637], it follows that the homotopy groups $\pi_p(\mathfrak{S}(N, 2)) = \pi_p(\mathbb{CP}^{N-2}) = \begin{cases} \mathbb{Z} & p = 2 \\ \pi_p(\mathbb{S}^{2N-3}) & \text{otherwise} \end{cases}$.

From [313], it follows that the homology and cohomology groups are

$$H_p(\mathfrak{S}(N, 2)) = H_p(\mathbb{CP}^{n-1}) = \begin{cases} \mathbb{Z} & p \text{ even up to } 2\{n - 2\} \\ 0 & \text{otherwise} \end{cases} = H^p(\mathfrak{S}(N, 2)) = H^p(\mathbb{CP}^{n-1}).$$

The first Stiefel–Whitney classes are trivial for all complex projective spaces [486], which implies that all the $\mathfrak{S}(N, 2)$ are orientable. The second Stiefel–Whitney classes are trivial for $N - 2$ an odd integer. Thus $\mathfrak{S}(N, 2)$ for N odd admit a nontrivial spin structure. On the other hand, they are nontrivial [equal to the generator of $H^2(\mathbb{CP}^{n-1}, \mathbb{Z})$] for $N - 2$ an even integer. Thus nontrivial spin structures do not exist for $\mathfrak{S}(N, 2)$ with N even due to topological obstruction. However, generalized ‘spin^c’ structures do exist [518] in these cases.

3.2.3 The simpler shape spaces as metric spaces

The \mathbb{R}^d inner product serves to have a notion of ‘localized in space’, which survives in some form for all the configuration spaces considered. This is useful as regards discussing observable configurations (see Secs 14 and 17).

One also has have a notion of localized in configuration space – i.e. of which configurations look alike (this is important as in physics one does not know precisely what configuration one has and also used in Sec 14). For this one has available the possibly-weighted \mathbb{R}^{nd} norm $\| \cdot \|$ as per Sec 2.1.1. Likewise, one has the possibly-weighted \mathbb{R}^{nd} norm $\| \cdot \|$ and the corresponding inner product for $\mathfrak{t}(N, d)$. (If the Jacobi coordinates themselves are mass-weighted, these norms are unweighted i.e. associated with the unit $nd \times nd$ matrix.) One could use the \mathbb{R}^{nd} norm in $\mathfrak{p}(N, d)$ too, as a chordal distance, but there are also intrinsic distances that could be used thereupon (based on angular separations in \mathbb{S}^{nd-1} that one has available once one has a notion of intrinsic metric); see below.

Structure A $\langle \mathfrak{q}(N, d), \| \cdot \| \rangle$ and $\langle \mathfrak{t}(N, d), \| \cdot \| \rangle$ are appropriate as metric spaces to work with. Then $\langle \mathfrak{p}(N, d), (\text{Chordal Dist}) \rangle$ is inherited as a metric space. This then turns out to be topologically equivalent (see p 13 of [368]) to the metric space $\langle \mathfrak{q}(N, d), (\text{Great Circle Dist}) \rangle$, which is a geodesic distance. Furthermore note that (Chordal Dist) and (Great Circle Dist) are related by (Chordal Dist) = sin(Great Circle Dist) (p 205 of [368]). Then (Great Circle Dist)(\bar{A}, \bar{B}) = arccos($\bar{A}\bar{B}$). This carries over to shape space: one has the metric space $\langle \mathfrak{S}(N, d), D \rangle$, for the quotient metric

$$\text{Dist}(Q(\bar{A}, \bar{B})) = \min_{T \in SO(d)} (\text{Great Circle Dist})(\bar{A}, T\bar{B}) = \min_{T \in SO(d)} \arccos(\bar{A}, T\bar{B}). \quad (205)$$

Appropriate topological spaces to work with are 1) $\langle \mathfrak{p}(N, d); \tau_p \rangle$ for τ_p the set of open sets (obeying topological space axioms) determined by (Great Circle Dist) or (Great Circle Dist) 2) $\langle \mathfrak{S}(N, d); \tau_s \rangle$ e.g. obtained from the preceding as the quotient topology corresponding to the map Q . Equivalently, it can be obtained as the set of open sets determined by (Great Circle Dist). [For later use, (Great Circle Dist) generalizes to (Geodesic Dist).]

3.3 Topological structure of O(pre)shape space

For all that one may wish to focus on the plain shapes, the O-shapes do provide some contrast, and characterizing their greater difficulty is of interest. Also, some of the Orelationalspaces feature as meaningful subspaces within plain relationalspaces, so at least a moderate understanding of the O-case in the study of the plain case is needed.

Note that at the topological level $\mathfrak{oa}(d) = \mathbb{R}^d/\mathbb{Z}_2 = \mathbb{R}_+^d$, $\mathfrak{oq}(N, d) = \mathbb{R}^{Nd}/\mathbb{Z}_2 = \mathbb{R}_+^{Nd}$, $\mathfrak{or}(N, d) = \mathbb{R}^{nd}/\mathbb{Z}_2 = \mathbb{R}_+^{nd}$. What about the various reduced configuration spaces? My choice here is to study the plain shape case due to its greater tractability. However, in the present set-up chapter, I do provide some non-laborious aspects of the O case’s working for contrast and possibly future treatment of the O case in parallel with the rest of the present article. Note that these spaces are going to be orbifolds at the level of differential geometry (see SSec 3.4). Some first-principles construction figures (along the lines of [20]) for O(pre)shape spaces are then as follows.

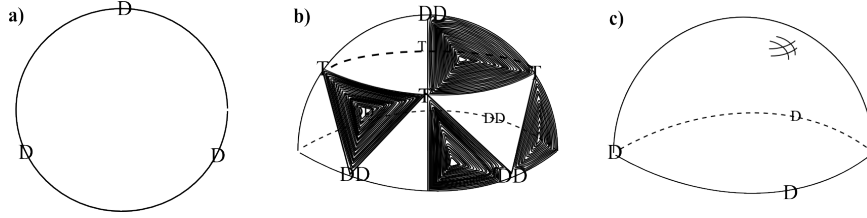


Figure 13: Repeating the procedure of Fig 11 in the O case leads to the following.

- a) $\mathbf{OS}(3, 1)$ can be represented as a semicircle and as a circle in double angles. In each case there are 3 D's (double collisions) so that the semicircle or circle is cut into 3 arcs. One can view this as the equator of c), or that of $\mathbf{S}(3, 2)$.
- b) $\mathbf{OS}(4, 1) = \mathbb{S}^2/\mathbb{Z}_2$ acting inversively $= \mathbb{RP}^2$ [20]. For this, there are 12 spherical triangle regions of shape space, bounded by 18 D (double collision) line segments, and 3 DD (double-double) and 4 T (triple) collision points. This also plays a role as the collinear configurations of $\mathbf{S}(4, 2)$.
- c) $\mathbf{OS}(3, 2) = \mathbb{S}^2/\mathbb{Z}_2$ acting reflexively.] This is a hemisphere whose equator edge is punctuated by 3 D points that split it up into 3 arcs.
- d) The excised regions from $\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$ are 6 \mathbb{RP}^2 Otrianglelands. that share 4 T points and 3 DD points, i.e. Fig 11 d).

3.3.1 The simpler Oshape spaces as topological surfaces

Corollary of ‘Partial Periodic Table’ Theorem In each case homeomorphically,

- i) $\mathbf{OS}(N, 1) = \mathbb{RP}^{N-2}$.
- ii) $\mathbf{OS}(N, 2) = \mathbb{CP}^{N-2}/\mathbb{Z}_2^{\text{conj}}$.
- iii) $\mathbf{OS}(A, B) = \mathbb{S}^{\{B-1\}\{B+2\}/2}/\mathbb{Z}_2^{\text{conj}}$ for $A \leq B + 1$ (and these are identified).

Proof For the last part, quotienting out twice is the same as quotienting out once. \square

Note 1) The superscript ‘conj’ denotes complex conjugate action at the level of the configuration space. N.B. that this is different from inversive action, as is clearest from the simplest case: $\mathbf{OS}(3, 2) = \text{hemisphere with edge} \neq \mathbb{RP}^2$. Demonstrating it is a reflection rather than an inversion follows from the transformation mapping a labelled shape to its mirror image being $z^a = \rho^a \exp(i\phi^a) \rightarrow \rho^a \exp(i\{\pi - \phi^a\})$, which is a reflection about the y-axis rather than an inversion.

Note 2) A notion of *weighted projective space*, that encapsulates the spaces of interest to us as particular weightings among many other possibilities, is present in the literature. These other possibilities correspond to the other possible ways in which the \mathbb{Z}_2 can act upon such a space. Unfortunately the examples of such that I have seen in the literature do not appear to coincide with the spaces of interest in this article, on account of having different weightings. Thus these would only serve as a rough guide to what deviations from \mathbb{CP}^{N-2} one might expect upon applying some \mathbb{Z}_2 action, and as such I relegate detailed discussion of this literature to Appendix 3.A

	$d =$	1	2	3	4	5
$N = 3$			$\mathbb{S}^2/\mathbb{Z}_2^{\text{conj}}$			
4		\mathbb{RP}^2	$\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$	$\mathbb{S}^3/\mathbb{Z}_2^{\text{conj}}$		
5		\mathbb{RP}^3	$\mathbb{CP}^3/\mathbb{Z}_2^{\text{conj}}$		$\mathbb{S}^4/\mathbb{Z}_2^{\text{conj}}$	
6		\mathbb{RP}^4	$\mathbb{CP}^4/\mathbb{Z}_2^{\text{conj}}$	missing triangle		$\mathbb{S}^5/\mathbb{Z}_2^{\text{conj}}$
		ON-stop metrolands \mathbb{RP}^{n-1}	ON-a-gon lands $\mathbb{CP}^{n-1}/\mathbb{Z}_2^{\text{conj}}$			QCasson Diagonal $\mathbb{S}^{\{d-1\}\{d+2\}/2}/\mathbb{Z}_2^{\text{conj}}$

Figure 14: The ‘Partial Periodic Table’ of discernible Oshape spaces at the topological level.

Corollary of Kuiper’s Theorem. From establishing that these are diffeomorphic by Kuiper’s theorem (see Sec 3.10.9 or [409, 410]), and diffeomorphic \implies homeomorphic, $\mathbf{OS}(4, 2) = \mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}} = \mathbb{S}^4$ at the topological level.

3.3.2 Some detailed topological properties of O(pre)shape spaces

We have identified $\mathbf{op}(N, d)$ and $\mathbf{OS}(N, 1)$ and $\mathbf{OS}(N, 2)$ as \mathbb{RP}^k and $\mathbb{CP}^k/\mathbb{Z}_2^{\text{conj}}$ manifolds. This permits the following classically and quantum-mechanically useful topological information about paths on and obstructions thereupon. 1) $\mathbf{op}(N, d) = \mathbb{RP}^{nd-1}$, $\mathbf{OS}(N, 1) = \mathbf{op}(N, 1)$ and $\mathbf{OS}(N, 2) = \mathbb{CP}^{n-1}/\mathbb{Z}_2^{\text{conj}}$ are compact without boundary.

2) Hausdorffness is not in general inherited in forming a quotient space, by the following basic counterexample. All open intervals in \mathbb{R} contain rationals and irrationals. Thus $\mathbb{Q}, \mathbb{R}/\mathbb{Q}$ is not open in \mathbb{R} . This can be viewed as a 2-class equivalence

relation that is the quotient of a Hausdorff space which has 2 points (the classes) and no open singleton, and so is not itself Hausdorff. Thus more specific results are required. \mathbb{RP}^n is straightforwardly Hausdorff [420]. So is $\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$, via it being homeomorphic with \mathbb{S}^4 .

Topological groups for Oprespaces are as follows. The homotopy groups corresponding to these spaces are $\pi_0(\mathbf{op}(N, d)) = \pi_0(\mathbb{RP}^{nd-1}) = 0$ for, trivially examining loops on the identified k -spheres, there is nothing for these to wrap around, so they are all contractible. $\pi_1(\mathbf{op}(N, d)) = \pi_1(\mathbb{RP}^{nd-1}) = \mathbb{Z}_2$ for $nd \geq 3$ (pp. 79-80 of [589]). $\pi_p(\mathbf{op}(N, d)) = \pi_p(\mathbb{RP}^{nd-1}) = \pi_1(\mathbb{S}^{nd-1})$ for $nd \geq 3$ by pp. 35-36 of [274], whereupon one can read them off (3.2.2). By

$$\mathbf{os}(N, 1) = \mathbf{op}(N, 1), \quad (206)$$

the corresponding results for the N -stop metrolands $\mathbf{s}(N, 1)$ are readily obtained from the above by setting $d = 1$: $\pi_0(\mathbf{os}(N, 1)) = 0$, $\pi_1(\mathbf{os}(N, 1)) = \mathbb{Z}_2$ for $n \geq 3$ and $\pi_p(\mathbf{op}(N, 1)) = \pi_p(\mathbb{RP}^{n-1}) = \pi_1(\mathbb{S}^{n-1})$.

The homology groups are $H_p(\mathbf{op}(N, d)) = H_p(\mathbb{RP}^{nd-1}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}_2 & p \text{ odd}, 0 < p < nd - 1 \\ 0 & \text{otherwise} \end{cases}$. Also, by (206) and the correspond-

ing results for the ON -stop metrolands $\mathbf{os}(N, 1)$, $H_p(\mathbf{os}(N, 1)) = H_p(\mathbb{RP}^{n-1}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}_2 & p \text{ odd}, 0 < p < n - 1 \\ 0 & \text{otherwise} \end{cases}$.

The cohomology groups are $H^p(\mathbf{op}(N, d)) = H^p(\mathbb{RP}^{nd-1}) = \begin{cases} \mathbb{Z} & p = 0 \text{ or } p \text{ odd and } 0 < p < nd \\ \mathbb{Z}_2 & p \text{ even}, 1 < p < nd \\ 0 & \text{otherwise} \end{cases}$. By (206), the corre-

sponding results for the N -stop metrolands $\mathbf{s}(N, 1)$ are then $H^p(\mathbf{os}(N, 1)) = \begin{cases} \mathbb{Z} & p = 0 \text{ or } p \text{ odd and } 0 < p < n - 1 \\ \mathbb{Z}_2 & p \text{ even}, 0 < p < n \\ 0 & \text{otherwise} \end{cases}$.

The first Stiefel–Whitney class of \mathbb{RP}^k is trivial for k odd and nontrivial for k even. Hence, setting $k = nd - 1$, for at least 1 of n and d even, the first Stiefel–Whitney class is trivial and so $\mathbf{sp}(N, d)$ is orientable. Conversely, if both n and d are odd, the first Stiefel–Whitney class is nontrivial and so $\mathbf{sp}(N, d)$ is non-orientable. In particular 1) for $d = 1$ and N odd one has an orientable Oprespace space, so odd-stop metroland is also orientable. 2) For $d = 1$ and N even, one has a non-orientable Oprespace space, so even-stop metroland is also nonorientable. 3) For $d = 2$ all Oprespace spaces are orientable.

The $\mathbf{os}(3, 2)$ hemisphere is also topologically standard. For Otriangleland, $\pi_1 = \mathbb{Z}_2$, and the rest of their homotopy groups coincide with those of \mathbb{S}^2 , as per Sec 3.2.2.

Next, if 2 spaces are diffeomorphic, they must they be homeomorphic. That $\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$ is homeomorphic to \mathbb{S}^4 implies the following. $\pi_p(\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}) = \pi_p(\mathbb{S}^4) = 0$ for $p < 4$, \mathbb{Z} for $p = 4$ and subsequently the third line of the table in Sec 3.2.2.

$H_p(\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}) = H_p(\mathbb{S}^4) = \begin{cases} \mathbb{Z} & p = 0 \text{ or } 4 \\ 0 & \text{otherwise} \end{cases} = H^p(\mathbb{S}^4) = H^p(\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}})$ The first and second Stiefel–Whitney classes for this are trivial, so this space is orientable and with nontrivial spin structure. I have not explored beyond this. A next modest step would be to heed what differences occur between \mathbb{CP}^{n-1} and the weighted projective spaces in Appendix 3.A).

3.3.3 Oshape spaces as metric spaces

In general, the quotient of a metric space only has a pseudometric on it, i.e. a structure Pseu such that

$$\text{Pseu}([x], [y]) = 0 \nRightarrow [x] = [y]. \quad (207)$$

For M compact, M/\sim has the quotient topology. One then needs to check if $(\mathbb{S}^2, \text{distance})/\mathbb{Z}_2$ is a metric space with respect to the corresponding notion of distance, and this is tied to \mathbb{RP}^2 's equivalents of chords and great circles.

Geodesic distance between 2 points x, y in \mathbb{RP}^n is

$$(\text{Geodesic Dist})_{\mathbb{RP}^2} = \min(\text{Geodesic Dist}_{\mathbb{S}^2}(x, y), \text{Geodesic Dist}_{\mathbb{S}^2}(x, -y)) \quad (208)$$

which gives a suitable definition of metric on \mathbb{RP}^n for the current level of structure. Also, the map $\mathbb{S}^n \rightarrow \mathbb{RP}^n$ sends great circles to what will be \mathbb{RP}^n geodesics at the level of Riemannian geometry. Also note (paralleling Lemma 3): there is an analogue of homogeneous coordinates on $\mathbf{os}(N, 2) = \mathbb{CP}^{n-1}/\mathbb{Z}_2^{\text{conj}}$, just with different coordinate ranges. For the triangleland case, define $\underline{y}^{\text{conj}} = (r\cos\theta, -r\sin\theta)$ corresponding to $\underline{y} = (r\cos\theta, r\sin\theta)$. Note that this is not just a reflection, but also a conjugation; it is the latter that extends to arbitrary particle number case. Then a geodesic distance is

$$(\text{Geodesic Dist})_{\mathbb{RP}^n} = \min^2((\text{Geodesic Dist})_{\mathbb{S}^n}(x, y), (\text{Geodesic Dist})_{\mathbb{S}^n}(x, y^{\text{conj}})) \quad (209)$$

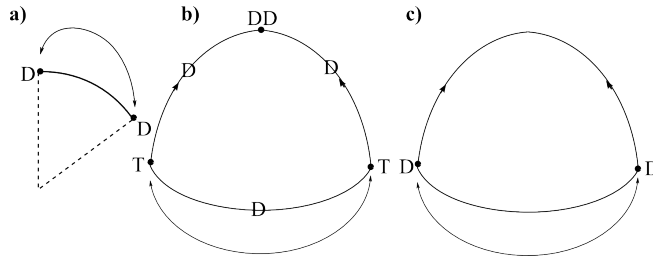


Figure 15: Fully indistinguishable particle (Leibniz space) cases of a) 3-stop metroland b) 4-stop metroland: \mathbb{S}^2/S_4 with the S_4 acting as one handedness of the cube group. and c) triangleland, all at the topological level: \mathbb{S}^2/S_3 with the S_3 acting as \mathbb{D}_3 . In each case, the A- instead of S-version is obtained by doubling the picture: to a 120 degree wedge in the first case, and to a full lune by adding a lower spherical triangle in the other two cases.

3.3.4 The case of (partly) indistinguishable particles

One could likewise set out to repeat this Chapter's study so far in this setting. This involves quotienting out larger groups and thus produces portions of 11's configuration spaces, as in Fig 15.

I use the prefix A for A_N quotients and the prefix S for S_N quotients just as I previously used the prefix O for $\mathbb{Z}_2^{\text{ref}}$ quotients. Thus I define *ARelative space* $\mathbf{ar}(N, d)$, *APreshape space* $\mathbf{ap}(N, d)$, *ARelational space* $\mathbf{aR}(N, d)$, *AShape space* $\mathbf{aS}(N, d)$, *SRelative space* $\mathbf{sr}(N, d)$, *SPreshape space* $\mathbf{sp}(N, d)$, *SRelational space* $\mathbf{sR}(N, d)$ and *SShape space*, which, both being hard to pronounce and the most very Leibnizian of the possible configuration spaces for mechanics with equal particle masses, I rename as *Leibniz space* $\text{Leib}(N, d) := \mathbf{s}(N, d)/S_N$

Question. What is known of \mathbb{S}^p/\mathbf{g} and \mathbb{CP}^p/\mathbf{g} where \mathbf{g} is (a subgroup of) a permutation group? [Do these occur elsewhere in mathematics or physics? In the first case, yes: they are a subcase of the spherical primes [257], and such spaces occur also in the study of the large-scale shape of space study [412] and in dynamical systems [619]. In the second case, I have not seen these 'complex projective primes' elsewhere, other than for $\mathbf{g} = \mathbb{Z}_2$, as per Appendix 3.16].]

3.4 Topological structure of (plain, O, A or S)relational spaces

3.4.1 Relational space as the cone over shape space

The third key step in understanding RPM's is that relational space $\mathbf{R}(N, d)$ can be viewed as the **cone**³³ over shape space, $C(\mathbf{s}(N, d))$. At the topological level, for $C(X)$ to be a cone over some topological manifold X ,

$$C(X) = X \times [0, \infty)/\sim, \quad (210)$$

where \sim means that all points of the form $\{p \in X, 0 \in [0, \infty)\}$ are 'squashed' or identified to a single point termed the *cone point*, and denoted by 0.

Cones are 'examples' of orbifolds (making them so can involve defining more structure than one would have otherwise). At the level of differential geometry, the additional extra structure for a real or complex *orbifold* [273, 191, 301] involves charts to quotients \mathbb{R}^k/\mathbf{g} and \mathbb{C}^k/\mathbf{g} . This parallels (and generalizes) how real and complex manifolds are defined in terms of charts to \mathbb{R}^k and \mathbb{C}^k .

Question* For the relational program, I would like to push the above definition of cone far as possible toward cases in which X is a stratified manifold (collection of manifolds of in general different dimensionality glued together in a particular way [See Sec 3.12's references for more detail] or orbifold

$$C(X) \text{ for } X = R/\mathbf{g}, \quad (211)$$

i.e. the cone over some topological manifold R quotiented by a group \mathbf{g} (which may be continuous, discrete or a mixture of both). E.g. $C(\mathbf{s}(N, d)) = C(\mathbb{R}^{nd}/SO(d) \times \text{Dil})$, including $C(\mathbf{s}(N, 1)) = C(\mathbb{S}^{n-1})$ and $C(\mathbf{s}(N, 2)) = C(\mathbb{CP}^{n-1})$, $C(\mathbf{os}(N, d)) = C(\mathbb{R}^{nd}/SO(d) \times \text{Dil} \times \mathbb{Z}_2)$, includes both $C(\mathbf{os}(N, 1)) = C(\mathbb{S}^{n-1}/\mathbb{Z}_2)$ and $C(\mathbf{os}(N, 2)) = C(\mathbb{CP}^{n-1}/\mathbb{Z}_2)$. E.g. further examples involving bigger discrete groups in the case of (partially) indistinguishable particles. Does the definition of cone indeed survive the weakening from 'X a manifold' to 'X an orbifold'? Which theorems survive such a weakening?

Partial Answer. Witten and Atiyah [643, 4] consider cones over weighted projective spaces (Sec 3.16) can, at least in some cases, possess orbifold singularities. More general such are developed by Emmerich and Römer in [224].

³³Early papers involving such a notion of cone are Lemaître [422], Deprit-Delie [204], and also implicitly in the 3-body Celestial Mechanics work of Moeckel [470] and of Saari [549] (see also [326, 476, 478]). (See also 3-body Celestial Mechanics work such as [470, 549, 478, 326, 476].)

3.4.2 Many of this article's specific examples' cones are straightforward

These being cones and these cones being mathematically straightforward is the third key to unlocking RPM's. Moreover, via the scale-shape split, the above pure-shape RPM study is useful again within scaled RPM's.

$C(\mathfrak{S}(n, 1)) = C(\mathbb{S}^{n-1}) = \mathbb{R}^n$ using just elementary results (c.f. e.g. [368, 535]). $C(\mathfrak{S}(3, 2)) = C(\mathbb{CP}^1) = C(\mathbb{S}^2) = \mathbb{R}^3$ (see e.g. [326]). $C(\mathbf{0}\mathfrak{S}(n, 1)) = C(\mathbb{S}^{n-1}/\mathbb{Z}_2) = \mathbb{R}_+^n$, the half-space i.e. half of the possible generalized deficit angle.

$C(\mathbf{0}\mathfrak{S}(3, 2)) = C(\mathbb{S}^2/\mathbb{Z}_2^{\text{conj}}) = \mathbb{R}_+^3$. These last two, however, have edge issues. In 3-stop metroland, the identification coincides with the gluing in constructing the 'everyday cone'. Oriented 4-stop metroland is \mathbb{RP}^2 . On the other hand, Otriangleland is *not* \mathbb{RP}^2 but rather the sphere with reflective rather than inversive \mathbb{Z}_2 symmetry about the equator; one can think of this space loosely as a 'half-onion'.

In the case of plain shapes, one has $\mathfrak{t}(N, d) = C(\mathbb{S}^{nd-1})$, $\mathfrak{R}(N, 1) = C(\mathbb{S}^{n-1})$ and $\mathfrak{R}(N, 2) = C(\mathbb{CP}^{n-1})$. $\mathfrak{R}(3, 2) = C(\mathbb{CP}^1) = C(\mathbb{S}^2)$ is then a special case of this studied in e.g. [326, 476]. It is said to be better-behaved than its O-counterpart in that collinearities can be included (see above reference for description of the desingularization)). In the O case, the spaces one has are $\mathbf{0}\mathfrak{t}(N, d) = C(\mathbb{RP}^{nd-1})$, $\mathbf{0}\mathfrak{R}(N, 1) = C(\mathbb{RP}^{n-1})$ and $\mathfrak{R}(N, 2) = C(\mathbb{CP}^{n-1}/\mathbb{Z}_2)$. $\mathbf{0}\mathfrak{R}(3, 2) = C(\mathbb{CP}^1/\mathbb{Z}_2) = C(\mathbb{S}^2/\mathbb{Z}_2) = C(\mathbb{RP}^2)$ is then a special case of this. I also note that Atiyah and Witten have studied $C(\mathbb{CP}^3)$, which study is directly relevant to this article's relational program.

N.B. 1) By the scale-shape split, shape space is both the entirety of the reduced configuration space for the pure-shape theory and the shape part of the scale-shape split of the corresponding scaled theory.

N.B. 2) Because of pathologies at the origin for certain purposes, relational space is in some ways a less advantageous intermediate to study than preshape space.

	$d =$	1	2	3	4	5
$N = 3$		\mathbb{R}^2	\mathbb{R}^3			
4		\mathbb{R}^3	$C(\mathbb{CP}^2)$	\mathbb{R}^6		
5		\mathbb{R}^4	$C(\mathbb{CP}^3)$		\mathbb{R}^{10}	
6		\mathbb{R}^5	$C(\mathbb{CP}^4)$	missing triangle		\mathbb{R}^{15}
		\cdot	\cdot	\cdot	\cdot	\cdot
		N -stop	N -a-gon			Casson
		metrolands	lands			Diagonal
		\mathbb{R}^{n-1}	$C(\mathbb{CP}^{n-1})$			$\mathbb{R}^{d(d+1)/2}$

Figure 16: 'Partial periodic table' of the discernible relational spaces at the topological level.

3.4.3 More detailed topological properties of this article's relational space cones

All the RPM shape spaces X over which I take cones in this article are compact and Hausdorff. Coning in general sends compact spaces to noncompact ones, due to the scale itself being noncompact. This article's relational space cones are all Hausdorff. They are also path-connected [476].

Cones have a tendency to be straightforward from the topological point of view. Certainly the main particular examples of this article, which reduce to \mathbb{R}^n (that plain triangleland is homeomorphic to \mathbb{R}^3 : [326]), and \mathbb{R}_+^n , are topologically straightforward. As additional results, cones are 1) contractible (pp. 23-24 of [535]), and so has the same homotopy type as the point, and 2) acyclic, by which they have no nontrivial homology groups (pp. 43-46 of [485]). The situation with cohomologies is also straightforwardly related to that of the shape space X [380].

That $C(\mathbb{S}^{n-1}) = \mathbb{R}^n$ finishes off the topological study of $\mathfrak{R}(N, 1)$. The further results that $C(\mathbb{CP}^1) = C(\mathbb{S}^2) = \mathbb{R}^3$ and $C(\mathbb{CP}^1/\mathbb{Z}_2^{\text{conj}}) = C(\mathbb{S}^2/\mathbb{Z}_2^{\text{conj}}) = \mathbb{R}_+^3$, and, since $\mathbb{CP}^2/\mathbb{Z}_2 = \mathbb{S}^4$, $C(\mathbb{CP}^2/\mathbb{Z}_2) = C(\mathbb{S}^4) = \mathbb{R}^5$ finish off some more cases.

Question. What are the other $C(\mathbb{CP}^{N-2})$ and $C(\mathbb{CP}^{N-2}/\mathbb{Z}_2^{\text{conj}})$ from a detailed topological perspective?

3.4.4 Relational spaces at the level of metric spaces

Question. Is there an analogue of chordal distance? Of great circle distance? These questions are meant to be applied in cases where the relational space is not just \mathbb{R}^n . Can one combine an a \mathbb{CP}^{n-1} distance with a radial distance to make a well-defined notion of distance?

A crude suggestion for now is to use the following extension of the pure-shape metric

$$\text{Dist}_{\mathfrak{t}(N, d)}(\rho_1, s_1^i; \rho_2, s_2^i) = (\rho_1 - \rho_2)^2 + \text{Dist}_{\mathfrak{S}(N, d)}(s_1^i, s_2^i), \quad (212)$$

which does indeed straightforwardly satisfy the distance axioms. It is not necessarily clear if this $\text{Dist}_{\mathbf{r}(N,d)}$ is the most natural measure however, e.g. due to its heterogeneity. Given Riemannian structure and the considerations of Sec 14, more satisfactory notions of distance for these spaces are forthcoming.

3.5 The collision set and singular potentials

The above results about cones being topologically straightforward do not, however, cover the following cases.

1) Some applications do require the cone point to be excised, i.e. *punctured cones*. As an example of how this changes topological properties, as regards homotopies, this effectively amounts to a return to the shape space by means of a retract, and thus now possesses the same nontrivial homotopies as for that.

2) The binary collisions define distinguished submanifolds of codimension d in \mathbf{q} . For N particles, there are obviously $\binom{N}{2}$ such. Let us label them \mathcal{C}_{IJ} for coincident masses m_I and m_J . Then the collision set $\mathcal{C} := \bigcup_{I,J} \mathcal{C}_{IJ}$. Then the collision-free configurations are $\mathbf{q}^* = \mathbf{q} \setminus \mathcal{C}$.

The study of singular potentials can require excision of some points. If these potentials involve negative powers of all the $|\mathbf{r}^{IJ}|$, what one must excise is the collision set \mathcal{C} (the set of configurations that include collisions, whether non-maximal or maximal). While the HO-type potentials I study the most are not of this form, some of the further cosmologically-motivated potentials in Sec (5.1) are.

To demonstrate that this wider class of excisions is indeed capable of altering – and substantially complicating – the topology, I give the following example. The topologically trivial $\mathbf{q}(N,d) = \mathbb{R}^{nd}$, upon excision, picks up the nontrivial homotopy group

$$\pi_1(\mathbf{q}(N,d) \setminus \mathcal{C}) = \{\text{coloured braid group}\} , \quad (213)$$

the colouring here being a rephrasing of the notion of particle distinguishability.³⁴ Additional work in dynamics involves

$$\text{projective coloured braid group} = \text{N}(B) := B/Z(B) , \quad (214)$$

where $Z(\mathfrak{g})$ and $\text{N}(\mathfrak{g})$ denote the centre and normal subgroup of a group \mathfrak{g} , so as to consider the case in which each path and hence each braid is to return to its starting point. For $N = 3$, the configuration space is homotopic to \mathbb{S}^2 with 3 points (the D-points) excised and then the π_1 of this is the ‘free group on two letters’. Braid group applications to configuration space are also mentioned in [615, 2].

Question* What are the other homotopy groups for these collision-excised configuration spaces?

3.6 Riemannian geometry of shape spaces

3.6.1 Preshape space and the shape space of N -stop metroland

Lemma 5 (‘Preshape space is metrically easy’). $\mathbf{p}(N,d)$ is isometric to the Riemannian geometry $\langle \mathbb{S}^{nd-1}, \mathbf{M}^{\text{sph}}(1/2) \rangle$. [I.e. the standard spherical metric on the sphere of radius $1/2$.]

Proof $\mathbf{p}(N,d)$ is described by $\sum_{A=1}^{nd} \bar{r}_A^2 = \text{constant}$ (normalization condition, which is clearly the \mathbb{S}^{nd-1} sphere embedded in the usual way in \mathbb{R}^{nd} . \square)

Lemma 6 Straightforwardly, in Lemma 2’s Beltrami coordinates, the standard metric on the $\{nd-1\}$ -sphere has arc element

$$ds^2 = M_{AB} d\mathcal{X}^A d\mathcal{X}^B = \{ \{1 + \|\mathcal{X}\|^2\} \|d\mathcal{X}\|^2 - (\mathcal{X}, d\mathcal{X})^2 \} / \{1 + \|\mathcal{X}\|^2\}^2 . \quad (215)$$

[N.B. these coordinates bring out the parallel with \mathbb{CP}^k , and occur in the reduction procedure in Sec 3.14) but are not themselves that useful to work with.]

Lemma 7 On $\mathbf{p}(N,d)$ the geodesics are great circles.

Proof By Theorem 3, $\mathbf{p}(N,d)$ is isometric to \mathbb{S}^{nd-1} . Then it is well-known that the geodesics of spheres are great circles. \square

Lemma 8 $\mathfrak{S}(N,1) = \mathbf{p}(N,1)$, $\mathfrak{R}(N,1) = \mathbf{r}(N,1)$ (both homeomorphically and isometrically).

Corollary $\mathfrak{S}(N,1) = \mathbf{p}(N,1) = \mathbb{S}^{n-1}$ homeomorphically.

Proof $\mathfrak{S}(N,1) = \mathbf{p}(N,1)$ by the rotations being trivial in $1-d$. Then use Lemma 2: homeomorphically, $\mathbf{p}(N,1) = \mathbb{S}^{nd-1} = \mathbb{S}^{n-1}$. \square

³⁴That these are isomorphic is clear given that the particles are assumed distinguishable and the orbits can wind around each of the binary collisions in whatever order but not intersect with them. This matches the definition of a coloured braid if each distinguishable particle is taken to represent a colour.) In fact, the mathematical structure nowadays known as the coloured braid group was first discovered by Hurwitz in 1891 [329] in this very mechanics context. Thus, the discovery of this mathematical structure preceded Artin’s realization in 1925 [61] of its braid interpretation; these two works were first interconnected in [237]. See e.g. [127] for a review and updates on the theory of the braid group. See [476] for further study of this in the case of 3 particles in $2-d$, and [615, 2] for further applications.

Corollary $\mathfrak{S}(N, 1)$ is isometrically $\mathfrak{p}(N, 1)$ which is isometrically $\langle \mathbb{S}^{N-2}, \mathbf{M}^{\text{sph}}(1/2) \rangle$.

Proof $\mathfrak{S}(N, 1) = \mathfrak{p}(N, 1)$ by rotations being trivial in 1-d. Then use Lemma 5: $\mathfrak{p}(N, 1)$ is isometrically $\mathbb{S}^{nd-1} := \mathbb{S}^{N-2}$. \square

My convention for naming coordinates now is ρ_1 to ρ_n and then $\mathcal{R}_{\bar{a}} = \rho_{\bar{a}}/\rho_n$. [I called these $\mathcal{X}_{\mathcal{A}}$ in the preshape space context.] These are clearly ‘projective’ rather than spherical coordinates.

Corollary The metric on 1-d shape space is the appropriate subcase of (215), corresponding to the arc element

$$d\mathfrak{s}_{N\text{-stop SRPM}}^{\text{relationspace}^2} = \{ \{1 + \|\mathcal{R}\|^2\} \|\mathbf{d}\mathcal{R}\|^2 - (\mathcal{R}, \mathbf{d}\mathcal{R})^2 \} / \{1 + \|\mathcal{R}\|^2\}^2 ; \quad (216)$$

alternatively, this is

$$d\mathfrak{s}_{N\text{-stop SRPM}}^{\text{relationspace}^2} = \{ \|\rho\|^2 \|\mathbf{d}\rho\|^2 - (\rho, \mathbf{d}\rho)^2 \} / \|\rho\|^2 \quad (217)$$

(the ρ^i are homogeneous coordinates to the \mathcal{R}^a being inhomogeneous coordinates).

Recast in the much more common ultraspherical coordinates by the transformation

$$\theta_a = \arctan \left(\sqrt{\sum_{A=1}^a \mathcal{R}_A^2} / \mathcal{R}_{a+1} \right) . \quad (218)$$

which diagonalizes the metric, by which these coordinates are often also much more useful. The corresponding arc element is

$$d\mathfrak{s}_{N\text{-stop SRPM}}^{\text{relationspace}^2} = \|\mathbf{d}\theta\|_{\mathbf{M}_{\text{sph}}}^2 = \sum_{a=1}^{n-1} \prod_{\hat{p}=1}^{a-1} \sin^2 \theta_{\hat{p}} d\theta_a^2 \quad (219)$$

(218), where $\prod_{i=1}^0$ terms are defined to be 1.

As subexamples, for 4-stop metroland, in either H or K-coordinates, the coordinate transformation is, in terms of ρ^i ,

$$\theta = \arctan \left(\sqrt{\rho_1^2 + \rho_2^2} / \rho_3 \right) , \quad \phi = \arctan (\rho_2 / \rho_1) , \quad (220)$$

or, inversely, the even more familiar form

$$\rho_1 = \rho \sin \theta \cos \phi , \quad \rho_2 = \rho \sin \theta \sin \phi , \quad \rho_3 = \rho \cos \theta . \quad (221)$$

The coordinate ranges are $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$, so these are geometrically the standard azimuthal and polar spherical angles on the unit shape space sphere \mathbb{S}^2 .

3.6.2 Explicit geometrical objects for N -stop metroland

For $\mathfrak{S}(N, 1) = \mathbb{S}^{n-1}$ and in ultraspherical coordinates $\{\theta^a\}$, the inverse metric is

$$N^{\text{pq}} = \delta^{\text{pq}} \prod_{A=1}^{p-1} \sin^{-2} \theta_A d\theta_p^2 . \quad (222)$$

and the square root of the determinant is

$$\sqrt{M} = \prod_{r=1}^{n-1} \prod_{A=1}^r \sin \theta_A . \quad (223)$$

The nonzero Christoffel symbols in these coordinates are

$$\Gamma_{\text{qq}}^{\text{p}} = -\sin \theta_{\text{p}} \cos \theta_{\text{p}} \prod_{A=1, A \neq \text{p}}^{r-1} \sin^2 \theta_{\text{q}} , \quad \Gamma_{\text{qp}}^{\text{p}} = \cos \theta_{\text{q}} / \sin \theta_{\text{q}} . \quad (224)$$

The Ricci tensor is

$$\text{Ric}(\mathcal{M})_{\text{pq}} = \{nd - 2\} \mathcal{M}_{\text{pq}} \quad (225)$$

(so \mathbb{S}^{nd-1} is Einstein) and hence have constant Ricci scalar curvature

$$\text{Ric}(\mathcal{M}) = \{nd - 1\} \{nd - 2\} . \quad (226)$$

Finally, these spaces are all conformally flat, as an easy consequence of their being maximally symmetric. These results obviously immediately extend to $\mathfrak{p}(N, d) = \mathbb{S}^{nd-1}$.

$\mathfrak{S}(N, 1)$ and $\mathfrak{p}(N, d)$ are real manifolds; $\mathbb{S}^2 = \mathbb{CP}^1$ alone among them is also a complex manifold. The Euler and Pontrjagin classes of these as real manifolds and the Chern classes and characters of \mathbb{S}^2 as a complex manifolds are readily computable. (These are defined in e.g. [486] and are important as obstructions to quantization, and as regards issues concerning instantons and magnetic charges). We need some Chern classes for the discussion of some global issues in quantization. $c_1(\mathbb{S}^2) = 2$ and $c_2(\mathbb{S}^2) = 0$.

3.6.3 Isotropy groups and orbits for $d > 1$ shape spaces

Associated with the nontrivial quotient map Q are the orbits $\text{Orb}(\bar{X}) = Q^{-1}(Q(\bar{X})) = \langle T\bar{X} | T \in SO(d) \rangle$ and the stabilizers $\text{Stab}(\bar{X}) = \langle T \in SO(d) | T\bar{X} = \bar{X} \rangle$. These can furthermore be thought of as fibres and isotropy groups. The orbits or fibres are, for \bar{X} of rank e , $Q(\bar{X}) = \begin{cases} SO(d), & e \geq d+1 \\ \text{Stie}(d, e), & e < d+1 \end{cases}$ for $\text{Stie}(d, e) = SO(d)/SO(d-e)$ the Stiefel manifolds [313] of orthonormal e frames in \mathbb{R}^d . [Note that $\text{Stie}(3,1) = SO(3)/SO(2) = \mathbb{S}^2$: spherical orbits.]

N -a-gonlands are homogeneous spaces. This follows from $\mathbb{CP}^n \cong SU(n+1)/\{SU(n) \times U(1)\}$, which is a composite of Hopf's $\mathbb{CP}^{n-1} = \mathbb{S}^{2n-1}/U(1)$ and $SU(n)/SU(n-1) = \mathbb{S}^{2n-1}$ (see e.g. p 219 of [625]).

Let $D_e(N, d)$ be the subset of $\mathfrak{p}(N, d)$ corresponding to rank $\leq e$ and let $D_e^c(N, d)$ be its complement. I will often drop the (N, d) from the notation. Note that restricting attention to the discernible shape spaces on or above the Casson diagonal, D_{d-2} is the set with nontrivial isotropy groups. $Q(D_{d-2})$ is the *singularity set* of $\mathfrak{S}(N, d)$, while $Q(D_{d-2}^c)$ is the *nonsingular part* of $\mathfrak{S}(N, d)$.

In 1- d case, note that, from the triviality of the rotations involved, nothing needs to be induced from the sphere, nor is any minimization required. Additionally, the singularity set D is empty in this case.

Lemma 7 On $Q(D_{d-2}^c)$ there is a unique Riemannian metric compatible with the differential structure and with respect to which Q is particularly well behaved is inherited from $\mathfrak{p}(N, d)$.

Proof 1) by Riemannian submersion [493].

Proof 2) Alternatively, from first principles according to the steps below up to and including Structure B. \square

Note that the geodesic joining \bar{X} and \bar{Z} takes the form

$$\Gamma_{\bar{Z}}(\mathbf{s}) = \bar{X} \cos \mathbf{s} + \bar{Z} \sin \mathbf{s} \quad (227)$$

parametrized by geodesic distance, for $0 \leq \mathbf{s} \leq \pi$. The tangent vector to the geodesic is $d\Gamma_{\bar{Z}}(\mathbf{s})/d\mathbf{s}|_{\mathbf{s}=0} = \bar{Z}$.

Next, define the *exponential map* by $T_{\bar{X}}(\mathfrak{p}(N, d)) \rightarrow \mathfrak{p}(N, d)$ $\bar{Z} \mapsto \Gamma_{\bar{Z}/\|\bar{Z}\|}(\|\bar{Z}\|)$. It restricts to a diffeomorphism of $\langle \bar{Z} \in T_{\bar{X}}(\mathfrak{p}(N, d)) | \|\bar{Z}\| < \pi \rangle$ onto $\langle \mathfrak{p}(N, d)/\text{antipode of } \bar{X} \rangle$.

$\hat{\Gamma}_A$ is a curve in $SO(d)$ starting from the unit matrix \mathbb{I} so that the tangent vector at $\mathbf{s} = 0$, $d\hat{\Gamma}_A/d\mathbf{s}|_{\mathbf{s}=0} = A$ is tangent to $SO(d)$ at \mathbb{I} . $\exp(\mathbf{s}A)$ is in $SO(d)$ iff $\exp(\mathbf{s}A)\exp(\mathbf{s}A)^T = \mathbb{I}$ iff $\exp(\mathbf{s}\{A + A^T\}) = \mathbb{I}$ iff $A + A^T = 0$. So any skew-symmetric matrix \bar{A} represents a vector tangent to $SO(d)$ at \mathbb{I} . As the space of $d \times d$ skew symmetric matrices has dimension $d\{d-1\}/2 = \dim(SO(d))$, that is the entire tangent space to $SO(d)$ at \mathbb{I} . As $\exp(\mathbf{s}\bar{A})$ lies in $SO(d)$ whenever $A^T = -A$, $\hat{\gamma}_A(\mathbf{s}) = \exp(\mathbf{s}A)\bar{X}$ lies in the fibre orbit through \bar{X} .

The subspace of tangent vectors $d\hat{\gamma}_A(\mathbf{s})/d\mathbf{s}|_{\mathbf{s}=0} = A\bar{X}$ to such curves at \bar{X} is the *vertical* tangent subspace at \bar{X} , $V_{\bar{X}} = \langle A\bar{X} | A^T = -A \rangle$. Its orthogonal complement $H_{\bar{X}}$ i.e. such that $T_{\bar{X}}(\mathfrak{p}(N, d)) = V_{\bar{X}}(N, d) \oplus H_{\bar{X}}(N, d)$ is the *horizontal* tangent subspace at \bar{X} .

Note that for $Q(\bar{X})$ nonsingular, i.e. $\bar{X} \notin D_{d-2}$, $V_{\bar{X}}(N, d)$ is isomorphic to $SO(d)$ at \mathbb{I} . On the other hand, at a singular point A is tangent to the isotropy subgroup, so $V_{\bar{X}}(N, d)$ is isomorphic to $\text{Stie}(d, e)$ at \mathbb{I} .

Proposition 1 i) If a geodesic in $\mathfrak{p}(N, d)$ starts out in a horizontal direction, then its tangent vectors remain horizontal along it.

ii) Distance-parametrization and horizontality are preserved under $SO(d)$.

Proof i) by definition, the geodesic $\Gamma_{\bar{Z}}(\mathbf{s})$ is horizontal at $\mathbf{s} = 0$ iff $\bar{X}\bar{Z}^T = \bar{Z}\bar{X}^T$. Then $\forall \mathbf{s}$,

$$\Gamma_{\bar{Z}} \left\{ \frac{d\Gamma_{\bar{Z}}}{d\mathbf{s}} \right\} = \frac{d\Gamma_{\bar{Z}}}{d\mathbf{s}} \Gamma_{\bar{Z}}(\mathbf{s})^T$$

via the explicit formula (227) for the geodesics and trivial algebra. Thus each tangent vector $d\Gamma_{\bar{Z}}/d\mathbf{s}$ complies with the definition of horizontal at $\Gamma_{\bar{Z}}(\mathbf{s})$.

ii) $T\Gamma_{\bar{Z}}(\mathbf{s}) = T\bar{X} \cos \mathbf{s} + T\bar{Z} \sin \mathbf{s}$ is a distance-preserving geodesic by (227).

$\text{tr}(T\bar{X}\{T\bar{Z}\}^T) = \text{tr}(T\bar{X}\bar{Z}^T T^T) = \text{tr}(T^T T\bar{X}\bar{Z}^T) = \text{tr}(\bar{X}\bar{Z}^T)$ by the cyclic identity and T orthogonal.

Next, $T\bar{X}\{T\bar{Z}\}^T = T\bar{X}\bar{Z}^T T^T = T\bar{X}\bar{Z}^T T^T = T\bar{Z}\{T\bar{X}\}^T$ (with the third step using horizontality of the untransformed geodesic), which reads overall that the transformed geodesic is then also horizontal. \square

(Riemannian) Structure B i) Thus the exp function restricted to $H_{\bar{X}}$, $\exp|_{H_{\bar{X}}}$, maps $\langle \text{vectors of length} < \pi \rangle$ onto the submanifold $\mathbb{H}_{\bar{X}}$ of $\mathfrak{S}(N, d)$ defined by $\mathbb{H}_{\bar{X}} = \langle \mathbf{s} \in \mathfrak{S}(N, d) | \text{all tangent vectors at } \bar{X} \text{ are horizontal} \rangle$.

ii) The tangent spaces to the fibre and to $\mathbb{H}_{\bar{X}}$ are clearly perpendicular. Thus there is a neighbourhood $U_{\bar{X}}$ such that $\forall Y \in U_{\bar{X}}$ the tangent spaces to the fibre and to $\mathbb{H}_{\bar{X}}$ remain transverse. Thus the fibre at Y meets $U_{\bar{X}}$ only at Y . Thus, one

has established that given X outside D_{d-2} , through each point $T\bar{X}$ of the fibre at \bar{X} there is a submanifold $U_{\bar{X}}$ traced out by local horizontal geodesics through $T\bar{X}$ with the following properties.

- a) $U_{T_1\bar{X}}$ and $U_{T_2\bar{X}}$ are disjoint if $T_1 \neq T_2$.
- b) Each submanifold $U_{T\bar{X}}$ is mapped by the quotient mapping Q bijectively and thus homeomorphically with respect to the quotient topology onto a neighbourhood of $Q(\bar{X}) \in \mathfrak{S}(N, d)$.
- c) The action of each $S \in SO(d)$ restricts to a diffeomorphism of $U_{T\bar{X}}$ that also preserves the Riemannian metric. I.e., it maps geodesics to geodesics of the same length and its derivative maps horizontal tangent vectors at $T\bar{X}$ to horizontal tangent vectors of the same length at $ST\bar{X}$. Thus one can use $U_{\bar{X}}$, $Q|_{U_{\bar{X}}}$ to determine a differential structure on the nonsingular part of shape space: $Q(D_{d-2})$. This is since, for any other choice $(U_{T\bar{X}}, Q|_{U_{T\bar{X}}})$, the composition $(Q|_{U_{T\bar{X}}})^{-1} \circ Q|_{U_{\bar{X}}}$ is just the diffeomorphism $T|_{U_{\bar{X}}}$.
- iii) The above ensures independence of which point on the fibre is used. Thus I have a Riemannian metric on the nonsingular part of shape space.

Note that this metric is naturally induced from $\mathfrak{p}(N, d) = \mathbb{S}^{nd-1}$. It has been defined such that $Q : H_{\bar{X}}(N, d) \longrightarrow \mathfrak{p}(N, d) \longrightarrow T_{Q(\bar{X})}(\mathfrak{S}(N, d))$, which is a Riemannian submersion (the second arrow is an isometric map).

A *geodesic* in $\mathfrak{S}(N, d)$ is the image of any horizontal geodesic in $\mathfrak{p}(N, d)$. Note that this permits geodesics to pass between strata. Thus geodesics can be extended beyond the nonsingular part of the space, and this serves to extend the above Riemannian structure (see [368] for these results, which not used in the present article).

Proposition 2 The geodesics of (227), Lemma 4 (or the associated Riemannian metric of Structure B) provide the same metric distance D as Structure A.

Proof By definition, the geodesic between two shapes $Q(\bar{X})$ and $Q(\bar{Y})$ is the image of a horizontal geodesic Γ from \bar{X} to some point $T\bar{Y}$ in fibre $Q(\bar{Y})$. Since Γ meets the fibres orthogonally at these points [Γ being horizontal and the fibres being vertical], so the induced distance that follows from the geodesics/associated Riemannian metric is indeed [c.f. (205)]

$$T \in SO(d) \left(\text{Dist}(\bar{X}, T\bar{Y}) \right) = \arccos \left(T \in SO(d) \text{tr}(\bar{X}^T T\bar{Y}) \right) := \text{Dist}(Q(\bar{X}), Q(\bar{Y})) \quad .\square$$

Note that what one has constructed thus is a Riemannian structure on $Q(D_{d-2}^c)$. In general, one would have to worry about the geometry on D_{d-2} – the name ‘singularity set’ does indeed carry curvature singularity connotations. But this article circumvents that by considering only 1- and 2- d , for which the singularity set is empty. Thus for these cases, what one has constructed above is a Riemannian structure everywhere on shape space (and one can then show computationally that there are no curvature singularities within these shape spaces).

3.6.4 Natural metric on the shape space of N -a-gonland

In 2- d rotations are simpler than in higher d while being nontrivial. It is this that lies behind Lemma 3’s straightforward complex representations for $\mathfrak{p}(N, 2)$ and $\mathfrak{S}(N, 2)$. The latter representation, $\langle Z^a, \{a = 1 \text{ to } n - 1\}$ has two manifest symmetries: \mathbb{Z}_2 conjugation and $U(N - 1)/U(1)$ permutations of coordinates. Each of these commutes with the $SO(2)$ action.

Other simplifications in 2- d are that $S(N, 2) = \mathbb{CP}^{n-1}$ topologically [ii] of the ‘Partial Periodic Table’ Theorem]. Also, the minimization in the quotient metric (in the metric space sense) of Structure A may be carried out explicitly in 2- d by use of the complex representation as follows.

Proposition 3 $\cos(\text{Dist}(Q(\mathbf{z}), Q(\mathbf{w}))) = |(\mathbf{z} \cdot \mathbf{w})_C| / \|\mathbf{z}\|_C \|\mathbf{w}\|_C$ for $(\cdot)_C$ the \mathbb{C}^n inner product [so $(\mathbf{z} \cdot \mathbf{w})_C = \sum_{a=1}^n \bar{z}^a \cdot w^a$ and $\|\cdot\|_C$ the corresponding norm].

Proof By the definition of Dist , the left-hand side is $\alpha \in [0, 2\pi) \text{tr}(|\mathbf{z}|^{-1} |\mathbf{w}|^{-1} e^{i\alpha} \mathbf{z} \mathbf{w}^T)$, the numerator of which contains $(\text{Re}(\mathbf{z}) \text{Im}(\mathbf{z})) \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \text{Re}(\mathbf{w}) \\ \text{Im}(\mathbf{w}) \end{pmatrix} = A \cos \alpha + B \sin \alpha$, where $A = \text{Re}(\mathbf{w} \cdot \mathbf{z})_C$ and $B = \text{Im}(\mathbf{w} \cdot \mathbf{z})_C$. The maximum condition which follows from this is then $\tan \alpha = B/A$, for which the maximum value is $\sqrt{A^2 + B^2} / \|\mathbf{z}\|_C \|\mathbf{w}\|_C$. \square

Kendall’s Theorem The corresponding Riemannian arc element is

$$d \text{Dist}^2 = \{ \|\mathbf{z}\|_C^2 \|\mathbf{d}\mathbf{z}\|_C^2 - |(\mathbf{z} \cdot \mathbf{d}\mathbf{z})_C|^2 \} / \|\mathbf{z}\|_C^4 = \{ \{1 + \|\mathbf{Z}\|_C^2\} \|\mathbf{d}\mathbf{Z}\|_C^2 - |\mathbf{Z} \cdot \mathbf{d}\mathbf{Z}|_C^2 \} / \{1 + \|\mathbf{Z}\|_C^2\}^2 =: d\mathbf{s}_{N-a\text{-gon SRPM}}^{\text{relationalspace}}{}^2. \quad (228)$$

in terms of homogeneous and inhomogeneous coordinates respectively.

Proof Consider $\mathbf{w} = \mathbf{z} + \delta\mathbf{z}$. Then

$$\delta \text{Dist}^2 = \sin^2 \delta \text{Dist} + O(\delta \text{Dist}^4) = 1 - \cos^2 \text{Dist}(Q(\mathbf{z}), Q(\mathbf{z} + \delta\mathbf{z})) + O(\delta \text{Dist}^4) \quad (229)$$

then use the Proposition 3, linearity, the binomial expansion and take the limit as $\delta\mathbf{z} \longrightarrow 0$ to get the first form. Then divide top and bottom by $\|z_i\|_C^4$ and use the definition of \mathbf{Z}^a to get the second form. \square

N.B. this arc element (which indeed is Riemannian, its positive-definiteness following from the Schwarz inequality) is the classical Fubini–Study [242, 595] arc element on \mathbb{CP}^{N-2} .³⁵ This is the natural arc element thereupon, such that its constant curvature is 4.³⁶

Thus the following has been proven.

Corollary $S(N, 2) = \langle \mathbb{CP}^{N-2}; \mathbf{M}^{\text{FS}} \rangle$ isometrically (with curvature constant 4).

Corollary $\mathfrak{S}(3, 2) = \langle \mathbb{S}^2; \mathbf{M}^{\text{sphe}} \text{ of radius } 1/2 \rangle$ isometrically.

Proof Now $\| \cdot \|_{\text{C}} = | \cdot |$, so two terms cancel in the second form of (228), leaving

$$ds_{\Delta\text{-SRPM}}^{\text{relationalspace}^2} = dZd\bar{Z}/\{1 + Z^2\}^2 = \{d\mathcal{R}^2 + \mathcal{R}^2 d\Phi^2\}/\{1 + \mathcal{R}^2\}^2 = \{d\Theta^2 + \sin^2\Theta d\Phi^2\}/4 \quad (230)$$

by using the polar form for the complex numbers and the coordinate transformation $\mathcal{R} = \tan \frac{\Theta}{2}$. \square

One can also use

$$ds_{\Delta\text{-SRPM}}^{\text{relationalspace}^2} = \dot{\mathcal{R}}^2 + \mathcal{R}^2 \dot{\Phi}^2 \quad (231)$$

by performing a PPSCT with conformal factor $\{1 + \mathcal{R}^2\}^2$. This is geometrically trivial, while the other above forms are both geometrically natural and mechanically natural (equivalent to \mathbf{E} appearing as an eigenvalue free of weight function). Also using the conformal factor $\{1 + \mathcal{R}^2\}$ is the conformally-natural choice.

Note 1) In 2- d , the $SO(d)$ action is free (i.e. if $gx = x$, then $g = \text{id}$). Thus the stabilizers $\{g \in \mathfrak{g} \mid gx = x\}$ are all trivial. Thus, by the orbit–stabilizer theorem, everything lies on the one orbit. I.e. there is no stratification/no singularity set. There are then no complications in considering geodesics. Also, the natural metric on shape space is everywhere-defined and everywhere of finite curvature.

Note 2) In contrast, in 3- d , for which, for collinear configurations, it is $SO(2)$ rather than $SO(3)$ that is relevant. This gives a number of further reasons why the 3- d case is harder.

Note 3) See Sec 3.10.4 for geometrical/mechanical interpretations of \mathcal{R} , Θ and Φ . Useful coordinates for pure-shape triangle and are then the azimuthal angle $\Theta = 2 \arctan \mathcal{R}$ and the ‘Swiss army knife’ angle between the 2 Jacobi vectors, $\Phi = \arccos(\rho_1 \cdot \rho_2 / |\rho_1| |\rho_2|)$ (c.f. Fig 6).

3.6.5 Explicit geometrical objects for pure-shape N -a-gonland

Using the multipolar form $Z^{\mathbf{a}} = \mathcal{R}^{\mathbf{a}} \exp(i\Theta^{\mathbf{a}})$, the configuration space metric can be written in two blocks: $M_{\mathbf{p}\bar{\mathbf{q}}} = 0$,

$$\mathcal{M}_{\mathbf{p}\bar{\mathbf{q}}} = \delta_{\mathbf{p}\bar{\mathbf{q}}}/\{1 + \|\mathcal{R}\|^2\} - \mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}/\{1 + \|\mathcal{R}\|^2\}^2, \quad (232)$$

$$\mathcal{M}_{\mathbf{p}\bar{\mathbf{q}}} = \{\delta_{\mathbf{p}\bar{\mathbf{q}}}/ - \mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}/\{1 + \|\mathcal{R}\|^2\}\} \mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}/\{1 + \|\mathcal{R}\|^2\} \text{ (no sum)}. \quad (233)$$

Then the inverse metric is $\mathcal{N}^{\mathbf{p}\bar{\mathbf{q}}}$ is

$$\mathcal{N}^{\mathbf{p}\bar{\mathbf{q}}} = \{1 + \|\mathcal{R}\|^2\} \{\delta^{\mathbf{p}\bar{\mathbf{q}}} + \mathcal{R}^{\mathbf{p}}\mathcal{R}^{\bar{\mathbf{q}}}\}, \quad \mathcal{N}^{\widetilde{\mathbf{p}}\bar{\mathbf{q}}} = \{1 + \|\mathcal{R}\|^2\} \{\delta^{\widetilde{\mathbf{p}}\bar{\mathbf{q}}}/\mathcal{R}_{\bar{\mathbf{p}}}^2 + 1^{\widetilde{\mathbf{p}}\bar{\mathbf{q}}}\} \text{ (no sum)}, \quad (234)$$

for $1^{\widetilde{\mathbf{p}}\bar{\mathbf{q}}}$ the matrix whose entries are all 1. Then the only nonzero first partial derivatives of the metric are (no sum)

$$\mathcal{M}_{\mathbf{p}\bar{\mathbf{q}}, \bar{\mathbf{r}}} = \{4\mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}\mathcal{R}_{\bar{\mathbf{r}}}/1 + \|\mathcal{R}\|^2 - \{2\mathcal{R}_{\bar{\mathbf{r}}}\delta_{\mathbf{p}\bar{\mathbf{q}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta_{\mathbf{p}\bar{\mathbf{r}}} + \mathcal{R}_{\bar{\mathbf{p}}}\delta_{\mathbf{q}\bar{\mathbf{r}}}\}/\{1 + \|\mathcal{R}\|^2\}^2\}, \quad (235)$$

$$\mathcal{M}_{\mathbf{p}\bar{\mathbf{q}}, \bar{\mathbf{r}}} = \frac{2\mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}}{\{1 + \|\mathcal{R}\|^2\}^2} \left\{ \frac{2\mathcal{R}_{\bar{\mathbf{p}}}\mathcal{R}_{\bar{\mathbf{q}}}\mathcal{R}_{\bar{\mathbf{r}}}}{1 + \|\mathcal{R}\|^2} - \{\mathcal{R}_{\bar{\mathbf{r}}}\delta_{\mathbf{p}\bar{\mathbf{q}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta_{\mathbf{p}\bar{\mathbf{r}}} + \mathcal{R}_{\bar{\mathbf{p}}}\delta_{\mathbf{q}\bar{\mathbf{r}}}\} \right\} + \frac{\delta_{\mathbf{p}\bar{\mathbf{q}}}\{\mathcal{R}_{\bar{\mathbf{p}}}\delta_{\bar{\mathbf{q}}\bar{\mathbf{r}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta_{\bar{\mathbf{p}}\bar{\mathbf{r}}}\}}{1 + \|\mathcal{R}\|^2}. \quad (236)$$

The only nonzero Christoffel symbols are

$$\Gamma^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}\bar{\mathbf{r}}} = -\{\mathcal{R}_{\bar{\mathbf{r}}}\delta^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta^{\bar{\mathbf{p}}}_{\bar{\mathbf{r}}}\}/\{1 + \|\mathcal{R}\|^2\}, \quad (237)$$

$$\Gamma^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}\bar{\mathbf{r}}} = \delta_{\bar{\mathbf{q}}\bar{\mathbf{r}}}\delta^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}} - \mathcal{R}_{\bar{\mathbf{q}}}\mathcal{R}_{\bar{\mathbf{r}}}\{\mathcal{R}_{\bar{\mathbf{r}}}\delta^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta^{\bar{\mathbf{p}}}_{\bar{\mathbf{r}}}\}/\{1 + \|\mathcal{R}\|^2\}, \quad (238)$$

$$\Gamma^{\bar{\mathbf{p}}}_{\bar{\mathbf{q}}\bar{\mathbf{r}}} = \left\{ \frac{\delta^{\bar{\mathbf{p}}\bar{\mathbf{s}}}}{\mathcal{R}_{\bar{\mathbf{p}}}^2} + 1^{\bar{\mathbf{p}}\bar{\mathbf{s}}} \right\} \left\{ \frac{\mathcal{R}_{\bar{\mathbf{s}}}\mathcal{R}_{\bar{\mathbf{q}}}}{\{1 + \|\mathcal{R}\|^2\}^2} \left\{ \frac{2\mathcal{R}_{\bar{\mathbf{s}}}\mathcal{R}_{\bar{\mathbf{q}}}\mathcal{R}_{\bar{\mathbf{r}}}}{1 + \|\mathcal{R}\|^2} - \{\mathcal{R}_{\bar{\mathbf{r}}}\delta_{\bar{\mathbf{q}}\bar{\mathbf{s}}} + \mathcal{R}_{\bar{\mathbf{s}}}\delta_{\bar{\mathbf{q}}\bar{\mathbf{r}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta_{\bar{\mathbf{s}}\bar{\mathbf{r}}}\} \right\} + \frac{\delta_{\bar{\mathbf{s}}\bar{\mathbf{q}}}\{\mathcal{R}_{\bar{\mathbf{s}}}\delta_{\bar{\mathbf{q}}\bar{\mathbf{r}}} + \mathcal{R}_{\bar{\mathbf{q}}}\delta_{\bar{\mathbf{s}}\bar{\mathbf{r}}}\}}{1 + \|\mathcal{R}\|^2} \right\}. \quad (239)$$

³⁵As regards appearance of these results in the N -body dynamics literature, that the Fubini–Study metric occurs for these was known to Iwai in 1987 [348], whilst Montgomery [475, 476, 477] states that this result was almost certainly known prior to that too. Indeed, I found that Kendall cites this result starting with a brief mention in 1977 [361] and built on this in his Whitehead Lecture given to the London Mathematical Society in 1980 and in his 1984 article [363]. Montgomery’s work and a large amount of literature from Molecular Physics [427] involves an absolute–relative split of Newtonian Mechanics, while Kendall’s work turns out to be more highly desirable as regards a whole-universe relational program.

³⁶Note that Fubini–Study metrics of this form are available for any N for ratios and relative angles paired together as complex coordinates on \mathbb{CP}^{N-2} . This is promising as regards the arbitrary- N 2- d case, as \mathbb{CP}^{N-2} is the reduced configuration space for scale, translation and rotation free shapes in 2- d [368].

These spaces have Ricci tensor

$$\text{Ric}_{\text{pq}}(\mathcal{M}) = 2n M_{\text{pq}} \quad (240)$$

so \mathbb{CP}^{n-1} is Einstein (hence its appearance in the literature on GR instantons [254, 497]), and thus these are also spaces of constant Ricci scalar curvature

$$\text{Ric}(\mathcal{M}) = 4n \{n-1\} . \quad (241)$$

However, for $N > 3$, they have nonzero Weyl tensor (as checked by Maple [440]) and so are not conformally flat. In fact, they have a very simple Weyl tensor [624]. None of the abovementioned curvatures, or curvature scalars constructed from them and the metric, blow up for finite $\mathcal{R}_{\bar{p}}$.

The second Chern class for quadrilateralland is $c_2(\mathbb{CP}^2) = 1$ [349].

3.6.6 Isometry groups for N -stop metroland

N -stop metroland's shape spaces $\mathfrak{S}(N, 1)$ are \mathbb{S}^{n-1} , which are the positively-curved maximally symmetric spaces. Thus they have the maximum possible isometry group dimension. For configuration space dimension q , the maximal isometry group has dimension $q\{q+1\}/2$, so here $n\{n-1\}/2$. $\text{Isom}(\mathfrak{S}(N, 1)) = \text{Isom}(\mathbb{S}^{n-1}) = \text{PSO}(n)$ (the n -dimensional projective special orthogonal group) $= \text{SO}(n)$. These results obviously immediately extend to $\mathfrak{p}(N, d) = \mathbb{S}^{nd-1}$. Also, $\text{Isom}(\mathbf{0}\mathfrak{S}(N, 1)) = \text{Isom}(\mathbb{RP}^{n-1}) = \text{the projective orthogonal group } \text{PO}(n) = \text{O}(n)/\text{Z}(\text{O}(n))$, which is $\text{SO}(n)$ again.

3.6.7 Isometry groups for pure-shape N -a-gonlands

$\text{Isom}(\mathfrak{S}(N, 2)) = \text{Isom}(\mathbb{CP}^{n-1}) = \text{the projective unitary group } \text{PSU}(n) = \text{SU}(n)/\text{Z}(\text{SU}(n)) = \text{SU}(n)/\mathbb{Z}_n$. Thus these are fairly symmetrical but not maximally symmetric for $N > 1$. For, there are $n^2 - 1$ Killing vectors associated with $\text{SU}(n)$, while the maximum number of Killing vectors for a configuration space of dimension $2\{n-1\}$ is $\{n-1\}\{2n-1\} = 2n^2 - 3n + 1$. One can in fact characterize \mathbb{CP}^{n-1} as the spaces of constant sectional curvature, which are in a sense the next most symmetric spaces after the maximally symmetric spaces of constant curvature.

E.g. for quadrilateralland, \mathbb{CP}^2 has the 8 Killing vectors [254] associated with $\text{SU}(3)$, rather than the maximal 10. This case's representation theory is particularly well-known due to its appearing in the particle physics' approximate flavour symmetry and exact colour symmetry. Then, for pentagonland, \mathbb{CP}^3 has 15 Killing vectors associated with $\text{SU}(4)$ rather than the maximum possible of 21. Next, for hexagonland, there are 24 compared to the maximum possible of 36. Hexagonland's representation theory and conserved quantities thus turn out to be particularly well-known through sharing the same mathematics with the well-known $\text{SU}(5)$ grand unified theory. However, mathematicians and physicists have studied the entirety of $\text{SO}(p)$ and $\text{SU}(q)$ well, so one is not compelled technically to stick to these particular cases.

One way of studying the Killing Vectors for \mathbb{CP}^{n-1} is that these can be obtained by projecting down the Hopf fibration of \mathbb{CP}^{n-1} by \mathbb{S}^1 with base manifold \mathbb{S}^{2n-1} [625].

	$d =$	1	2	3	4	5
$N = 3$		$\text{SO}(2)$	$\text{SO}(3) = \text{SU}(2)/\mathbb{Z}_2$			
4		$\text{SO}(3)$	$\text{SU}(3)/\mathbb{Z}_3$			
5		$\text{SO}(4)$	$\text{SU}(4)/\mathbb{Z}_4$			
6		$\text{SO}(5)$	$\text{SU}(5)/\mathbb{Z}_5$	slightly bigger missing triangle		
		$\text{SO}(n)$	$\text{SU}(n)/\mathbb{Z}_n$			

Figure 17: The ‘partial Periodic Table of Periodic Tables’ illustrating the two series of known isometry groups that coincide with the well-known families of Lie groups $\text{SO}(p)$ and $\text{SU}(p)$. Taking out \mathbb{Z}_n amounts to rendering a connected component, a step which, of course, makes no difference infinitesimally. In any case, there is a good (and relational!) reason for colour modelling of the strong force to involve this connected component also (namely that the colour labels red, green and blue are frivolous, in the sense that the theory is invariant under permuting these names). Thus these isometry groups and the subsequent representation theory are widely known both in mathematics and in physics. The Casson diagonal is, as far as I know, only a simple series at the topological level.

Finally, I note that $\mathbb{CP}^k/\mathbb{Z}_2$ has in general a different number of Killing vectors from \mathbb{CP}^k , at least under some of some the weighted projective space group actions (see SSec 3.16). A toy model (see e.g. Sec 3.8.8) for this is how identifications can cause loss of some Killing vectors from \mathbb{R}^2 . This is due to these becoming discontinuous at the identification points, so that the identified topology does not globally support such Killing vectors. The triangleland case's number of Killing vectors

is unaffected by whether one identifies mirror images (that case keeps all 3 of the $SU(2)$ Killing vectors). I have not yet considered the specific case of $\mathbb{CP}^k/\mathbb{Z}_2^{\text{conj}}$ for $k > 2$.

3.7 Riemannian geometry of O(pre)shapespace (mostly open questions)

3.7.1 Oriented N -stop metroland: \mathbb{RP}^n

Metric geometry is local, so this is as in the previous section as regards metric, inverse, determinant, Christoffel symbols and curvature tensors.³⁷

3.7.2 Question: what is the Riemannian geometry of N -a-gonland's $\mathbb{CP}^n/\mathbb{Z}_2^{\text{conj}}$?

As a partial answer **Corollary of Kuiper's theorem** ([409, 410], Sec 3.10.9) $\mathbb{CP}^2/\mathbb{Z}_2$ is diffeomorphic to \mathbb{S}^4 . The conj action is a means of representing reflection. Moreover, the metric here is not the standard spherical one. For, locally the metric is as for \mathbb{CP}^2 : Fubini–Study (8 local Killing vectors) whilst the 4-sphere's spherical metric has 10.

3.7.3 Questions outside scope of this article on higher- d Oshape spaces

Question. What is the mathematics of pure-shape ‘ Nd -simplexland’ for $d \geq 3$?

It is considerably harder [368] than for $d \leq 2$ though at least pure-shape RPM is free of maximal collisions. Though at least pure-shape RPM is free of collisions that are *maximal* (i.e. between all the particles at once). For increasing N , even the topology quickly cease to be standard or even well-studied, while the singular set begins to play a prominent and obstructory role. [368] does however describe and provide references for a partial study of (easier subcases of) these shape spaces.

3.8 Riemannian geometry of relational space

Relational space $\mathfrak{R}(N, d)$ can then be envisaged in scale–shape split form as the *cone* $C(\mathfrak{S}(N, d))$ over the corresponding shape space $\mathfrak{S}(N, d)$ [35].

3.8.1 Parametrizing scale

In GR, one uses e.g. the scalefactor a such that $h_{\mu\nu} = a^2 u_{\mu\nu}$ for $\det(u_{\mu\nu}) = 1$; then the alternative $\sqrt{h} = \sqrt{\det(h_{\mu\nu})} = a^3$, which is local volume. There is also the Misner variable $\Omega = \ln(\sqrt{h}) = 3\ln(a)$ (which is sometimes called ‘Misner time’ in the literature; see Sec 11.5 for how this and other scale variables fare as candidate times).

Analogy 34), the RPM counterparts of these are, respectively, ρ or I as analogues of scalefactor. The above shape analysis can be uplifted to above to scaled RPM, alongside a further size variable, by the cone construction/scale–shape split. This further variable can be taken to be $\rho = \sqrt{I}$ for N -stop metroland, or I for triangleland. ρ is termed the *hyperradius* in the literature; its use apparently dates back to Jacobi [56]. In the present article, I usually choose to call this by the more descriptive name **configuration space radius**. It was used in QM at least as far back as the 50's by Fock [235] and by Morse and Feshbach [479]; see also e.g. [581, 475, 326]. I is used e.g. in [205, 348, 428] though these are for θ running over half of the range most often used in the present article (but there is an analogous use of I in the case involving the whole range too). I term $\rho^{\dim(\mathfrak{R}(N, d))} = \rho^{nd-d\{d-1\}/2}$ *pseudo-volume scales* due to their parallels with GR's local volume scale. Finally, I term quantities proportional to $\ln \rho$ *pseudo-Misner scales* due to their parallel with GR's Misner variable.

3.8.2 Shape–scale split as a cone at the level of metric geometry

At the level of Riemannian geometry, a **cone** $C(\langle M, \mathbf{g} \rangle)$ over a k -dimensional Riemannian space $\langle M, \mathbf{g} \rangle$ is a $(k + 1)$ -dimensional metric space with 1) topological properties as in the preceding Sec. 2) Now, it additionally possesses also a arc element

$$ds^2 = d\sigma^2 + \sigma^2 ds^2, \quad (242)$$

where ds^2 the arc element of X itself and σ is a suitable ‘radial variable’ running over $[0, \infty)$, which represents the distance from the cone point. This metric is smooth everywhere except (in general) at the troublesome cone point. The above arc element is the second half of the third key point, and has the shape–scale split application, by which shape space geometry, dynamics, QM is re-applicable as a subproblem of the scaled case.

Note 1) The everyday-life cone can indeed be viewed as a simple example of these constructs, using $X = (\text{a piece of}) \mathbb{S}^1$.

Note 2) At the Riemannian level, there is a notion of distance and hence (for sufficiently nontrivial dimension) of angle, so that one can talk in terms of deficit angle. $C(\mathbb{S}^1)$ itself has no deficit angle, while using a p -radian piece entails a deficit angle of $2\pi - p$. The presence of deficit angle, in turn, gives issues about ‘conical singularities’, e.g. [360].

³⁷Note as an analogy also within Mechanics that \mathbb{RP}^2 has also been used as a simple model of the falling cat [474]. This consists of 2 jointed rods, for which one can see how the number of degrees of freedom matches the present problem's 2 Jacobi vectors. However, the cat model is physically/geometrically distinct, so this analogy is at most partial.

Note 3) Notes 1) and 2) (generalized to ‘deficit solid angle’) straightforwardly generalize to \mathbb{S}^k .

Note 4) (**Lemma 10.**) The simplest few cases of RPM’s involve very straightforward cones. $C(\mathfrak{S}(N, 1)) = C(\mathbb{S}^{n-1}) = \mathbb{R}^n$ using just elementary results (c.f. e.g. [368, 535]). $C(\mathfrak{S}(3, 2)) = C(\mathbb{CP}^1) = C(\mathbb{S}^2)$ of radius $1/2 = \mathbb{R}^n$ (see e.g. [326, 494]) up to a conformal factor at the metric level, which can be ‘passed’ to the potential, c.f. Sec 4.

Note 5) $C(\mathbf{OS}(n, 1)) = C(\mathbb{S}^{n-1}/\mathbb{Z}_2) = \mathbb{R}_+^n$, the half-space i.e. half of the possible generalized deficit angle. $C(\mathbf{OS}(3, 2)) = C(\mathbb{S}^2/\mathbb{Z}_2) = \mathbb{R}_+^3$. These last two, however, have edge issues. 3-stop metroland: the identification coincides with the gluing in constructing the ‘everyday cone’. Oriented 4-stop metroland is \mathbb{RP}^2 . Finally, triangleland might include the boundary or might not, depending on the application, and should not be glued as antipodal points in the $z = 0$ plane do not match. One can loosely think of this as a half-onion.

3.8.3 Scaled N -stop metroland as a cone

The plain N -stop metroland case is straightforward as $C(\mathbb{S}^{n-1})$ is \mathbb{R}^n (c.f. [494], also the topology Sec and below). The O case can be viewed as having a deficit angle of π for 3 particles and a deficit solid angle of 2π for 4 particles. This involves coordinate ranges rather than form of arc element, so below arc elements serve for both. [Oriented cases do have further issues as regards whether conical singularities show up. These could lead to having to excise the cone point.]

$C(\mathbb{CP}^k)$ is mentioned by Montgomery [476], alongside consideration of the corresponding collision set.

Minus the cone point, plain triangleland is diffeomorphic to \mathbb{R}^3 [326]. Is the version I consider in [30]. If one keeps the $1/4I$ factor, the geometry is curved, with a curvature singularity at 0, and, obviously, conformally flat where the conformal transformation is defined (i.e. elsewhere than 0). In any case the E/I ‘potential term’ in [30, 37] is singular there too.

Clearly

$$ds_{N\text{-stop ERP}}^{\text{relationalspace } 2} = d\rho^2 + \rho^2 ds_{\text{sph}}^2 . \quad (243)$$

3.8.4 Scaled N -a-gonland as a cone

For N -a-gonland,

$$ds_{N\text{-a-gon ERP}}^{\text{relationalspace } 2} = d\rho^2 + \rho^2 ds_{\text{FS}}^2 . \quad (244)$$

Then e.g. for scaled triangleland, as the shape space is the sphere of radius $1/2$,

$$ds_{\triangle \text{ERP}}^{\text{relationalspace } 2} = d\rho^2 + \{\rho^2/4\}\{d\Theta^2 + \sin^2\Theta d\Phi^2\} , \quad (245)$$

which inconvenience in coordinate ranges can be overcome³⁸ by using, instead, I as the configuration space radius variable,

$$ds_{\triangle \text{ERP}}^{\text{relationalspace } 2} = \{1/4I\}\{dI^2 + I^2\{d\Theta^2 + \sin^2\Theta d\Phi^2\}\} , \quad (246)$$

which metric itself is conformally flat; the flat metric itself is

$$ds_{\text{flat } \triangle \text{-ERP}}^{\text{relationalspace } 2} = dI^2 + I^2\{d\Theta^2 + \sin^2\Theta d\Phi^2\} \quad (247)$$

(spherical polar coordinates with I as radial variable). This using of I as radial variable: the start of significant differences between the triangleland and 4-stop metroland spheres and half-spheres. In the O case, one can also use a double angle variable running over the usual range of angles e.g. [581, 425].

Question* I leave details of the metric and topological structure of $C(\mathbb{CP}^n)$ $n > 1$, $C(\mathbb{RP}^n)$ $n > 2$ and $C(\mathbb{CP}^n/\mathbb{Z}_2)$ $n > 1$ as problems for a further occasion.

Minus the cone point, plain triangleland is diffeomorphic to \mathbb{R}^3 [326]. Is the version I consider in [30]. If one keeps the $1/4I$ factor, the geometry is curved, with a curvature singularity at 0, and, obviously, conformally flat where the conformal transformation is defined (i.e. elsewhere than 0). In any case the E/I ‘potential term’ in [30, 37] is singular there too.

3.8.5 Aside: comments on the pyramidal presentation of scaled triangleland

I mention this due to it appearing in discussions in Sec 14. Parametrizing triangleland by the three sides gives the pyramidal presentation of the relational configuration space in [80, 83] (also mentioned in [428]). Note that this is limited in its usefulness compared to the spherical presentation, because it lacks naturality at the metric level.

3.8.6 Question: what is the geometry of scaled ON-a-gonland?

I.e. of the cone $C(\mathbb{CP}^{n-1}/\mathbb{Z}_2^{\text{conj}})$. Of possible relevance to this study, Witten, Acharya and Joyce [643, 66, 3, 359] (see Sec 3.16) consider cones over both \mathbb{CP}^3 and a WCP^3 .

³⁸See [357] for a distinct way of doing this. Also, from now on I upgrade ‘configuration space radius’ to mean the portmanteau of ‘usually ρ but $\rho^2 = I$ in the case of the spherical presentation of triangleland. By this subtlety, I am not in the end taking ‘configuration space radius’ to be entirely synonymous to ‘hyperradius’. One use for the spherical presentation of triangleland is use that it allows the shape part to be studied in \mathbb{S}^2 terms which more closely parallel the Halliwell–Hawking [296] analysis of GR inhomogeneities over \mathbb{S}^3 .

3.8.7 Explicit geometrical objects for scaled N -a-gonland

Using the multipolar form; then the configuration space metric can be written in three blocks (with all other components zero):

$$\mathcal{M}_{00} = 1, \quad \mathcal{M}_{\bar{p}\bar{q}} = \rho^2 \{ \{1 + \|\mathcal{R}\|^2\}^{-1} \delta_{\bar{p}\bar{q}} - \{1 + \|\mathcal{R}\|^2\}^{-2} \mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}} \}, \quad (248)$$

$$\mathcal{M}_{\bar{p}\bar{q}} = \rho^2 \{ \{1 + \|\mathcal{R}\|^2\}^{-1} \delta_{\bar{p}\bar{q}} - \{1 + \|\mathcal{R}\|^2\} \mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}} \} \mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}} \text{ (no sum)}, \quad (249)$$

$$\mathcal{M}_{\bar{p}\bar{q}} = 0. \quad (250)$$

Then the inverse metric's nonzero components are

$$\mathcal{N}^{00} = 1, \quad \mathcal{N}^{\bar{p}\bar{q}} = \{1/\rho^2\} \{ \{1 + \|\mathcal{R}\|^2\} \{ \delta^{\bar{p}\bar{q}} + \mathcal{R}^{\bar{p}} \mathcal{R}^{\bar{q}} \} \} \quad (251)$$

$$\mathcal{N}^{\bar{p}\bar{q}} = \{1/\rho^2\} \{ \{1 + \|\mathcal{R}\|^2\} \{ \delta^{\bar{p}\bar{q}} / \mathcal{R}_{\bar{p}}^2 + 1^{\bar{p}\bar{q}} \} \} \text{ (no sum)}. \quad (252)$$

Then the only nonzero first partial derivatives of the metric are (no sum)

$$\mathcal{M}_{\bar{p}\bar{q}, \bar{r}} = \frac{1}{\{1 + \|\mathcal{R}\|^2\}^2} \left\{ \frac{4\mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}} \mathcal{R}_{\bar{r}}}{1 + \|\mathcal{R}\|^2} - \{2\mathcal{R}_{\bar{r}} \delta_{\bar{p}\bar{q}} + \mathcal{R}_{\bar{q}} \delta_{\bar{p}\bar{r}} + \mathcal{R}_{\bar{p}} \delta_{\bar{q}\bar{r}}\} \right\}, \quad (253)$$

$$\mathcal{M}_{\bar{p}\bar{q}, \bar{r}} = \frac{2\mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}}}{\{1 + \|\mathcal{R}\|^2\}^2} \left\{ \frac{2\mathcal{R}_{\bar{p}} \mathcal{R}_{\bar{q}} \mathcal{R}_{\bar{r}}}{1 + \|\mathcal{R}\|^2} - \{ \mathcal{R}_{\bar{r}} \delta_{\bar{p}\bar{q}} + \mathcal{R}_{\bar{q}} \delta_{\bar{p}\bar{r}} + \mathcal{R}_{\bar{p}} \delta_{\bar{q}\bar{r}} \} \right\} + \frac{\delta_{\bar{p}\bar{q}} \{ \mathcal{R}_{\bar{p}} \delta_{\bar{q}\bar{r}} + \mathcal{R}_{\bar{q}} \delta_{\bar{p}\bar{r}} \}}{1 + \|\mathcal{R}\|^2}. \quad (254)$$

The only nonzero Christoffel symbols are (no sum except over \bar{s}) the same as for \mathbb{CP}^n with the following extra nonzero ones:

$$\Gamma_{\bar{p}\bar{q}}^\rho = \mathcal{M}_{\bar{p}\bar{q}}/\rho, \quad \Gamma_{\bar{p}\rho}^{\bar{p}}(\text{no sum}) = 1/\rho. \quad (255)$$

These spaces have Ricci tensor

$$\text{Ric}_{\bar{p}\bar{q}}(\mathcal{M}) = 3\mathcal{M}_{\bar{p}\bar{q}}/\rho^2 \quad (256)$$

(run from 1 to 4) with all other components zero, and

$$\text{Ric}(\mathcal{M}) = 6n\rho^2, \quad (257)$$

which is *not* constant (reported in [20] for triangleland case). Additionally, for $N > 3$, they have nonzero Weyl tensor (as checked by Maple [440]) and so are not conformally flat.

Then the Ricci scalar blows up as $\rho \rightarrow 0$, so this is a curvature singularity. The Ricci tensor and Weyl tensor's components are finite (remember $\mathcal{M}_{\bar{p}\bar{q}}$ already has an ρ^2 factor).

As regards $C(\mathbb{RP}^k)$ and $C(\mathbb{CP}^k/\mathbb{Z}_2^{\text{conj}})$ geometries, these are locally as before. However, they have scope for new singularities due to the identifications... The latter is now the coning of a metrically nonstandard 4-sphere. It is also not clear how one associates a \mathbb{R}^5 with a 6- d system's objects (three 2- d relative Jacobi vectors). This is like triangleland's \mathbb{R}^3 for a 4- d system's objects, but now there is no longer a Hopf map to provide the answer? [The 4- d case trivially contains \mathbb{S}^3 , which Hopf-maps to \mathbb{S}^2 which is trivially contained in 3- d . Now, however the 6- d case is \mathbb{S}^5 which does not Hopf-map to \mathbb{S}^4 . It is \mathbb{S}^7 which does. That gives a small chance to 8- d of absolute space trivially containing \mathbb{S}^7 , which Hopf-maps to \mathbb{S}^4 which sits trivially in \mathbb{R}^5 . In any case, there are only 3 Hopf maps of this kind, so one would run out eventually. However, the Hopf map admits a generalization via its $\mathbb{S}^3 \rightarrow \mathbb{CP}^1$ aspect to $\mathbb{S}^{2n+1} \rightarrow \mathbb{CP}^n$. This might be exploitable in generalizing the Dragt coordinates to the general N -a-gon, the first stop being for $\mathbb{S}^5 \rightarrow \mathbb{CP}^2$ of quadrilateralland.]

3.8.8 Isometry groups for scaled N -stop metrolands

$\text{Isom}(\mathfrak{A}(N, 1)) = \text{Isom}(\mathbb{R}^n) = \text{Eucl}(n)$, which is the flat case of maximally symmetric space, and thus possesses $n\{n+1\}/2$ independent Killing vectors. Next, $\text{Isom}(\mathfrak{OA}(N, 1)) = \text{Isom}(\mathbb{R}^n/\mathbb{Z}_2)$. For $n = 3$, this gives $\text{SO}(3) \times \mathbb{R}^2$ ($\partial/\partial z$ being discontinuous). This result also extends easily to arbitrary n .

3.8.9 Isometry groups for scaled N -a-gonlands

The N -stop metroland case is somewhat of a fluke in terms of computability. N -a-gonland requires more general considerations.

Lemma 11 Coning i) preserves Killing vectors and ii) gives a new conformal Killing vector. (The proof is straightforward).

Question* Does coning \mathbb{CP}^{n-1} and $\mathbb{CP}^{n-1}/\mathbb{Z}_2^{\text{conj}}$ produce any further (conformal) Killing vectors? As this concerns the N -a-gonland relational spaces, I view understanding it as crucial for the future of this research program.

We do control the 3-particle case of course, through it being conformal to the flat metric. We also know that $C(\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}})$ is diffeomorphic to \mathbb{R}^5 as a corollary of Kuiper's theorem. However the metric on this is not the standard flat one, as its Ricci scalar is proportional to $1/\rho^2$. It is not conformally flat either, since its Weyl tensor does not vanish.

3.9 Edges and singularities of configuration space

Nontrivial group orbit structure is in general relevant here. Also, one sometimes needs to excise due to singular potentials, though detail of this is diverted to Sec 5.

3.10 Some more useful coordinatizations of specific RPM's configuration spaces

3.10.1 \mathbb{S}^k within \mathbb{R}^{k+1}

Spherical coordinates $[(\alpha, \chi)]$ for the most common case of the 2-sphere, see Overall Appendix A.4) are good for solving many aspects of the dynamics of the sphere no matter what its physical interpretation is to be. Working at the level of a general joint treatment, one can recast the k -sphere in terms of $k+1$ variables u^Γ such that $\sum_{\Gamma=1}^k u^{\Gamma^2} = 1$. These describe a Euclidean $\{k+1\}$ -space that surrounds the sphere; it is often convenient to use u_x, u_y and u_z for the components of u^Γ in the case of the 2-sphere in 3- d Euclidean space. Then the u^Γ are related to the α and χ through being the components of the corresponding unit Cartesian vector in spherical polar coordinates:

$$u_x = \sin \alpha \cos \chi, \quad u_y = \sin \alpha \sin \chi, \quad u_z = \cos \alpha. \quad (258)$$

For \mathbb{S}^2 in 'actual space', the \mathbb{R}^3 is 'actual space' with the physical radius r in the role of the radial coordinate.

3.10.2 Coordinates for N -stop metroland

For 4-stop metroland, I make subsequent use of

$$\mathcal{R} = \tan \frac{\theta}{2} = \sqrt{\{1 - n_z\} / \{1 + n_z\}}, \quad (259)$$

which is a radial stereographic coordinate corresponding to the tangent plane at the 'North' DD-pole. As regards surrounding coordinates for the pure-shape N -stop metroland configuration (or coordinates for the scaled N -stop metroland configuration space), here, the ρ^i serve straightforwardly as Cartesian coordinates. These are subject to the on-hypersphere condition

$$\sum_{i=1}^n \mathbf{I}^i = \sum_{i=1}^n \rho^{i^2} = \rho^2 = \mathbf{I} \quad (\text{constant}), \quad \text{or} \quad \sum_{i=1}^n \mathbf{N}^i = \sum_{i=1}^n \mathbf{n}^{i^2} = 1. \quad (260)$$

The \mathbf{n}^i are then the components of the unit Cartesian vector [e.g. $\mathbf{n}^i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ in spherical polar coordinates in the 4-stop case]. I call passing from x^α in n - d space to ρ^i in N -stop metroland configuration space the *Cartesian correspondence*.

3.10.3 N -stop metrolands relationally interpreted

There are pure-shape versions and scaled versions of these, for which my notation is lower case for the former and upper-case for the latter. The former are related to the latter by multiplication in the obvious way by the applicable size quantity $\rho := \text{Size}$ or $\mathbf{I} := \text{SIZE}$.

Scaled N -stop metroland's Cartesian components are all cluster-dependent ratio quantities. These concern how large a given cluster is relative to the whole model or how well-separated the two clusters are in the latter case that has enough particles to build up such a quantity. For 3-stop metroland [50], $n_1^{(\text{Hb})}$ is a measure of how large the universe is relative to cluster bc. $n_2^{(\text{Hb})}$ is a measure of how large the universe is relative to the separation between cluster bc and particle a. For 4-stop metroland [50], in Jacobi H-coordinates or K-coordinates, my convention is to position the ρ^i as in (220). In the H-coordinates case, these are, respectively a measure of the size of the universe's contents relative to the size of the whole model universe, and a measure of inhomogeneity among the contents of the universe (whether one of the constituent clusters is larger than the other one.) Also, $n_3^{(\text{Hb})} := \text{RelSize}(1b, cd)$ is a measure of how large the universe is relative to the separation between clusters 1b and cd. This being large means physically that clusters $\{1b\}$ and $\{cd\}$ each cover but a small portion of the model universe, and corresponds geometrically to the polar caps. $n_1^{(\text{Hb})} := \text{RelSize}(1b)$ is a measure of how large the universe is relative to cluster 1b. This being small means physically that cluster $\{1b\}$ is but a speck in the firmament, and corresponds geometrically to a belt around the 'Bangladeshi' meridian. This being large means physically that cluster $\{1b\}$ engulfs the rest of the model universe, and corresponds geometrically to an antipodal pair of caps around each of the intersections of the equator and the Greenwich meridian. $n_2^{(\text{Hb})}$ is a measure of how large the universe is relative to cluster cd. This being small corresponds to clusters $\{1b\}$ and $\{cd\}$ being merged, and corresponds geometrically to the equatorial belt.

In the K-coordinates case, these are now respectively a measure the sizes of the $\{12\}$ and $\{T3\}$ clusters relative to the whole model universe and of the sizes of the $\{12\}$ and $\{T3\}$ clusters relative to each other. $n_1^{(\text{Ka})} = \text{RelSize}(ab)$ is a measure of how large the universe is relative to cluster ab. $n_2^{(\text{Ka})} = \text{RelSize}(ab, c)$ is a measure of how large the universe is relative to the separation between subcluster ab and particle c. $n_3^{(\text{Ka})} = \text{RelSize}(abc, d)$ is a measure of how large the universe is relative to the separation between cluster abc and particle d. Note that these have a more symmetric meaning in H coordinates than in K coordinates or for 3 particles.

3.10.4 Stereographic and Dragt coordinate systems for triangleland

A sometimes convenient coordinatization is in terms of $\mathcal{R} = \tan \frac{\Theta}{2}$. This is geometrically the stereographic coordinate in the tangent plane to the North Pole, and is physically another coordinatization of the non-angle ratio that is the tallness of the triangle. Two further useful coordinatizations for this are as in Fig 18.

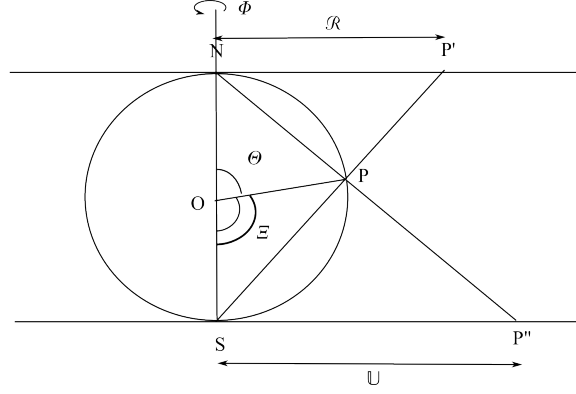


Figure 18: Interrelation between the following useful coordinates \mathcal{R} and $\mathcal{U} = 1/\mathcal{R}$, and its supplementary angle $\Xi = \pi - \Theta$. I term the underlying operation between each of these pairs the *duality map*; in ρ_i or I_i variables, it takes the form of interchanging the 1-indices and the 2-indices. N is the North Pole, S is the South Pole, O is the centre. Sometimes N and S are E and \bar{E} , and sometimes they are D and M (see Sec 3.11.3 for this article's axis systems for triangleland).

Like for 4-stop metroland, the shape space sphere for triangleland can be taken to sit in embedding relational space. However, for triangleland, one no longer has the above straightforward Cartesian map. Pure-shape triangleland's ρ_1 and ρ_2 are not related to the Cartesian coordinates of the surrounding relational space in the above familiar Cartesian way. Rather, they are related in the less straightforward extended Hopf map/Dragt correspondence way. For the particular case of triangleland, this is the fourth key step in understanding the physics, and again is a kinematical manoeuvre with precedent in the Theoretical Molecular Physics literature.

That corresponds to having to use not ρ but I as radial variable in triangleland. (This is due to triangleland's shape sphere arising from \mathbb{CP}^1 , which gives its natural radius an unusual factor of $1/2$, which is absorbed by the coordinate transformation to radial variable I . This makes triangleland quite unlike 4-stop metroland or actual space as regards the physical meaning of its u^Γ .) I.e. the triangleland u^Γ are Dragt coordinates [205] are related to the configuration space's coordinates, rather, by the mathematics of the $S^3 \rightarrow S^2$ Hopf-type map.³⁹ I then term substituting x^α by Dra^Γ , or, alternatively, substituting the u^α of space for: dra^Γ the *Dragt correspondence*. Dropping (a) labels,

$$\text{dra}_x = \sin \Theta \cos \Phi = 2n_1 n_2, \cos \Phi = 2\{\underline{n}_1 \times \underline{n}_2\}_3, \quad (261)$$

$$\text{dra}_y = \sin \Theta \sin \Phi = 2n_1 n_2 \sin \Phi = 2\underline{n}_1 \cdot \underline{n}_2, \quad (262)$$

$$\text{dra}_z = \cos \Theta = N_2 - N_1. \quad (263)$$

The 3 component in the first of these indicates the component in the fictitious third dimension of this cross-product. In these depending on squared quantities, one can see consequences of $\rho^2 = I$ and not ρ being the radial coordinate. So one goes from (α, χ) to u^Γ , $\Gamma = 1$ to 3, and then, for 4-stop metroland, to $n_{(\text{Hb})}^i$ via the Cartesian correspondence, and, for triangleland to dra via the Dragt correspondence. The metric is then

$$ds_{\Delta \text{ ERPM}}^{\text{relational space } 2} = \sum_{\Gamma=1}^3 d\text{Dra}^\Gamma{}^2. \quad (264)$$

Note: the usual Dragt coordinates are related to ours by $\text{Dra}^\Gamma = I \text{dra}^\Gamma$; moreover in the literature it is usually the O-case's half-space for which these are presented.

3.10.5 My geometrical interpretation of Dragt-type coordinates

I drop (a) labels. Now, dra_z is the 'ellipticity'

$$\text{ellip} = N_2 - N_1 \quad (265)$$

of the two 'normalized' partial moments of inertia involved in the (a)-clustering. This (and Θ itself) is a function of a pure ratio of relative separations rather than of relative angle. It is closely related to $\text{sharp} = N_2$, a sharpness quantity – how

³⁹See [279, 581] for earlier literature and e.g. [57, 467, 348, 494, 428] for some applications of these coordinates. Note that in the plain case, Hopf applies. Iwai [348] then wrote down the O-counterpart of this. Also note that Dragt's own work is for 3-cornerland's \mathbb{R}_+^3 ; outside of this context, I call the analogous coordinates "of Dragt type". There also being some slight differences here between 2- d and 3- d , plain and O triangles, and the scaled and pure-shape cases, I refer to the coordinates used as of 'Dragt-type'. (Most e.g. Dragt [205], Iwai [348], Littlejohn and Reinsch [428] have the O-case's range for angles, though e.g. Hsiang [326] and I [30, 34] do not [which is natural from the intrinsically 2- d perspective].).

sharp the triangle is with respect to the (a)-clustering, and flat = N_1 , which is likewise a flatness quantity. Moreover sharp and flat are but linear functions of ellip:⁴⁰

$$\text{sharp} = \{1 + \text{ellip}\}/2 \quad \text{and} \quad \text{flat} = \{1 - \text{ellip}\}/2. \quad (266)$$

The rearrangement $\text{ellip} = 1 - 2\text{flat}$ is furthermore useful interpretationally as flat = (length of the {bc} base) per unit I , which is an obviously primary quantity within the mass-weighted triangle. Θ can also be considered as a ratio variable. That ellip is $\cos \Theta$ subsequently plays an important role in this article.

Φ is the relative angle ‘rightness variable’ right corresponding to each clustering. Triangles with $\Phi = \pi/2$ and $3\pi/2$ are (a)-isosceles, which is as (a)-right as possible, while 0 and π are collinear – which is as unright as possible. Thus, dra_x and dra_y provide mixed ratio and relative angle information. The ratio information for both of these of these is a $2n_1n_2 = \sqrt{1 - \text{ellip}^2}$ factor. On the other hand, the relative angle information is in the $\cos \Phi$ and $\sin \Phi$ factors. The relation $\cos \Phi = \text{ellip}/\sqrt{1 - \{4 \times \text{area}\}^2}$ is useful in the below analysis.

One can view dra_x as a measure of ‘anisoscelesness’ [i.e. departure from (a)’s notion of isoscelesness, c.f. anisotropy as a departure from isotropy in GR Cosmology], so I denote it by aniso [with an (a)-label]. It is specifically a measure of anisoscelesness in that $\text{aniso(a)} \times I$ per unit base length in mass-weighted space is the $l_1 - l_2$ indicated in Fig 3. I.e., it is the amount by which the perpendicular to the base fails to bisect it (which it would do if the triangle were isosceles).

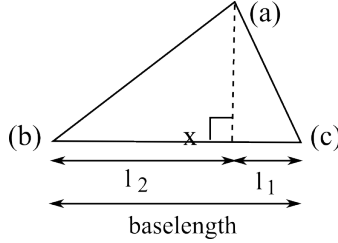


Figure 19: Definitions of l_1 and l_2 in interpretation of this article’s notion of anisoscelesness.]

One can likewise view $\text{dra}_y^{(a)}$ as a measure of noncollinearity. Moreover this is actually clustering-independent/a democracy invariant [662, 56, 427]. It is furthermore equal to $4 \times \text{area}$ (the area of the triangle per unit I in mass-weighted space). In contrast, pure-shape 4-stop metroland has no democratic invariant.

$$\text{physical area} = I \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}} \frac{4 \times \text{area}}{4} = \frac{I\sqrt{3}}{m} \text{area} \quad (267)$$

(with the second equality holding for the equal-mass case) and collinearity retains its precise meaning (and area-minimizing property) in the arbitrary-mass case. However, the meanings of some of the quantities in this Subsection change. Equilateral becomes, in all clusterings, to equal-magnitude mass-weighted Jacobi vectors (i.e. equal moments of inertia) which are perpendicular to each other; it remains area-maximizing. (a)-isosceles retains this perpendicularity with respect to the (a) clustering, so that anisoscelesness generalizes to unrightness in this sense. The definition of ellipticity is unchanged since that already involved moments of inertia.

Finally, the Dra(a)^Γ are $I \text{dra}^\Gamma$, $\text{Ellip(a)} := I \text{ellip(a)}$, $\text{Aniso(a)} := I \text{aniso(a)}$, and $\text{Area} := I \text{area}$.

3.10.6 Democratic invariants

Moving between different clusterings involves linear transformations $u^\Gamma = D^\Gamma_\Lambda u^\Lambda$, termed ‘democracy transformations’ in e.g. [427]. The present article’s notion of ‘clustering invariant’ thus coincides with the Theoretical Molecular Physics literature’s notion of ‘democracy invariant’.

Note: these are d -independent transformations but the number and nature of democracy invariants is dimension-dependent. The area Dragt coordinate is a democratic invariant $\text{demo}(3)$ and is useable as a measure of uniformity [38], its modulus running from maximal value at the most uniform configuration (the equilateral triangle) to minimal value for the collinear configurations.

Note that this is new to triangleland: N -stop metroland had no shape variables of this nature (the configuration space radius is always of this nature.) See [38] for derivation of the $\text{demo}(4)$ counterpart of this for quadrilateralland.

3.10.7 Parabolic-type coordinates for triangleland

One sometimes also swaps the dra_2 for the scale variable I in the non-normalized version of the coordinates to obtain the $\{I, \text{aniso}, \text{ellip}\}$ system (see next SSec for further interpretation). Then a simple linear recombination of this is $\{I_1, I_2, \text{aniso}\}$, i.e. the two partial moments of inertia and the dot product of the two Jacobi vectors. This is in turn closely related [30] to

⁴⁰Barbour calls sharp configurations ‘needles’ and Kendall refers collectively to considerably sharp and flat configurations as ‘splinters’.

the parabolic coordinates on the flat \mathbb{R}^3 conformal to the triangleland relational space, which are $\{I_1, I_2, \Phi\}$. The arc element is then

$$ds^2 = \frac{dI_1^2}{4I_1} + \frac{dI_2^2}{4I_2} + \frac{I_1 I_2}{I} d\Phi^2 = \frac{1}{I} \left\{ \frac{I}{4} \left\{ \frac{dI_1^2}{I_1} + \frac{dI_2^2}{I_2} \right\} + I_1 I_2 d\Phi^2 \right\} . \quad (268)$$

i.e. conformal to the (confocal) parabolic coordinates. This is a useful identification to make as regards solving and understanding the classical dynamics and quantum mechanics of the system.

N.B. the existence of coordinate systems of this kind is irrespective of whether one is making the plain or O choice. I note also the following inversions from the definitions of the Dragt coordinates,

$$\text{Flat} := I_1 = \{I + \text{Dra}_z\}/2 = \{I + \text{Ellip}\}/2 = I\{1 + \cos\Theta\}/2 , \quad \text{Sharp} := I_2 = \{I - \text{Dra}_z\}/2 = \{I - \text{Ellip}\}/2 = I\{1 - \cos\Theta\}/2 . \quad (269)$$

These invert to

$$I = I_1 + I_2 , \quad \Theta = \arccos(\{I_2 - I_1\}/\{I_1 + I_2\}) . \quad (270)$$

A physical interpretation for these is that I_1, I_2 are the partial moments of inertia of the base and the median, with Φ the ‘Swiss army knife’ angle between these. They are clearly a sort of subsystem-split coordinates and thus useful in applications concerning subsystems/contextual ideas (see Sec 14).

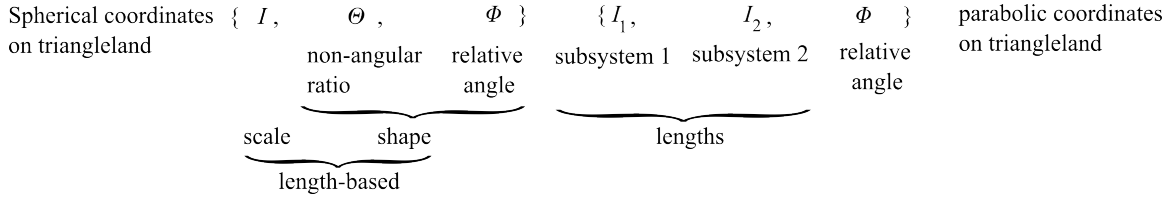


Figure 20: The shape, scale, length ratio, relative angle and subsystem statuses of the spherical and parabolic-type coordinate systems.

For triangleland, spherical, Dragt and parabolic coordinates are keys 4, 5 and 6 to unlocking the mathematics of the model.

3.10.8 Useful coordinate systems for quadrilateralland. I. Shape and Kuiper coordinates

This is a first departure from standard kinematics, which would here, rather follow its 3- d prejudice and consider tetrahedronland. Nevertheless, quadrilateralland possesses \mathbb{CP}^2 mathematics and coordinate systems for this have been worked out in detail for a number of other applications of this space to theoretical physics.

The Dragt correspondence does not straightforwardly carry though to larger 2- and 3- d models; the situation in my 2- d case of interest is rather different from that in the more usually studied 3- d case. Even for 4 particles, there are multiple ways of building analogues of the Dragt coordinates in 3- d . For 2- d , see [38].

Further useful coordinate systems for quadrilateralland are as follows (see also Sec 3.10.10).

1) $\{s^\Gamma\}$ are a redundant set of six shape coordinates (see [38] for their explicit forms), which, according to the construction in [427, 38], are a quadrilateralland analogue of triangleland’s Dragt coordinates

2) The coordinate system $\{I, s^\gamma\}$ turns out to be more useful in this case [38]. These are the counterpart of triangleland’s $\{I, \text{aniso}, \text{ellip}\}$ albeit they are now redundant.

3) Kuiper’s coordinates are then a simple linear combination of 2) (mixing the I, s_1 and s_5 coordinates). These consist of all the possible inner products between pairs of Jacobi vectors, i.e. 3 magnitudes of Jacobi vectors per unit MOI N^e , alongside 3 $\mathbf{n}_f \cdot \mathbf{n}_g$ ($e \neq f$), which are very closely related to the three relative angles. As such, they are, firstly, very much an extension of the parabolic coordinates for the conformally-related flat \mathbb{R}^3 of triangleland [30]. Secondly, in the quadrilateralland setting, they are a clean split into 3 pure relative angles (of which any 2 are independent and interpretable as the anioscesenesses of the coarse-graining triangles) and 3 magnitudes (supporting 2 independent non-angular ratios). I therefore denote this coordinate system by $\{N^e, \text{aniso}(e)\}$. Thirdly, whilst they clearly contain 2 redundancies, they are fully democratic in relation to the constituent Jacobi vectors and coarse-graining triangles made from pairs of them. This coordinate system is a close analogue of the parabolic coordinates for triangleland (c.f. Fig 20 and Fig 23), albeit now redundant. The very closest parabolic coordinate analogue involves the three partial moments of inertia alongside the three relative angles themselves. Note: the above can be viewed as quadrilateralland counterparts of the Dragt coordinates, the $I, \text{aniso}, \text{ellip}$ coordinate system and the parabolic coordinate system (i.e. key steps 4 and 5).

In Kuiper coordinates, each of the three on- \mathbb{S}^2 conditions for whichever of H or K coordinates involves losing one magnitude and two inner products. Thus the survivors are two magnitudes and one inner product. If the Kuiper coordinates are recombined to form the $\{I, s_\gamma\}$ system, and swap the I for the s_6 then the survivors of the procedure are the Dragt coordinates (since the procedure kills two of the three area contributions to the s_6). This gives another sense in which the $\{s^\Gamma\}$ system is a

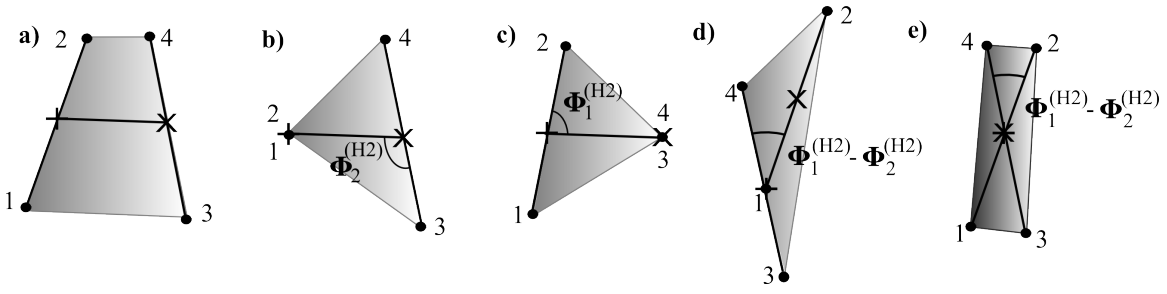


Figure 21: Figure of the coarse-graining triangles. a) For Jacobi H-coordinates coordinates for a given quadrilateral, collapsing each of $\rho_1^{(H2)}$, $\rho_2^{(H2)}$ and $\rho_3^{(H2)}$ in turn gives the coarse-graining triangles b), c) and d) [38]. [One can use d) to interpret the associated anisoscelesness, but the true physical situation is that of e) i.e. a parallelogram.]

natural extension of the Dragt coordinate system. For extension of above analysis to considerations of democratic invariants for quadrilateralland, see [38].

3.10.9 The split of quadrilateralland into hemi- \mathbb{CP}^2 's of oriented quadrilaterals

Definition: The *Veronese surface* Ver [303] is the space of conics through a point (parallel to how a projective space is a set of lines through a point). It is the embedding $\mathbb{CP}^2 \rightarrow \mathbb{CP}^5$ Ver: $[x, y, z] \rightarrow [x^2, y^2, z^2, yz, xz, xy]$ where x, y, z are homogeneous coordinates [303] (and can be taken to formalize the statement that 5 points determine a conic).

Kuiper's Theorem i) The map

$$\eta : (\mathbb{CP}^2) \xrightarrow{\quad} \xrightarrow{\quad} \mathbb{E}^5 \quad (|z_1|^2, |z_2|^2, |z_3|^2, \{z_2 \bar{z}_3 + z_3 \bar{z}_2\}/2, \{z_3 \bar{z}_1 + z_1 \bar{z}_3\}/2, \{z_1 \bar{z}_2 + z_2 \bar{z}_1\}/2) \quad (271)$$

induces a piecewise smooth embedding of $\mathbb{CP}^2/\mathbb{Z}_2^{\text{conj}}$ onto the boundary of the convex hull of Ver in \mathbb{E}^5 , which moreover has the right properties to be the usual smooth 4-sphere [409].

Restricting $\{z_1, z_2, z_3\}$ to the real line corresponds in the quadrilateralland interpretation to considering the collinear configurations, which constitute a \mathbb{RP}^2 space as per Fig 11. Moreover the above embedding sends this onto Ver itself. Proving this proceeds via establishing that, as well as the on- \mathbb{S}^5 condition $N_1 + N_2 + N_3 = 1$, a second restriction holds, which in the quadrilateralland interpretation, reads $\sum_e \text{aniso}(e)^2 N^e - 4N_1 N_2 N_3 + \text{aniso}(1) \text{aniso}(2) \text{aniso}(3) = 0$. (Knowledge of this restriction should also be useful in kinematical quantization [332], and it is clearer in the $\{N^e, \text{aniso}(e)\}$ system, which is both the quadrilateralland interpretation of Kuiper's redundant coordinates and a simple linear recombination of the $\{I, s^\gamma\}$ coordinates obtained in [38], than in these other coordinates themselves.)

Another form for Kuiper's theorem [409]. Moreover, \mathbb{CP}^2 itself is topologically a double covering of \mathbb{S}^4 branched along the \mathbb{RP}^2 of collinearities which itself embeds onto \mathbb{E}^5 to give Ver.

Here, branching is meant in the sense familiar from the theory of Riemann surfaces [107]. Moreover, the \mathbb{RP}^2 itself embeds *non-smoothly* into Ver.

Then the quadrilateralland interpretation of these results is in direct analogy with the plain shapes case of triangleland consisting of two hemispheres of opposite orientation bounded by an equator circle of collinearity, the mirror-image-identified case then consisting of one half plus this collinear edge. Thus plain quadrilateralland's distinction between clockwise- and anticlockwise-oriented figures is strongly anchored to this geometrical split, with the collinear configurations lying at the boundary of this split.

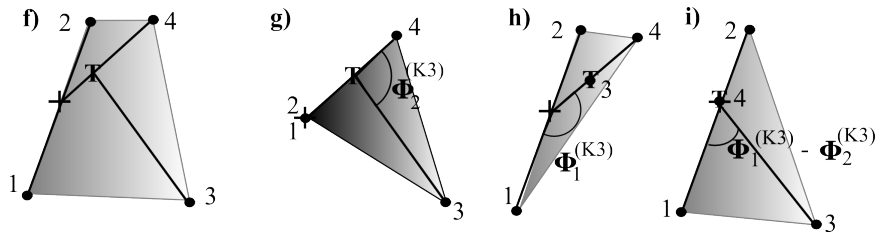


Figure 22: Figure of the coarse-graining triangles in Jacobi K-coordinates instead, then collapsing each of $\rho_1^{(K3)}$, $\rho_2^{(K3)}$ and $\rho_3^{(K3)}$ in turn gives the coarse-graining triangles g), h) and i).

3.10.10 Useful coordinate systems for quadrilateralland. II. Gibbons–Pope type coordinates.

Useful intrinsic coordinates (which extend the spherical coordinates on the triangleland and 4-stop metroland shape spheres) are the Gibbons–Pope type coordinates $\{\chi, \beta, \phi, \psi\}$ [254, 518] are also useful for the study of quadrilateralland. These are, in some senses, a generalization of triangleland’s Θ and Φ . Their ranges are $0 \leq \chi \leq \pi/2$, $0 \leq \beta \leq \pi$, $0 \leq \phi \leq 2\pi$ (a reasonable range redefinition since it is the third relative angle), and $0 \leq \psi \leq 4\pi$. These are related to the bipolar form of the Fubini–Study coordinates by

$$\psi' = -\{\Phi_1 + \Phi_2\}, \quad \phi' = \Phi_2 - \Phi_1, \quad (272)$$

with then $\psi = -\psi'$ (measured in the opposite direction to match Gibbons–Pope’s convention) and ϕ is taken to cover the coordinate range 0 to 2π , which is comeasurate with it itself being the third relative angle between the Jacobi vectors involved. Also,

$$\beta = 2 \arctan(\mathcal{R}_2/\mathcal{R}_1), \quad \chi = \arctan(\sqrt{\mathcal{R}_1^2 + \mathcal{R}_2^2}). \quad (273)$$

However now in each of the conventions I use for H and K coordinates, a different interpretation is to be attached to these last two formulae in terms of the ρ^e . For H-coordinates in my convention,

$$\beta = 2 \arctan(\rho_2/\rho_1), \quad \chi = \arctan(\sqrt{\rho_1^2 + \rho_2^2}/\rho_3), \quad (274)$$

whilst for K-coordinates in my convention,

$$\beta = 2 \arctan(\rho_1/\rho_3), \quad \chi = \arctan(\sqrt{\rho_1^2 + \rho_3^2}/\rho_2). \quad (275)$$

Note 1) By their ranges, β and ϕ parallel azimuthal and polar coordinates on the sphere. [In fact, β , ϕ and ψ take the form of Euler angles on $SU(2)$, with the remaining coordinate χ playing the role of a compactified radius. Thus Gibbons–Pope coordinates are a particular example of $SU(2)$ -adapted coordinates, for the $SU(2)$ in question being a subgroup of the natural $SU(3)$, by which such coordinates are useful for a number of applications.

Note 2) In the quadrilateralland interpretation, β has the same mathematical form as triangleland’s azimuthal coordinate Θ . Additionally, χ parallels 4-stop metroland’s azimuthal coordinate θ , except that it is over half of the range of that, reflecting that the collinear 1234 and 4321 orientations have to be the same due to the existence of the second dimension via which one is rotateable into the other.

Note 3) These coordinates represent the simplest available choice of block structure (a weakening of diagonalization, which is not here possible: a 2 by 2 block and two diagonal entries), with the Gibbons–Pope choice of radii further simplifying the formulae within each of these blocks. In this way, Gibbons–Pope coordinates are analogous to spherical coordinates, Fubini–Studi coordinates themselves being more like stereographic coordinates (projective, not maximally block-simplified).

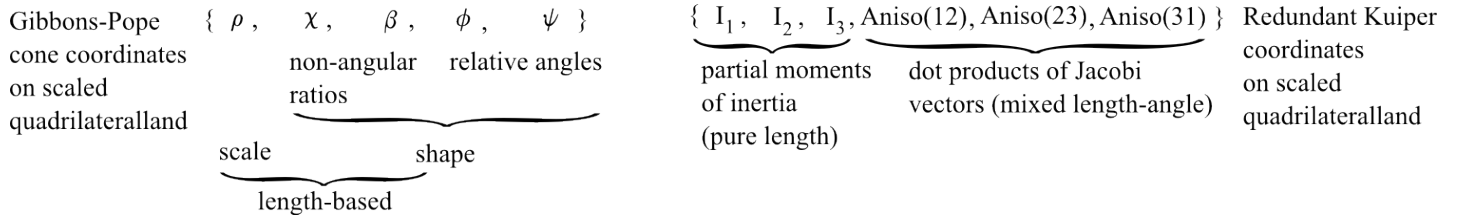


Figure 23: The shape, scale, length ratio, relative angle and subsystem statuses of the Gibbons–Pope type and Kuiper coordinate systems.

The Gibbons–Pope type coordinates have the following quadrilateralland interpretations. In an H-clustering, the ϕ is the difference of the relative angles [see Fig 2e)], so the associated momentum represents a counter-rotation of the two constituent subsystems (\times relative to $\{12\}$ and $+$ relative to $\{34\}$). The ψ is minus the sum of the relative angles, so the associated momentum represents a co-rotation of these two constituent subsystems (with counter-rotation in \times relative to $+$ so as to preserve the overall zero angular momentum condition). The β is a measure of contents inhomogeneity of the model universe: the ratio of the sizes of the 2 constituent subclusters. Finally, the χ is a measure of the selected subsystems’ sizes relative to that of the whole-universe model. These last two are conjugate to quantities that involve relative distance momenta in addition to relative angular momenta.

On the other hand, in a K-clustering, the ϕ is the difference of the two relative angles [see Fig 2f)], but by the directions these are measured in, the associated momentum now represents a co-rotation of the two constituent subsystems (which are now $\{12\}$ relative to 4 and $\{+4\}$ relative to 3, and with counter-rotation in T relative to $+$ so as to preserve the overall zero angular momentum condition). The ψ is minus the sum of the two relative angles, so the associated momentum represents a counter-rotation of these two constituent subsystems. The β is now a comparer between the sizes of the $\{12\}$ subcluster and the separation between the non-triple cluster particle 3 and T. Finally, the χ is a comparer between the sizes of the above two contents of the universe ($\{12\}$ and $\{T3\}$) on the one hand, and the separation between them on the other hand ($\{4+\}$, which is a measure of separation of $\{12\}$ and $\{T3\}$). These last two are again conjugate to quantities that involve relative distance momenta in addition to relative angular momenta.

Note how in both H and K cases, the Gibbons–Pope type coordinates under the quadrilateralland interpretation involve a split into two pure relative angles and two pure non-angular ratios of magnitudes.

In Gibbons–Pope type coordinates, the Fubini–Study metric then takes the form

$$ds^2 = d\chi^2 + \sin^2\chi\{d\beta^2 + \cos^2\chi\{d\phi^2 + d\psi^2 + 2\cos\beta\,d\phi\,d\psi\} + \sin^2\chi\sin^2\beta\,d\phi^2\}/4. \quad (276)$$

3.10.11 The inclusion of trianglelands and 4-stop metroland within quadrilateralland

In Gibbons–Pope type coordinates based on Jacobi H-coordinates, when $\rho_3 = 0$, $\chi = \pi/2$ and the metric reduces to

$$ds^2 = \{1/2\}^2\{d\beta^2 + \sin^2\beta\,d\phi^2\} \quad (277)$$

i.e. a sphere of radius $1/2$, which corresponds to the conformally-untransformed \mathbb{CP}^1 . When $\rho_2 = 0$, $\beta = 0$ and the metric reduces to

$$ds^2 = \{1/2\}^2\{d\Theta_1^2 + \sin^2\Theta_1\,d\Phi_1^2\} \quad (278)$$

for $\Theta_1 = 2\chi$ having the correct coordinate range for an azimuthal angle. Finally, when $\rho_1 = 0$, $\beta = 0$, the metric reduces to

$$ds^2 = \{1/2\}^2\{d\Theta_2^2 + \sin^2\Theta_2\,d\Phi_2^2\} \quad (279)$$

for $\Theta_2 = 2\chi$ again having the correct coordinate range for an azimuthal angle. The first two of these spheres are a triangleland shape sphere included within quadrilateralland as per Sec 3.10.11. The first is for $\{12\}$, 4 and 3 as the particles. The second is for 1, 2 and 34 as the particles. The third of these spheres corresponds, rather, to a **merger**, of $+$ and \times , i.e. a merger of type M^{DD} – the space of parallelograms labelled as in Fig 21e). [See Sec 3.11 for detailed definition and classification of types of merger.]

In Gibbons–Pope type coordinates based on the Jacobi K, when $\rho_3 = 0$, $\chi = \pi/2$ and the metric reduces to

$$ds^2 = \{1/2\}^2\{d\beta^2 + \sin^2\beta\,d\phi^2\} \quad (280)$$

i.e. a sphere of radius $1/2$, which corresponds to the conformally-untransformed \mathbb{CP}^1 . When $\rho_2 = 0$, $\beta = 0$ and the metric reduces to

$$ds^2 = \{1/2\}^2\{d\Theta_1^2 + \sin^2\Theta_1\,d\Phi_1^2\} \quad (281)$$

for $\Theta_1 = 2\chi$ again having the correct coordinate range for an azimuthal angle. Finally, when $\rho_1 = 0$, $\beta = 0$ and the metric reduces to

$$ds^2 = \{1/2\}^2\{d\Theta_2^2 + \sin^2\Theta_2\,d\Phi_2^2\} \quad (282)$$

for $\Theta_2 = 2\chi$ yet again having the correct coordinate range for an azimuthal angle. The second of these is a triangleland shape space sphere with 12, 4 and 3 as the particles. The first and third correspond rather to mergers, of $+$, T and 4 in the first case, and of T and 3 in the second case.

3.10.12 N -a-gonland generalization of the quadrilateralland coordinates used so far

Extending the Dragt/parabolic/shape/Kuiper type of redundant coordinates is itself straightforward, though it is not clear the extent to which the resulting coordinates will retain usefulness for the study of each \mathbb{CP}^{N-2} . Certainly the number of Kuiper-type coordinates (based on inner products of pairs of Jacobi vectors, of which there are $n\{n-1\}/2$) further grows away from $2\{n-1\} = \dim(\mathbb{CP}^{n-1})$ as N gets larger. The present paper also finds a surrounding space of just *one* dimension more for \mathbb{CP}^2 . The N -a-gonland significance of two half-spaces of different orientation separated by an orientationless manifold of collinearities gives reason for double covers to the $N-2 \geq 3$ \mathbb{CP}^{N-2} spaces to exist for all N . However, there is no known guarantee that these will involve geometrical entities as simple as or tractable as quadrilateralland's \mathbb{S}^4 for the half-spaces, or of the Veronese surface V as the place of branching. However, one does have the simple argument of Sec 3.10.12 that the manifold of collinearities within N -a-gonland's \mathbb{CP}^{N-2} is \mathbb{RP}^{N-2} , so at least that is a known and geometrically-simple result for the structure of the general N -a-gonland. Whether the intrinsic Gibbons–Pope type coordinates can be extended to N -a-gonland in a way that maintains their usefulness in characterizing conserved quantities [53] and via separating the free-potential time-independent Schrödinger equation, remains to be seen.

3.11 Tessellation by the physical/shape-theoretical interpretation

The configuration space splits up into a number of physically/shape-theoretically significant regions. I restrict myself to the equal-mass case in this article⁴¹ and this case definitely confers a large amount of symmetry in the allocation of physical

⁴¹This applies in mass-weighted space. This is a further stipulation since equal masses does not imply equal Jacobi masses (c.f. 2.2.4), so if one unweights the mass-weighted relative Jacobi coordinates, there is somewhat of a distortion of the tessellation. This distortion does preserve the perpendicularity of the axes, but stretches each by a different amount so that angles interior to each quadrant/octant are indeed distorted (e.g. the 120 degree angle in the next SSSec changes by a few percent under this distortion). However, I principally work with mass-weighted variables in this article (so my notion of shape is, strictly, of mass-weighted shape). Then the below tessellations are exact.

significance to these regions, by splitting it up into a number of *equal* regions. Such a split is known as a **tessellation** [439], i.e. a tiling of the configuration space with equal tiles that completely cover it. Tessellations are most well-known for flat space, but have also been studied for e.g. the sphere (see e.g. Magnus’ book [439]), which is significant to us as the shape space of both 4-stop metroland and triangleland. This method is indeed particularly useful for low configuration space dimension [50, 34, 35, 36, 37], for which pictorial representation of the whole configuration space is straightforward. Faces, edges and vertices therein are physically significant (and not in all cases physically equivalent), so that one really has a **labelled tessellation**.

One is then to interpret classical trajectories as paths upon these, and classical potentials and quantum-mechanical probability density functions as height functions over these. Thus this SSec’s figures will often feature as an ‘interpretational back-cloth’ in subsequent Secs. This parallels Kendall’s technique for the visualization of shape statistics of triangles in e.g. [364]. See [426] for a distinct application of tessellations in Theoretical Molecular Physics.

Note 1) that some such features were already meaningful at the topological level, and so have already been covered in Secs 3.2, 3.3 and 3.4. This SSec considers a number of further metrically-significant notions, as well as providing an overall picture of the physical/shape-theoretical interpretation.

Note 2) At the topological level, one does have a notion of coincidence and hence of collision. However, further concepts such as equilateral, isosceles and the various mergers of subsystems require additional metric structure to be in place.

Note 3) The tessellations in question are underlied by the $S_N \times \mathbb{Z}_2$ symmetry group of particle label permutations and orientation allocation (of order $2N!$). There is still an issue of how these act geometrically, which I treat case by case below.

3.11.1 3-stop metroland case

In the plain case [19], the shape space for this is the circle with 6 regularly-spaced double collision (D) points on it. [C.f. the rim of Fig 11a) but now these are metrically-meaningfully symmetrically placed]. One can furthermore fine-tile.⁴²

Here, the $S_3 \times \mathbb{Z}_2$ symmetry group acts as⁴³ $\mathbb{D}_6 \cong \mathbb{D}_3 \times \mathbb{Z}_2$. [Ignoring the labelling, the ‘clock-face’ fine-tiling has the larger symmetry group \mathbb{D}_{12} , but this is broken down to the preceding group by the M and D labels (which are indeed physically distinct as well as labelled distinctly).]

Next, by forming diametrically-opposite pairs, the 6 D’s pick out 3 preferred D-axes, and the 6 M’s likewise pick out 3 further preferred M-axes, each perpendicular to a D-axis. Each orthogonal (D, M) axis system corresponds to one of the 3 permutations of Jacobi coordinates. The polar angle $\varphi^{(a)}$ about each of these axes is then natural for the study of the clustering corresponding to that choice of Jacobi coordinates, $\{a, bc\}$.

The O-counterpart (Fig 24c) has 3 D’s and 3 M’s and the symmetry group $\mathbb{D}_3 \cong S_3$. This can also be represented using double angles as the rim of a whole pie (Fig 24c). Now each D is opposite to an M, so these D–M pairs pick out 3 preferred axes. Forming a Cartesian system now requires a second axis that has no particular extra physical significance; I thus term the points on this second axis *spurious*, and denote them with an S (Fig 24d).

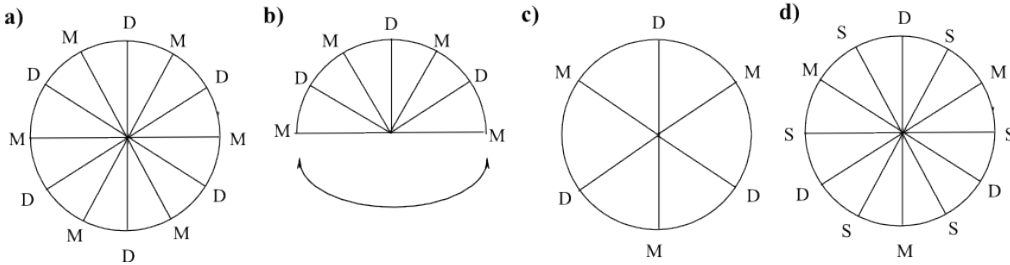


Figure 24: A sketch of the tessellation of 3-stop metroland’s relational spaces (the corresponding shape spaces are the rims of these).

The relational spaces for the scaled 3-stop metroland theories are then the cones over these decorated shape spaces. I.e., they are (the infinite extension of) 1) Fig 24a)’s pie of 12 slices with 6 D half-lines and 6 M half-lines in the plain case. 2) likewise Fig 24b)’s half-pie of 6 slices (or Fig 24c)’s pie of 3 slices) with 3 D half-lines and 3 M half-lines in the O case. All of these emanate from the triple collision at the cone point, 0.

3.11.2 4-stop metroland case

There are 8 triple collision (T) points and 6 double-double (DD) collision points [Fig 25; c.f. also Fig 25a) but now these are metrically-meaningfully symmetrically placed]. Each DD is attached to 4 T’s, and each T to 3 T’s and 3 DD’s. This forms a tessellation with 24 identical spherical isosceles triangle faces, 36 single double collision (D) edges and 14 vertices.

The T’s and DD’s form respectively the vertices of a cube [Fig 25a)] and the octahedron dual to it [Fig 25b)], so that the $S_4 \times \mathbb{Z}_2$ group acts as the symmetry group of the cube/octahedron (which order-48 group is indeed isomorphic to this as

⁴²C.f. the use of the terms ‘fine graining’ and ‘coarse graining’ in statistical mechanics and Histories Theory (Sec 11.9). I likewise call the opposite notions of these ‘coarse-tiling’ and ‘coarse-labelling’.

⁴³ \mathbb{D}_p is the dihaedral group of order $2p$.

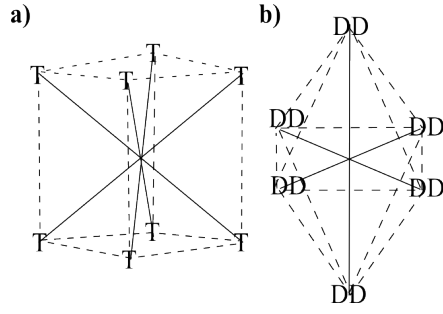


Figure 25: a) The octahedron of DD-vertices and the non-perpendicular diagonals between them. b) The cube of T-vertices and the set of perpendicular axes between them.

is clear by the action of the permutations on the four diagonals between opposite vertices).⁴⁴ See p 72-75 of [439] for further mathematical discussion of this particular tessellation and SSSec 3.5, [426] for other occurrences of it in mechanics. I note that in this case the labelling does not break the symmetry group of the unlabelled tessellation.

The T's and DD's form 7 antipodal pairs, thus picking out 7 preferred axes. The 3 axes corresponding to antipodal DD pairs are related to the 3 permutations of Jacobi H-coordinates, which are thus adapted to 'seeing' DD collisions. Likewise, the 4 axes corresponding to antipodal T pairs are related to the 4 permutations of Jacobi K-coordinates. Being perpendicular, (DD, DD, DD) is a suitable Cartesian axis system. Extending each T to such an axis system requires more work below.

Then the spherical polar coordinates about each of these axes – a $(\phi_{(Hb)}, \theta_{(Hb)})$ or a $(\phi_{(Ka)}, \theta_{(Ka)})$ coordinate system – are natural for the study of the corresponding H or K structure. I.e. each choice of H or K has a different natural spherical polar coordinate chart. Any two of these natural charts suffice to form an atlas for the sphere (each goes bad solely at its poles, where its axial angle ceases to be defined). To look very close to a pole, one can 'cartesianize' e.g. after projecting the relevant hemisphere onto the equatorial disc.

Note the inclusion of six copies of 3-stop metroland (with each's 6 D's labelled as DD,T,T,DD,T,T). These correspond to the 6 ways of fusing 2 of the 4 particles to produce a 3-particle problem. In the O case (Fig 25 d), one has this picture with antipodal identification. Or, correspondingly, the (without loss of generality) Northern hemisphere portion (so the numbers of faces and vertices down by a factor of 2, and the corresponding group is now S_4). (O)relational spaces are then the solid cones made from each corresponding shape space decorated by its tessellation. In making this a cone, points go to radial half-lines, arcs go to 2-d sectors and faces go to 3-d sectors of solid angle. All emanate from the maximal (quadruple) collision at the cone point, 0.

Fine-tessellation by notions of merger. Since this further complicates the picture, I present it at the end as a fine-tiling and fine-labelling of one of the preceding diagram's tiles (Fig 26). For 4-stop metroland, there are 2 kinds of great circles of mergers, and these furthermore contain 2 kinds of special points that are both a merger and a double collision. In fact, there are 2 kinds of merger: one particle at the centre of mass of the other 3 (which I denote by M^T) and the centres of mass of two pairs of particles coinciding (which I denote by M^{DD}). I denote the special points on these lines by appending a D; the diagram is consistent since $M^{DD}D$ can also be viewed as two different M^TD 's, so I give these triple confluence points a new notation M^*D . Then there are in total 50 vertices (now also 12 $M^{DD}D$ as these subdivide each edge of the cube and 24 M^TD as there are 4 per face of the cube) and 144 edges. (Each old D is cut into 2 new D's and there are also 48 M^T edges – 8 per face of the cube – and 24 M^{DD} edges – 4 per face of the cube). It has 96 faces (by each old tile being subdivided into 4). These tiles are not all labelled the same, however. The fundamental unit is one pair of such tiles, i.e. a T-DD- M^*D triangle. There are then 48 of these, and considering them amounts to not using the M^TD as vertices or the M^T as edges, in which case there remain 26 vertices and 72 edges. These are also the types of merger present in quadrilateralland. As can be seen from how the new vertices democratically decorate the cube, the above two fine-tilings with labellings continue to respect the symmetry group of the cube. These merger points define a number of further axes, allowing for the T-axes to be extended to form a Cartesian system of the form (T, M^*D , M^*D).

3.11.3 Triangleland case

Beyond the previously encountered topologically-meaningful points of double collision (D), triangleland's shape space's further notions of distance and angle at the level of metric geometry furnish the following [Fig 19]].

- 1) a notion of collinearity C that comprise the equator (split into 3 equal arcs by the D's, so that these lie at $2\pi/3$ to each other. N.B. that this equator of collinearity is the same as O3-stop metroland, the rotation in the second dimension forcing the O-choice. This equator of collinearity is orientationless and separates the 2 hemispheres of distinct orientation. These correspond to clockwise- and anticlockwise labelled triangles [Fig 27 a, d]).
- 2) A notion of equilaterality E (and its labelled-triangle mirror image \bar{E} .) which occur at the poles.
- 3) The half-meridians I that join the D's to the E's correspond to the 3 distinct notions of isosceles triangles that a labelled triangle possesses (3 choices of base). Each of these is associated with a particular clustering (the one that picks out the

⁴⁴Cubes and octahedra are also involved in quadrilateralland, e.g. in the form of the collision set, c.f. Fig 13d).

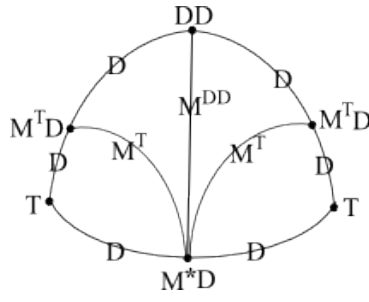


Figure 26: There are M-points for 4-stop metroland also. This figure illustrates the notions of merger within a single triangular face.

particular base with respect to which one is to have mirror symmetry about). On the other hand, the notions of collinearity and equilaterality are clearly clustering-independent. These are the plane of zero area and points of extremal area per given I, i.e. the E-direction perpendicular to the collinearity plane is proportional to the area vector.

However, it is not just half, but the whole of, the abovementioned great circles that correspond to isosceles triangles. There are planes of zero anisoscelesness, the perpendiculars to each of which are proportional to the corresponding aniso(a) vector. That a) only captured half of each isoscelesness great circle I also attracts attention to the other intersection points of this and the collinearity equator C (which are antipodal to the D's). These are mergers, M: configurations such that one particle is at the centre of mass of the other two. I depict this fine-tiling of the preceding paragraph in Fig 27 b, e). Finally, the meridian of isoscelesness separates hemispheres of left(a) [left-slanting of (a)'s notion of isoscelesness] and right(a) [right-slanting of (a)'s notion of isoscelesness].

The meridians perpendicular to the I's also have lucid physical meaning. These are the *regular configurations*, R (given by $I_1^{(a)} = I_2^{(a)}$, i.e. median = base in mass-weighted length), of which, indeed, a labelled triangle also possesses three different permutations. These are planes of zero ellipticity, the perpendiculars to each of which are proportional to the corresponding ellip(a) vector. They separate two hemispheres of clustering-dependent notions, sharp(a) (median > base, i.e. 'sharp' triangles) from flat(a) (median < base, i.e. 'flat' triangles) [Fig 27 c, f)]. I then know of no further interesting feature possessed by the intersection points of the equator of collinearity C and the meridians of regularness R, so, again, I use the letter S (for 'spurious') for these.

Thus there are 12 vertices on the equator forming an hour-marked clock-face labelled D,S,M,S,D,S,M,S,D,S,M,S (with the hours fully distinguished by further clustering-labels), and there are 14 vertices in total. There are then 12 edges of collinearity, 12 of isoscelesness and 12 of regularness, and thus 36 edges in total. This corresponds to a tessellation by (Fig 19c) 24 equal isosceles spherical triangles in the 'austroboreal zodiac' shape (the split of the night sky into the 12 signs of the Zodiac and then further partitioned into boreal and austral skies). The symmetry group corresponding to this unlabelled tessellation is $\mathbb{D}_{12} \times \mathbb{Z}_2$, of order 48.⁴⁵

However, the labels partly break down this group to $\mathbb{D}_3 \times \mathbb{Z}_2$, which is indeed isomorphic to the expected $S_3 \times \mathbb{Z}_2$ of order 12. The 48-triangle tessellation's constituent triangles are labelled either D,I,E,R,S,C or M,I,E,R,S,C rather than all being labelled the same. Thus the 'fundamental region' if some or all labels are included is any tile of the tessellation with 12 faces, 8 vertices (2 E's, 3 D's and 3 M's) and 18 edges (6 C's, 6 I's and 6 R's).

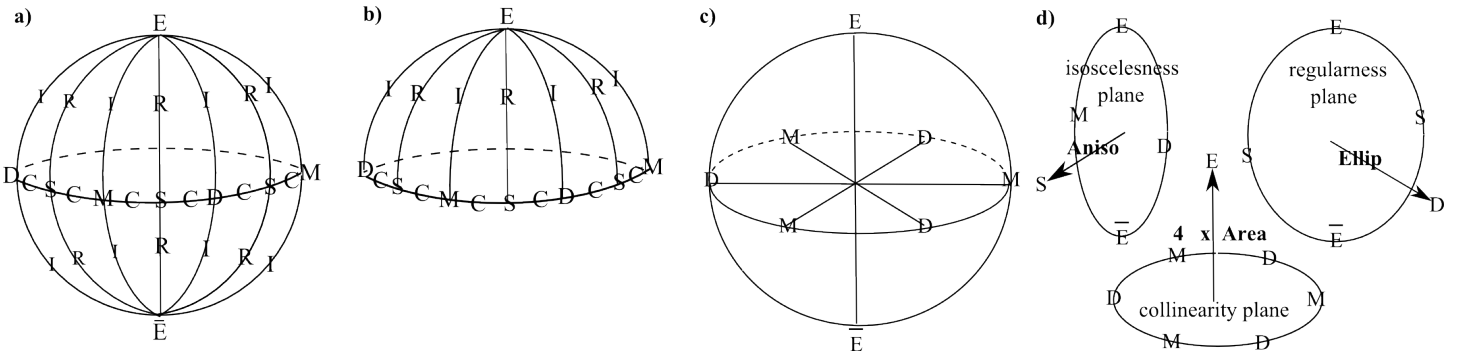


Figure 27: a) Tessellation of triangleland. The fundamental region is 1/3 of a hemisphere.

b) Tessellation of Otriangleland.

c) Axes through the equilateral triangle poles (E, \bar{E}) and through pairings of double collisions (D) and mergers (M).

d) The relation between triangleland's basis of Cartesian vectors and the planes perpendicular to each, clearly illustrating that $4 \times \text{area}$ is a departure from the plane of collinearity, that anisoscelesness is indeed a departure from the plane of isoscelesness, and that ellipticity is a departure from the plane of regularness.

⁴⁵Note that these are all members of the family of dihaedral tessellations studied on p 71-72 of [439]. Moreover, $\mathbb{D}_3 \times \mathbb{Z}_2$ is trigonal bipyramidal to the 4-stop metroland's $S_4 \times \mathbb{Z}_2$ that is tetragonal bipyramidal and regular and hence octahedral and so dual to the cube. Also note the contrast between the extra labelling breaking the tessellation group here with how the 4-stop metroland's fine labelling preserves that case's 'octahedral-cubic' symmetry group.

Next, note that the D and M vertices pick out 3 particularly distinguished axes at $\pi/3$ to each other in the collinearity plane. Each of these corresponds to a different choice of particle permutations in building the 3-particle Jacobi coordinates. Each such axis corresponds to an $\text{ellip}(a)$. The axes perpendicular to each of these correspond to the 3 notions of $\text{aniso}(a)$. I denote spherical polar coordinates about each of these as principal axis by $\{\Theta_{(a)}, \Phi_{(a)}\}$, where $\Theta_{(a)}$ is the azimuthal spherical angle as measured from each D. Also, I denote spherical polar coordinates with E as North pole and a as the second axis by $\{I, \Theta_{[a]}, \Phi_{[a]}\}$. I.e. a (DM, EE , SS) axis system or (D, E, S) for short for a particular choice of orientation for the axes.

One can take EE as principal axis by interchanging the roles of the second and third coordinate axes. I.e. the (EE , DM, SS) axis system or likewise (E, D, S) for short. I denote the subsequent quantities by square brackets $[a]$ for a the axis system's choice of second axis. The principal axis, however, now measures $4 \times \text{area}$ independently of any clustering considerations. On the other hand, each $\Phi_{[a]}$ is a regularness and isoscelesness quantity.

The O-counterpart (Fig 18) is the preceding with antipodal identification. Thus it consists of 12 equal isosceles forming the signs of the zodiac but now just for the boreal sky. There is now but one equilateral configuration E. The other 12 vertices are as for the O case. There are then the obvious subset of 24 edges of the preceding, with the same interpretation. The symmetry group for the labelled tessellation is then The corresponding physical group for this tessellation additionally labelled in this way is $\mathbb{D}_3 \cong S_3$ (order 6). This corresponds to 3 labelling permutations but now without additionally ascribing an overall orientation.

Triangleland's relational space is the corresponding cone over the above tessellation-decorated shape space. E, D, M, S become half-lines, C, I, R become sectors of $2\text{-}d$ angle, and faces become sectors of solid angle, all emanating from the triple collision at the cone point, 0.

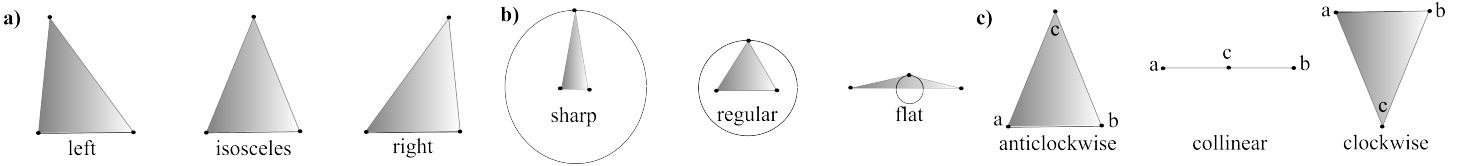


Figure 28: The most physically meaningful great circles on the triangleland shape correspond to the isosceles, regular and collinear triangles. These respectively divide the shape sphere into hemispheres of right/left triangles, sharp/flat triangles, and anticlockwise/clockwise triangles.

3.11.4 Quadrilateralland case

Beyond the topological detail given in Sec 3.2, at the metric level, collinearities become meaningful. These form \mathbb{RP}^2 in the distinguishable particle cases and \mathbb{RP}^2/A_4 in the indistinguishable particle cases. Here is a demonstration that \mathbb{RP}^{N-2} plays this role within the general N -a-gonland \mathbb{CP}^{N-2} case. Collinear configurations involve the relative angle coordinates being 0 or multiples of π , by which the complex projective space definition collapses to the real projective space definition (for which the \mathcal{R}^c are Beltrami coordinates). Moreover, $abcd\dots$ can be rotated via the second dimension into $\dots dcba$ (and that there is no further such identification) and so the real submanifold of collinear configurations is \mathbb{RP}^{N-2} . The above spaces of collinearity are in each case like an equator in each case, e.g. separating two S^4 's in \mathbb{E}^5 in the first case (see below for why the two halves are, topologically, S^4 and for further issues of the geometry involved).

Finally, there are also 6, 3, 2 and 1 distinguishable labelled squares in each case. This motivates a new action on quadrilateralland, in which e.g. the left-most particle is preserved and the other 3 are permuted or just evenly-permuted. Note: it is also easy to write conditions in these coordinates for rectangles, kites, trapezia, rhombi... but these are less meaningful 1) from a mathematical perspective (e.g. they are not topologically defined). 2) From a physical perspective (the square is additionally a configuration for which various notions of uniformity are maximal). However, squares are not the only notion of maximal uniformity by [38]'s $s_6 = \text{demo}(4)$ quantifier described in [38, 41].

3.11.5 Configuration spaces for the indistinguishable particle cases at the metric level

3.12 GR counterpart: configuration space study

At the level of dimension-counting, GR configuration spaces are obviously all infinite-dimensional.

Moreover [Smale]⁴⁶, superspace's points are not of equal dimension. In particular, points corresponding to metrics with maximal and non-maximal symmetry differ in dimension. For, different 3-metrics $h_{\mu\nu}$ have isometry groups $\text{Isom}(h_{\mu\nu})$ of different dimension: $\dim(\text{Isom}(\delta_{\mu\nu})) = \dim(\text{Eucl}) = 6$, $\dim(\text{Isom}(\text{generic } h_{\mu\nu})) = 0$. I note that this argument extends to CS and CS + V using conformal isometries and VP conformal isometries.

⁴⁶Wheeler [633] credits Smale with all points marked with his name here.

Leutwyler and Wheeler [633] appear to have been the first to ask about initial/boundary conditions on superspace. DeWitt suggested [203] that when one reaches the edge of one of the constituent manifolds (where the next stratum starts), one could reflect the path in Superspace that represents the evolution of the 3-geometry. C.f. also on p 352 of [230], with examples in [463]. Fischer subsequently alternatively proposed to deal with the extensions of these motions by working instead with a nonsingular extended space which no longer has the stratified manifold's problems with differential equations becoming questionable on edges between strata. He explicitly built a such in [231] based on the theory of fibre bundles.

3.12.2 Some further GR–RPM analogies at the level of configuration space differential geometry

The GR configuration space supermetric that features in the GR action and whose inverse, the DeWitt supermetric, features in the Hamiltonian constraint, are defined on $\text{Riem}(\Sigma)$: (47).

Analogy 37) RPM's Relationalspaces and GR's Riem both have curved configuration space metrics.

Difference 16) RPM's have positive-difference kinetic arc elements, whilst GR's is indefinite.

We shall see that this helps a number of applications and hinders a number of other ones.

Analogy 38) Additionally, CS is also like shape space in being a space of shapes! Though now with 'shape' being taken to have a different, more complicated meaning than in the study of RPM's: that of conformal 3-geometries. Each notion of shape is then a good candidate for true degrees of freedom of the theory in question.

Analogy 39) **Shape-definiteness alignment Lemma 12** This is based firstly on DeWitt's observation [201] that the part of the GR configuration space metric that causes it to be indefinite is the overall scale part of the metric. In particular, DeWitt established that CRiem is positive-definite, and then CS must also be since it is contained within CRiem . The analogy part of this is with shape space also being positive-definite. The rest of the Lemma is that in the homogeneous/inhomogeneous split, some shape parts – the anisotropies – are present within the homogeneous part of the split, as are homogeneous matter modes. In this way, this split differs in GR from the scale–shape split. Another common split is into 'scale part and homogeneous mode of scalar field matter' and the remainder. Finally, one can nest these splits hierarchically. One can conclude the following.

The split-out parts other than the scale can be taken to be positive-definite. In particular, subconfiguration spaces built out of (the GR notion of conformal-geometric) shapes and conventional matter fields have positive-definite kinetic metrics. I will be putting this fact to various good uses in Part III.

Possible Analogy 40) Relational space containing a better-behaved shape space might give hope that CS is better-behaved than superspace. Whether CRiem is better behaved than Riem is a more accessible pre-test of this speculation, and DeWitt already found some evidence for this [201].

N.B. the geometrical nature of superspace and conformal superspace is extremely complicated, placing limitations on what insights one can get from their study.

As regards putting a metric on superspace itself, investigating whether the DeWitt supermetric has a vertical–horizontal split is relevant [259]. Infinite dimensionality means that transversality alone is insufficient to make a direct sum split of this kind. Also one needs to check that the summands are topologically closed subspaces, which typically involves regularity properties of elliptic operators, and which is necessary for the projection maps to be continuous. Because of the nature of the metric at a point corresponding to a geometry with isometries, the tangent space does not exist. By a metric at such a point, what is meant is the induced metric in the horizontal subspaces of the covering points.

Difference 17) The GR counterpart of relational space being the cone over shape space in RPM's is the configuration space $\{\text{CS} + \text{V}\}(\Sigma) = \text{Riem}(\Sigma)/\text{Diff}(\Sigma) \times \text{VPCConf}(\Sigma)$, where the V stands for global volume. Now clearly the 2 roles played by relational space are played by obviously different (eg dimensionally different) spaces superspace and $\text{CS} + \text{V}$. Additionally, I have shown that $\text{CS} + \text{V}$ is not the cone over CS.

Question. Is $\{\text{CRiem} + \text{V}\}(\Sigma)$ the cone over $\text{CRiem}(\Sigma)$?

$\text{CS} + \text{V}$ is the closest thing known to a 'space of true dynamical degrees of freedom for GR' [652, 654, 49].

Analogy 41) Each of relational space and superspace [397] has a conformal Killing vector associated with scale.

3.12.3 Shape variables in GR

Analogy 42) Scale in scaled RPM's is a direct counterpart of scale (and homogeneous matter modes in cases in which these are present) as slow, heavy h degrees of freedom. Pure shape is a direct counterpart of anisotropies, inhomogeneities and corresponding matter modes as fast, light l degrees of freedom. (In GR, anisotropy could be an alternative l or part of

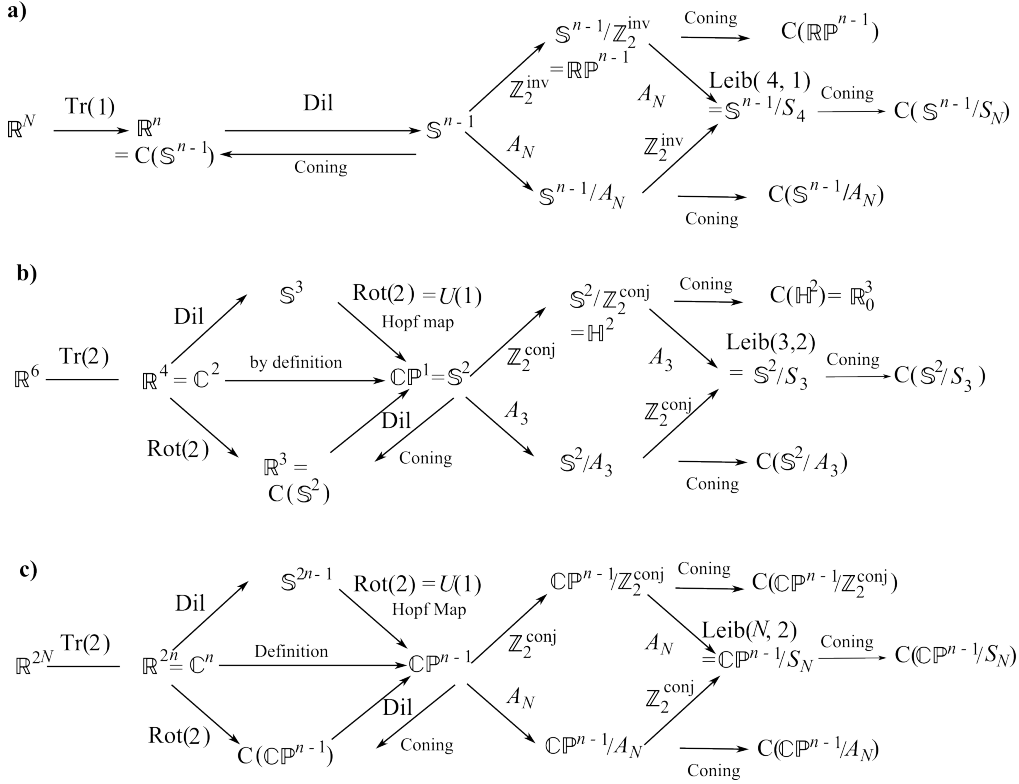


Figure 31: The sequence of configuration spaces for a) N -stop metroland, b) the exceptional case of triangleland (\mathbb{H}^2 is the hemisphere with edge and \mathbb{R}_0^3 the half-space with edge), and c) N -a-gonland.

This is explicitly directly constructible for the following of Kendall's shape space geometries and the cones thereover. The shape space of N -stop metroland is $\langle \mathbb{S}^{n-1}, \mathbf{M}_{\text{sph}} \rangle$, so that the natural pure-shape RPM associated with this is

$$\mathbf{S}_{N\text{-stop SRPM}}^{\text{relationspace}} = \sqrt{2} \int d\lambda \sqrt{W} ds_{N\text{-stop SRPM}}^{\text{relationspace}} \quad (284)$$

for $ds_{N\text{-stop SRPM}}^{\text{relationspace}2}$ as given by (219) in terms of ultraspherical coordinates.

The shape space of N -a-gonland is $\langle \mathbb{CP}^{n-1}, \mathbf{M}_{\text{FS}} \rangle$, so that the natural pure-shape RPM associated with this is

$$\mathbf{S}_{N\text{-a-gon SRPM}}^{\text{relationspace}} = \sqrt{2} \int d\lambda \sqrt{W} ds_{N\text{-a-gon SRPM}}^{\text{relationspace}} \quad (285)$$

for $ds_{N\text{-a-gon SRPM}}^{\text{relationspace}2}$ given by (228) in inhomogeneous coordinates. In the case of triangleland, this simplifies to (230) in polar coordinate form.

The relational space of N -stop metroland is $C(\mathbb{S}^{n-1}, \mathbf{M}_{\text{sph}}) = \langle \mathbb{R}^n, \mathbf{M}_{\text{flat}} \rangle$, so that the natural scaled RPM associated with this is

$$\mathbf{S}_{N\text{-stop ERPM}}^{\text{relationspace}} = \sqrt{2} \int \sqrt{W} ds_{N\text{-stop ERPM}}^{\text{relationspace}} \quad (286)$$

for $ds_{N\text{-stop ERPM}}^{\text{relationspace}2}$ given by (243) in the scale-shape split's ultraspherical polar coordinates.

The relational space of N -a-gonland is $C(\langle \mathbb{CP}^{n-1}, \mathbf{M}_{\text{FS}} \rangle) = \langle C(\mathbb{CP}^{n-1}), \mathbf{M}_{C(\text{FS})} \rangle$, so that the natural scaled RPM associated with this is

$$\mathbf{S}_{N\text{-a-gon ERPM}}^{\text{relationspace}} = \sqrt{2} \int \sqrt{W} ds_{N\text{-a-gon ERPM}}^{\text{relationspace}} \quad (287)$$

with $ds_{N\text{-a-gon ERPM}}^{\text{relationspace}2}$ given by (244) in ρ alongside inhomogeneous coordinates Z^r . In the case of triangleland, this simplifies to (246) in polar coordinate form. This triangleland case is also then castable in terms of $\langle \mathbb{S}^2, \mathbf{M}_{\text{sph}}(\text{radius } 1/2) \rangle$, so that e.g.

$$\mathbf{S}_{\Delta\text{-ERPM}}^{\text{relationspace}} = \sqrt{2} \int \sqrt{\tilde{W}} d\tilde{s}_{\Delta\text{-ERPM}}^{\text{relationspace}}, \quad (288)$$

with $d\tilde{s}_{\Delta\text{-ERPM}}^{\text{relationspace}2}$ given by (247) (away from $I = 0$ in which place this conformal transformation is invalid) and with $\tilde{W} = W/4I$ Cartesianizing that, one ends up using Dragt coordinate form (264).

3.14 Scheme C.I) reduction approach

In the indirect approach of Sec 2 using mass-weighted Jacobi coordinates, $\underline{\mathcal{L}}$ and \mathcal{D} give, in Lagrangian form,

$$\underline{\mathcal{L}} = \sum_{i=1}^n \underline{\rho}^i \times \{ \underline{\rho}^{i*} - \underline{B}^* \times \underline{\rho}^i + C^* \underline{\rho}^i \} = 0, \quad \mathcal{D} = \sum_{i=1}^n \underline{\rho}^i \cdot \{ \underline{\rho}^{i*} - \underline{B} \times \underline{\rho}^i + C^* \underline{\rho}^i \} = 0. \quad (289)$$

Now the third term of the first equation and the second term of the second equation are 0 by symmetry-antisymmetry, so eliminating ('Routhian reduction', see e.g. [415, 262]) $d\underline{B}$ from the first and dC from the second in no way interfere with each other. The first equation then gives, $\sum_{i=1}^n \underline{\rho} \times \{ \underline{\rho} \times d\underline{B} \} = -\sum_{i=1}^n \underline{\rho}^i \times d\underline{\rho}^i$. This is recastable, at least formally, as $d\underline{B} = -\underline{I}^{-1} d\underline{\mathcal{L}}$ for \underline{L} the stationary frame version of $\underline{\mathcal{L}}$ and \underline{I} the barycentric inertia tensor,

$$I_{\alpha\beta} = \sum_{i=1}^n \{ |\underline{\rho}^i|^2 \delta_{\alpha\beta} - \rho_{\alpha}^i \rho_{\beta}^i \}. \quad (290)$$

This is realizable in $2-d$ away from the cone-point $I = 0$. However, it has further singularities in $3-d$ on the collinear configurations. These are due to these having a zero principal moment factor for collinearities in the $3-d$ case, and these not being cases one has any particular desire to exclude on physical grounds. The second equation gives $dC = -I^{-1} d\mathcal{D}$ (realizable away from $I = 0$, which is never in any case included in pure-shape RPM).

Then

$$ds_{\text{SRPM}}^{\text{red}}{}^2 = \{ I \{ ds^2 - dA^2 \} - d\mathcal{D}^2 \} / I^2 \quad (291)$$

for $dA^2 = d\underline{L} \underline{I} d\underline{L}$ (twice the rotational kinetic arc element). So, for $1-d$, as $\underline{L} = 0$, $dA = 0$ and the expressions for I , ds^2 and $d\mathcal{D}$ give this to be the ultraspherical kinetic arc element in Beltrami coordinates,

$$ds_{N\text{-stop SRPM}}^{\text{red}}{}^2 = \{ \{ |\underline{\rho}|^2 |d\underline{\rho}|^2 - (\underline{\rho}, d\underline{\rho})^2 \} / \{ |\underline{\rho}|^2 \}^2 \}, \quad (292)$$

which suffices to identify $ds_{N\text{-stop SRPM}}^{\text{red}}{}^2$ to be the same as $ds_{N\text{-stop SRPM}}^{\text{relationalspace}}{}^2$. For $2-d$, A has another form by the inertia tensor collapsing to just a scalar, the definition of \underline{L} and the Kronecker Delta Theorem, $dA^2 = I^{-1} \sum_i \sum_j \{ (\underline{\rho}^i, \underline{\rho}^j) (d\underline{\rho}^i, d\underline{\rho}^j) - (\underline{\rho}^i, d\underline{\rho}^j) (\underline{\rho}^j, d\underline{\rho}^i) \}$. Then using multipolar Jacobi coordinates,

$$d^2 A = I^{-1} \sum_{i=1}^n \sum_{j=1}^n \rho^{i2} \rho^{j2} d\theta^i d\theta^j \quad \text{and} \quad \mathcal{D}^2 = \sum_{i=1}^n \sum_{j=1}^n \rho^i d\rho^i \rho^j d\rho^j, \quad (293)$$

so

$$I d^2 A + d\mathcal{D}^2 = |(\underline{z} \cdot d\underline{z})_c|^2 \quad (294)$$

in complex notation. Then as also $I = |\underline{\rho}|^2 = |\underline{z}|_c^2$ and $ds = |d\underline{\rho}| = |d\underline{z}|_c$, one obtains the kinetic arc element to be the Fubini-Study one in inhomogeneous coordinates, thus identifying $ds_{N\text{-a-gon SRPM}}^{\text{red}}$ to be the same as $ds_{N\text{-a-gon SRPM}}^{\text{relationalspace}}$. The triangleland case is then simpler and rearrangeable as per the preceding SSec.

Next, since there is now no dilational constraint, and in the mechanical rather than geometrical PPSCT representation,

$$ds_{\text{ERPM}}^{\text{red}}{}^2 = ds^2 - d^2 A = \{ d\mathcal{D} / \sqrt{I} \}^2 + I ds_{\text{SRPM}}^{\text{reduced}}{}^2. \quad (295)$$

[The second equality is by (291).] Then $d\mathcal{D} = (\underline{\rho}, d\underline{\rho}) = dI/2 = \rho d\rho$, so $d\mathcal{D}/\sqrt{I} = d\rho$, and so

$$ds_{\text{ERPM}}^{\text{red}}{}^2 = d\rho^2 + \rho^2 ds_{\text{SRPM}}^{\text{red}}{}^2, \quad (296)$$

i.e. twice the reduced scaled RPM kinetic term is the one whose metric is the cone over the metric corresponding to twice the reduced pure-shape RPM kinetic term. Note furthermore that this derivation is independent of the spatial dimension. Next, in the cases for which $ds_{\text{SRPM}}^{\text{red}}$ has been specifically derived, $ds_{\text{ERPM}}^{\text{red}}$ can then also immediately be derived. Namely, in $1-d$, $ds_{N\text{-stop ERPM}}^{\text{red}} = ds_{N\text{-stop ERPM}}^{\text{relationalspace}}$. Of course, this case could have been established more trivially, since in this case there are no constraints to eliminate, leaving this working being but a change to the scale-shape-split-abiding ultraspherical coordinates. However, the $2-d$ case is less trivial, amounting to $ds_{N\text{-a-gon SRPM}}^{\text{red}}$ being identified to be the same as the $ds_{N\text{-a-gon SRPM}}^{\text{relationalspace}}$. The simpler triangleland example is furthermore rearrangeable as per the preceding SSec.

3.15 Comparison of schemes A), B), C), and D) at the classical level

Thus I have obtained the following theorem (which holds for the O-case as well, due to this just changing coordinate ranges, but these play no part in the above proof).

Direct = Best-Matched Theorem: For $1-$ and $2-d$ RPM's, the direct relationalspace implementation of configurational relationalism A) is equivalent to the 'indirect implementation B) followed by reduction C.I)' (the whole of which package is termed 'Best Matching').

This can be expressed, in each case for $d = 1, 2$, as a connection of research programs,

$$\text{JS(Kendall)} = \text{Red(Barbour 03)}, \quad (297)$$

$$\text{JS}(\text{C}(\text{Kendall})) = \text{Red}(\text{Barbour-Bertotti 82}) , \quad (298)$$

as maps

$$\text{JS} \circ \mathbf{q}(N, d)\text{-Quotient} = \text{Sim}(d)\text{-Red} \circ \text{JS} \circ \text{Sim}(d)\text{-Bundle} , \quad (299)$$

$$\text{JS} \circ \mathbf{q}(N, d)\text{-Quotient} = \text{Eucl}(d)\text{-Red} \circ \text{JS} \circ \text{Eucl}(d)\text{-Bundle} . \quad (300)$$

and as actual formulae,

$$\text{JS}(\mathbf{S}(N, d)) = \underset{\underline{A}, \underline{B}, C \in \text{Sim}(d)}{\text{extremum}} \text{JBB}(\mathbf{q}(N, d), \text{Sim}(d)) , \quad (301)$$

$$\text{JS}(\mathbf{R}(N, d)) = \underset{\underline{A}, \underline{B} \in \text{Eucl}(d)}{\text{extremum}} \text{JBB}(\mathbf{q}(N, d), \text{Eucl}(d)) , \quad (302)$$

with the O versions of these also applying throughout. Thus indeed Kendall already knew the answer to ‘what is the non-redundant configuration space for RPM’s?’, but, since he never crossed paths with Barbour, it was left to me to make this connection between these two research programs, with the consequence of greatly strengthening the RPM program.

Note 1) To celebrate this result, I henceforth jointly abbreviate the demonstratedly equivalent ‘relationalspace’ and ‘reduced’ labels by ‘r’.

Note 2) I later comment on the generalization of this ‘Direct = Best-Matched’ theorem in its map form

$$\text{JS} \circ \mathbf{q}\text{-Quotient} = \mathbf{g}\text{-Red} \circ \text{JS} \circ \mathbf{g}\text{-Bundle} , \quad (303)$$

and actual formulaic form

$$\text{JS}(\mathbf{q}/\mathbf{g}) = \underset{\mathbf{g} \in \mathbf{g}}{\text{extremum}} \text{JBB}(P(\mathbf{q}, \mathbf{g})) . \quad (304)$$

I will sometimes speculate that this result holds more widely than in the case of this article’s specific models for which I proved it above. I will term this (and its further generalization in Sec 14) the **Direct = Best-Matched conjecture**.

Note 3) NRPM has relationalspace = reduced for all dimensions, and is always explicitly evaluable (no Best Matching Problem and thus always a computable r-action and $\tau^{\text{em}(\text{JBB})}$).

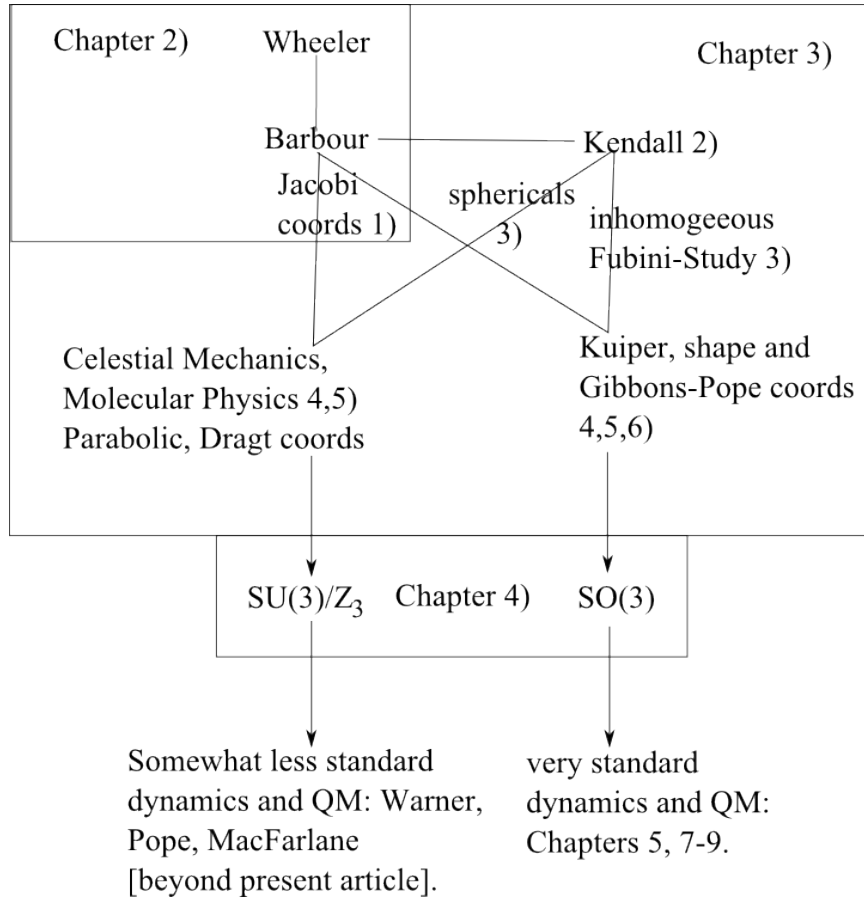


Figure 32: Summary of the key moves of Secs 2 to 4 in unlocking RPM’s.

3.16 Appendix A: weighted projective spaces

These are valuable as examples of departures from how \mathbb{CP}^p behaves, albeit not exactly the spaces of direct interest in this article. A simple example of weighted projective space is $\mathbb{WCP}_{2,1,1}^2$. [The bracketed numbers are the weights: here identify (z_1, z_2, z_3) with $(\lambda^2 z_1, \lambda z_2, \lambda z_3)$]. This occurs as a moduli space in field theory and string theory, in which context it is studied geometrically e.g. by Auzzi et al [69]. E.g. this has only an $SU(2)$'s worth of Killing vectors and a \mathbb{Z}_2 orbifold singularity. Witten, Acharya and Atiyah [643, 66, 4], Acharya [3] and Joyce [359] consider a \mathbb{WCP}^3 in the context of M-theory (it is a 6- d compact space). Collinucci [179] mentions a \mathbb{WCP}^4 . Auzzi et al. [69] also mention $\mathbb{WCP}_{2,2,2,1,1,1}^5$, Witten [644] considers a \mathbb{WCP}^p (including its cohomology) and Eto et al. [225] state that some such spaces are singular. Witten and Atiyah [643] also consider $C(\mathbb{WCP}_{q,q,1,1}^3)$ for $q \in \mathbb{N}$.

3.17 Appendix B: parallels to Barbour–Bertotti 1977 from deeper layers of structure

Note 1) The BB77 ansatz is exposed as not being general enough by perusal of the relative Lagrangian coordinate forms of ERPM and SRPM breaking its mathematical simplicities whilst remaining just as relational.

Note 2) BB77 no longer looks as simple as it did in relative Lagrangian coordinates once one passes to relative Jacobi coordinates. Thus some of that simpleness was coordinate-dependent. Moreover, in e.g. Jacobi coordinates and in Dragt coordinates for the triangle land case, one can re-apply the BB77 mathematical simplicities to obtain other simple-looking theories in terms of each of these. (Again, these only retain that simple form in that particular coordinate system, and the three of them are indeed not inter-convertible into each other and so constitute three separate theories.) E.g. clusterwise analogue of BB77's kinetic arc element is

$$ds^2 = \frac{\mu_i \delta_{\alpha\beta} \delta_{ij}}{||\underline{\mathbf{R}}^i||} d\mathbf{R}^{\alpha i} d\mathbf{R}^{\beta j} \quad (305)$$

which does not reduce to BB77 as the $\underline{\mathbf{R}}^i$ become linear combinations of $\underline{\mathbf{r}}^{IJ}$'s inside the mod signs. On the other hand, the Dragt analogue of BB77 for a triangle universe has kinetic arc element

$$ds^2 = \frac{\delta_{\Gamma\Lambda}}{||\underline{\mathbf{Dra}}||} d\mathbf{Dra}^\Gamma d\mathbf{Dra}^\Lambda . \quad (306)$$

Note 3) On the other hand, the argumentation leading to BB82 and B03 is unaffected by changes of coordinate system and of amount of reducedness. This furnishes a further theoretical reason to favour these over BB77 (and its above two counterparts).

3.18 Appendix C: Geometry, conformal geometry and just relative angles

Riemannian geometry has all of distance, relative distance and angle, whilst conformal geometry has just the last two of these notions. [I use g_{AB} for a general Riemannian metric, the unit-determinant factor of which is the conformal metric u_{AB} .] v^A and w^A are vectors associated with the space in question.

$$\text{distance} = ||v|| = \sqrt{g_{AB} v^A v^B} , \quad (307)$$

$$\text{relative distance} = \frac{||v||}{||w||} = \sqrt{\frac{g_{AB} v^A v^B}{g_{CD} w^C w^D}} = \sqrt{\frac{u_{AB} v^A v^B}{u_{CD} w^C w^D}} , \quad (308)$$

$$\text{angle} = \arccos \left(\frac{(v, w)}{||v|| ||w||} \right) = \arccos \left(\frac{g_{AB} v^A w^B}{\sqrt{g_{CD} v^C v^D} \sqrt{g_{EF} w^E w^F}} \right) = \arccos \left(\frac{u_{AB} v^A w^B}{\sqrt{u_{CD} v^C v^D} \sqrt{u_{EF} w^E w^F}} \right) . \quad (309)$$

I let these definitions apply both to space and to configuration space, and be possibly modulo involving mass-weighted quantities. ρ is a special case of distance.

Barbour has at times argued in terms of just relative angles, as a general possibility and in connection with the specific example of the celestial sphere in astronomy. However, the celestial sphere modelled in this way entails adopting a fixed vantage point and using the corresponding projective model as one's picture of the world [Fig 33a)]. Also, the split into relative angle information and information in ratios of relative separations is not irreducible, as is clear from how any of 2 angles, the ratio of 2 sides and the angle between them or whichever 2 ratios of 2 sides all characterize triangles modulo scale (i.e. the shape alias similarity class of the triangle).

3.19 Appendix D. Further r-formulation of the scaled triangle land action

This is the form obtained by Legendre transformation to the almost-Hamiltonian variables (in parallel to Sec 2.11.4). This case is most cleanly presented in Dragt coordinates. It is needed for Sec 15.1.5.

$$\begin{aligned} \mathbf{S}_{\Delta-\text{ERPM}}^r &= \int \{ \mathbf{Dra}^\Gamma \Pi_F^{\text{Dra}} - \mathbf{A}(\mathbf{Dra}^\Gamma, \Pi_F^{\text{Dra}}, \dot{\mathbf{I}}) \} d\lambda = \int \{ \mathbf{Dra}^\Gamma \Pi_F^{\text{Dra}} - \dot{\mathbf{I}} \mathcal{E} \} d\lambda = \int \{ \check{\mathbf{Dra}}^\Gamma \Pi_F^{\text{Dra}} - \mathcal{E} \} d\tilde{t}^{\text{em}(\text{JBB})} . \quad (310) \\ &= \int \{ d\mathbf{Dra}^\Gamma \Pi_F^{\text{Dra}} - d\mathbf{I} \mathcal{E} \} \end{aligned}$$

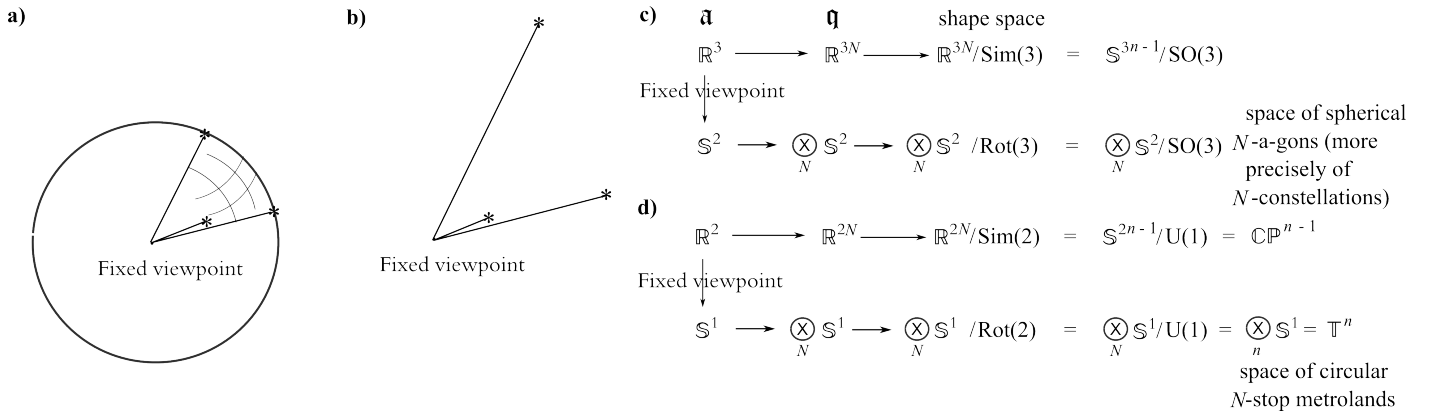


Figure 33: The geometrical difference between a) the projective single vantage point celestial sphere model and b) the usual model that also allows for relative distances. c) If one adopts \mathbb{S}^2 rather than \mathbb{R}^3 as one's incipient notion of space \mathbf{A} , one obtains the indicated model. I also include a more tractable toy model d). These constitute a new type of RPM's within the general scheme laid out in Secs 1 to 3. More generally, albeit less motivatedly, one can consider the counterpart of RPM's formulated indirectly on whichever other spatial Riemannian geometry.

3.20 Appendix E. The O3-cornerland fallacy

This model exists, but is less relational as per Sec 2.1. O3-Cornerland has 3 moment of inertia components while (O)Triangleland other has just 1. The argument is that relationally only 2 dimensions are meaningful so there is no sense in building a secondary object that has components that come about via supposing a third dimension. Consequences of this 2- d versus 3- d distinction are differences in the mathematics of the collinearities, collisions, configuration space singularities, classical stability, and in the (obstruction to the) passage to the reduced form (Sec 3.14). These issues carry over to pure-shape RPM too. In 3-cornerland, the collinear configurations entail passing to a smaller isotropy group (i.e. to belonging to a distinct orbit), causing a jump that usually leads to the collinear configurations being excluded. This does not happen for triangleland, as in 2- d the collinears are no different in symmetry. One often has to exclude the collinear configurations in the O case (particularly in the 3- d version). This leaves one with rather less structure in the tessellation, but, on the other hand, it does mean that a boundary condition is provided.

Some of these differences proceed via assumption of tensor components beyond the maximum number of such supported by the relational physics of 3 particles. E.g. this then enters into the non-invertibility of larger than relationally maximal inertia tensors. This occurs e.g. in Gergely [249] and Gergely–McKain [250]'s work in the 3-particle case, by which what they are studying is clearly O3Cornerland and not the yet more relational triangleland. These authors did note that the 3-particle case had qualitatively different properties. They obtain the metric on the space of orbits of group of translations and rotations rigged by the generators (this sense of 'rigged' is advocated more widely in [568]). They then compute the Riemann tensors of the $3N$ space and the $3N - 6$ space and their interrelation by a (higher codimension generalization of the) Gauss equation. They then compute the tensor for the rigged space in terms of sums of products of pairs of vorticity tensors (these having a complicated form, see the original paper for their definition). They then contract to form the Ricci scalar, expressing it in terms of the principal moments of inertia and the number of particles:

$$R = 6\{N - 2\} \sum_{\mu} /I_{\mu} - \{3/2I_1I_2I_3\} \sum_{\mu} I_{\mu}^2 \quad (311)$$

This permits one to figure out whether the collinear configurations can be included. For, without loss of generality $I_1 = 0$, $I_2 = I_3 = I$,

$$R = 6\{2N - 5\}/I + 6\{N - 3\}/I_1 \quad (312)$$

which is infinite hence revealing a curvature singularity, unless $N = 3$. They show that for O3-cornerland the metric is conformally flat (a result in fact already noted by Iwai [348]). They also demonstrate that 3-cornerland the metric has a conformal symmetry, enabling computation of extrinsic curvature of ellipsoids orthogonal to the conformal Killing vector. They furthermore consider a prescription for allowing geodesics to reach and leave the boundaries.

Questions The above study should have an interesting, distinct and fully relational 2- d counterpart, and also a pure-shape RPM counterpart in both 2- d and 3- d . I encourage some PhD student with a keen interest in differential geometry to perform these calculations in parallel to Gergely's and Gergely–McKain's work. Inclusion of coincident configurations in such a study is claimed to be done in [72]. However, this is controversial insofar as [615] asserts not knowing how to give such a space topological or differential structure.

Sec 10 contains a footnote on further O3-Cornerland–Triangleland differences at the quantum level; one of these is anticipated in Sec 6.10.

4 Dynamics: variation of reduced actions

The next two sections explore RPM's at the classical level, which is an important prequel to Parts II and III's studies of QM and time. For a dynamical system, conserved quantities correspond to isometries of the kinetic metric that are also respected by the potential; the isometries for RPM's were given in Secs 3.6.6, 3.6.7, 3.8.8 and 3.8.9.

4.1 Dynamical equations for pure-shape RPM

4.1.1 4-stop metroland

One can immediately obtain the momenta, constraints and equations of motion from Sec 17.4 under $\alpha \rightarrow \theta$, $\chi \rightarrow \phi$, $\mathcal{S}_A \rightarrow \mathcal{Dil}_\Gamma$ [these are $SO(3)$ objects]. If V is ϕ -independent, $\mathcal{Dil} = \mathcal{Dil}_3$ is conserved (which I term the *special case*). If V is also θ -independent and thus constant, all 3 \mathcal{Dil}_Γ are conserved (which I term the *very special case*). N.B. special is a clustering-dependent property but very special is not (having no preferred axis is a basis-independent property).

This is in clear analogy with well-known rotor and planar problems in ordinary mechanics. A first analogy involves

$$\theta \text{ and } \phi \text{ in place of their spatial counterparts } \theta_{\text{sp}}, \phi_{\text{sp}}, \text{ (moment of inertia of the rotor)}, I_{\text{rot}} \rightarrow 1 \quad . \quad (313)$$

A second analogy involves transforming θ to the radial stereographic coordinate $\mathcal{R} = \tan \frac{\theta}{2}$ and passing to the 'barred' PPSCCT representation. Then the kinetic term becomes that for the flat plane in polar coordinates and one can read off the analogies \mathcal{R} to r , ϕ to ϕ_{sp} , (test particle mass) $m \rightarrow 1$. These analogies will be furthermore fruitful in analyzing 4-stop metroland's equations of motion and conserved quantities in the next 2 subsections, as well as when further specifics about the potential are brought in (see Sec 5.1).

Finally, $\tilde{V}_{\text{eff}} := \tilde{V} + \mathcal{Dil}^2/\sin^2\theta - \tilde{E}$ is the potential quantity that is significant for motion in time. On the other hand, combining (1031) and (1032), $\tilde{U}_{\text{orb}} := -\sin^4\theta \tilde{V}_{\text{eff}}$ is the potential quantity that is significant as regards featuring in the integrals for the shapes of the classical orbits. The whole-universe aspect of RPM modelling means that it is desirable to study these with E fixed and S a free parameter, which is somewhat unusual.

4.1.2 N -stop metroland generalization in ultraspherical coordinates

In straightforward extension, the conjugate momenta are

$$P_q = \left\{ \prod_{A=1}^{q-1} \sin^2\theta_p \right\} \theta_q^* \quad . \quad (314)$$

These momenta obey as a primary constraint the quadratic 'energy equation'

$$\mathcal{E} := \frac{1}{2} \sum_{r=1}^{n-1} P_q^2 / \prod_{A=1}^{q-1} \sin^2\theta_p + V(\theta_p) = E \quad , \quad (315)$$

the middle expression of which also serves as the Hamiltonian. The evolution equations are

$$\left\{ \prod_{A=1}^{q-1} \sin^2\theta_p \theta_q^* \right\}^* - \sum_{r=q+1}^{n-1} \left\{ \prod_{p=1, p \neq q}^{r-1} \sin^2\theta_p \sin\theta_q \cos\theta_q \right\} \theta_r^{*2} = -\partial V / \partial \theta_q \quad , \quad (316)$$

one of which can be supplanted by the Lagrangian form of (315),

$$\sum_{r=1}^{n-1} \prod_{A=1}^{r-1} \sin^2\theta_A \theta_r^{*2} / 2 + V(\theta_r) = E \quad . \quad (317)$$

If V is θ_1 -independent, there is one $SO(2)$ conserved quantity \mathcal{Dil} . If V is θ_1 and θ_2 independent, there are three $SO(3)$ conserved quantities. This 'very^k special' tower continues through to V being independent of all the θ_r i.e. constant for which there are $n\{n-1\}$ $SO(n)$ quantities correspond to the full underlying isometry group.

4.1.3 Triangleland in (Θ, Φ) spherical coordinates

One can immediately obtain the momenta, constraints and equations of motion from Sec 17.4 under $\alpha \rightarrow \Theta$, $\chi \rightarrow \Phi$, $\mathcal{S}_A \rightarrow \mathcal{S}_\Gamma$ [these are $SO(3)$ objects]. Also note that, to make analogy with ordinary mechanics on the sphere,

$$1 \longleftrightarrow m = \text{(test particle mass)} \quad . \quad (318)$$

If V is ϕ -independent, $\mathcal{J} = \mathcal{S}_3$ is conserved (which, again, I term the *special case*). If V is also θ -independent and thus constant, all 3 \mathcal{Dil}_Γ are conserved (which, again, I term the *very special case*).

4.1.4 Triangleland in (\mathcal{R}, Φ) stereographic coordinates

The above analogy furthermore points to the well-known $u = 1/r$ substitution of ordinary mechanics being paralleled by the $\mathcal{U} = 1/\mathcal{R}$ one (Fig 18), which turns out to be exceedingly useful in my study below. The momenta are

$$P_{\mathcal{R}} = \mathcal{R}^{\bar{*}} \quad , \quad P_{\Phi} = \mathcal{R}^2 \Phi^{\bar{*}} . \quad (319)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\bar{\mathcal{E}} = \frac{1}{2} \left\{ P_{\mathcal{R}}^2 + \frac{P_{\Phi}^2}{\mathcal{R}^2} \right\} + \bar{\mathcal{V}}(\mathcal{R}, \Phi) = \bar{\mathcal{E}}(\mathcal{R}) , \quad (320)$$

the middle expression of which also serves as the Hamiltonian. The evolution equations are

$$\mathcal{R}^{\bar{*}\bar{*}} - \mathcal{R} \Phi^{\bar{*}2} = \partial\{\bar{\mathcal{E}} - \bar{\mathcal{V}}\}/\partial\mathcal{R} \quad , \quad \{\mathcal{R}^2 \Phi^{\bar{*}}\}^{\bar{*}} = -\partial\bar{\mathcal{V}}/\partial\Phi , \quad (321)$$

one of which can be supplanted by the Lagrangian form of (320),

$$\mathcal{R}^{\bar{*}2}/2 + \mathcal{R}^2 \Phi^{\bar{*}2}/2 + \bar{\mathcal{V}}(\mathcal{R}, \Phi) = \bar{\mathcal{E}}(\mathcal{R}) . \quad (322)$$

If $\tilde{\mathcal{V}}$ is Φ -independent,

$$\mathcal{J} = \mathcal{R}^2 \Phi^{\bar{*}} = P_{\Phi} , \quad (323)$$

and the energy equation becomes

$$\{d\mathcal{R}/d\bar{t}\}^2/2 + \mathcal{J}^2/2\mathcal{R}^2 + \tilde{\mathcal{V}} = \tilde{\mathcal{E}} . \quad (324)$$

or

$$P_{\mathcal{R}}^2/2 + \mathcal{J}^2/2\mathcal{R}^2 + \bar{\mathcal{V}} = \bar{\mathcal{E}} . \quad (325)$$

Finally, $\bar{\mathcal{V}}_{\text{eff}} := \bar{\mathcal{V}} + \mathcal{J}^2/\mathcal{R}^2 - \bar{\mathcal{E}}$ is the potential quantity that is significant for motion in time. Also, combining (323) and (322), $\bar{\mathcal{U}}_{\text{orb}} := -\mathcal{R}^4 \bar{\mathcal{V}}_{\text{eff}}$ is the potential quantity that is significant as regards featuring in the integrals for the shapes of the classical orbits. Translating into spherical coordinates and the breved PPST presentation, the corresponding quantities are $\check{\mathcal{V}}_{\text{eff}} = \check{\mathcal{V}} + \check{\mathcal{J}}^2/\sin^2\Theta - \check{\mathcal{E}}$ and $\check{\mathcal{U}}_{\text{orb}} = -\sin^4\Theta \check{\mathcal{V}}_{\text{eff}}$.

4.1.5 N -a-gonland extension in multi-polar inhomogeneous coordinates

For pure-shape N -a-gonland, the conjugate momenta are

$$\mathcal{P}_{\bar{\mathbf{p}}}^{\mathcal{R}} = \left\{ \frac{\delta_{\bar{\mathbf{p}}\bar{\mathbf{q}}}}{1 + \|\mathcal{R}\|^2} - \frac{\mathcal{R}_{\bar{\mathbf{p}}} \mathcal{R}_{\bar{\mathbf{q}}}}{\{1 + \|\mathcal{R}\|^2\}^2} \right\} \mathcal{R}_{\bar{\mathbf{q}}}^{\star} \quad , \quad \mathcal{P}_{\bar{\mathbf{p}}}^{\Theta} = \left\{ \frac{\delta_{\bar{\mathbf{p}}\bar{\mathbf{q}}}}{1 + \|\mathcal{R}\|^2} - \frac{\mathcal{R}_{\bar{\mathbf{p}}} \mathcal{R}_{\bar{\mathbf{q}}}}{\{1 + \|\mathcal{R}\|^2\}^2} \right\} \mathcal{R}_{\bar{\mathbf{p}}} \mathcal{R}_{\bar{\mathbf{q}}}^{\Theta\star} . \quad (326)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{H} := \frac{1}{2\{1 + \|\mathcal{R}\|^2\}} \left\{ \{\delta^{\bar{\mathbf{p}}\bar{\mathbf{q}}} + \mathcal{R}^{\bar{\mathbf{p}}} \mathcal{R}^{\bar{\mathbf{q}}}\} \mathcal{P}_{\bar{\mathbf{p}}}^{\mathcal{R}} \mathcal{P}_{\bar{\mathbf{q}}}^{\mathcal{R}} + \left\{ \frac{\delta^{\bar{\mathbf{p}}\bar{\mathbf{q}}}}{\mathcal{R}_{\bar{\mathbf{p}}}^2} + 1|\bar{\mathbf{p}}\bar{\mathbf{q}} \right\} \mathcal{P}_{\bar{\mathbf{p}}}^{\Theta} \mathcal{P}_{\bar{\mathbf{q}}}^{\Theta} \right\} + \mathcal{V}(\mathcal{R}^{\bar{\mathbf{p}}}, \Theta^{\bar{\mathbf{p}}}) = \mathcal{E} , \quad (327)$$

the middle expression of which also serves as the Hamiltonian. The corresponding evolution equations are (for $Q^A = \{\mathcal{R}^{\bar{\mathbf{p}}}, \Theta^{\bar{\mathbf{p}}}\}$)

$$D^2 Q^A / Dt^2 := Q^{A\star\star} + \Gamma_{\text{BC}}^A Q^{B\star} Q^{C\star} = \nabla^A \mathcal{W} , \quad (328)$$

with the geometrical objects of Sec 3.6.5 substituted in.

4.1.6 Quadrilateralland in Gibbons–Pope-type coordinates

These are particularly useful coordinates as regards dynamics on \mathbb{CP}^2 and its quadrilateralland interpretation. The conjugate momenta are

$$p_{\psi} = \sin^2 \chi \cos^2 \chi \{\psi^{\star} + \cos \beta \phi^{\star}\}/4 \quad , \quad p_{\phi} = \sin^2 \chi \{\cos^2 \chi \{\phi^{\star} + \cos \beta \psi^{\star}\} + \sin^2 \chi \sin^2 \beta \phi^{\star}\}/4 \quad , \quad p_{\beta} = \sin^2 \chi \beta^{\star}/4 \quad , \quad p_{\chi} = \chi^{\star} . \quad (329)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} = \frac{p_{\chi}^2}{2} + \frac{2}{\sin^2 \chi} \left\{ p_{\beta}^2 + \frac{1}{\sin^2 \beta} \{p_{\phi}^2 + p_{\psi}^2 - 2p_{\phi} p_{\psi} \cos \beta\} \right\} + \frac{2}{\cos^2 \chi} p_{\psi}^2 + \mathcal{V} = \mathcal{E} , \quad (330)$$

the middle expression of which also serves as the Hamiltonian. The evolution equations are

$$\{\sin^2 \chi \cos^2 \chi \{\psi^{\star} + \cos \beta \phi^{\star}\}/4\}^{\star} = -\partial \mathcal{V} / \partial \psi , \quad (331)$$

$$\{\sin^2 \chi \{\cos^2 \chi \{\phi^{\star} + \cos \beta \psi^{\star}\} + \sin^2 \chi \sin^2 \beta \phi^{\star}\}/4\}^{\star} = -\partial \mathcal{V} / \partial \phi , \quad (332)$$

$$\{\sin^2 \chi \beta^{\star}/4\}^{\star} = \sin^2 \chi \sin \beta \{\sin^2 \chi \cos \beta \phi^{\star} - \cos^2 \chi \psi^{\star}\} \phi^{\star}/4 - \partial \mathcal{V} / \partial \beta , \quad (333)$$

$$\chi^{\star\star} = \sin \chi \cos \chi \{\beta^{\star 2} + \cos 2\chi \{\phi^{\star 2} + \psi^{\star 2} + 2\phi^{\star} \psi^{\star} \cos \beta\} + 2\sin^2 \chi \sin^2 \beta \phi^{\star 2}\}/4 - \partial \mathcal{V} / \partial \chi , \quad (334)$$

one of which can be supplanted by the Lagrangian form of (330),

$$\chi^{\star 2}/2 + \sin^2 \chi \{\beta^{\star 2} + \cos^2 \chi \{\phi^{\star 2} + \psi^{\star 2} + 2\phi^{\star} \psi^{\star} \cos \beta\} + \sin^2 \chi \sin^2 \beta \phi^{\star 2}\}/8 + \mathcal{V} = \mathcal{E} . \quad (335)$$

4.2 Dynamical equations for scaled RPM

4.2.1 3-stop metroland in Cartesian coordinates

This is the $k = 2$, $X^A \rightarrow \rho^i$ case of Sec 17.1.

4.2.2 3-stop metroland in (ρ, φ) polar coordinates

This is the $R, \chi \rightarrow \rho, \varphi$ and $\mathcal{S} \rightarrow \mathcal{Dil}$ case of Sec 17.2.

4.2.3 4-stop metroland in Cartesian coordinates

This is the $k = 3$, $X^A \rightarrow \rho^i$ case of Sec 17.1.

4.2.4 4-stop metroland in (ρ, θ, ϕ) spherical polar coordinates

This is the $R, \alpha, \chi \rightarrow \rho, \theta, \phi$ and $\mathcal{S}_A \rightarrow \mathcal{Dil}_\Gamma$ case of Sec 17.3.

4.2.5 N -stop metroland extension

In Cartesian coordinates, this is the $k = n$ counterpart of the above. On the other hand, in ultraspherical polar coordinates, the conjugate momenta are

$$P_q = \rho^2 \left\{ \prod_{A=1}^{q-1} \sin^2 \theta_p \right\} \theta_q^* , \quad p_\rho = \rho^* . \quad (336)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} := \frac{p_\rho^2}{2} + \frac{1}{2\rho^2} \sum_{r=1}^{n-1} P_q^2 / \prod_{A=1}^{q-1} \sin^2 \theta_A + V(\theta_p) = E , \quad (337)$$

the middle expression of which also serves as the Hamiltonian. The evolution equations are now

$$\left\{ \rho^2 \prod_{A=1}^{q-1} \sin^2 \theta_p \theta_q^* \right\}^* - \rho^2 \left\{ \sum_{r=q+1}^{n-1} \left\{ \prod_{A=1, A \neq q}^{r-1} \sin^2 \theta_p \right\} \right\} \sin \theta_q \cos \theta_q \theta_r^{*2} = -\frac{\partial V}{\partial \theta_q} , \quad (338)$$

one of which can be supplanted by the Lagrangian form of (337),

$$\dot{\rho}^2/2 + \rho^2 \sum_{r=1}^{n-1} \prod_{p=1}^{r-1} \sin^2 \theta_p \theta_r^{*2}/2 + V(\theta_r) = E . \quad (339)$$

4.2.6 Triangleland in Cartesian (i.e. Dragt) coordinates

This is the $k = 3$, $X^A, P_A \rightarrow \text{Dra}^\Gamma, \Pi_\Gamma^{\text{Dra}}$ case of Sec 17.1; I reproduce the energy constraint for this since I need it for future reference:

$$\sum_\Gamma \text{Dra}^{\Gamma*2} + V(\text{Dra}^A) = E . \quad (340)$$

I also comment that the Dragt correspondence $\underline{\rho}^i \rightarrow \text{Dra}^\Gamma$ does preserve the form of a number of objects, though it is not quite as nice as the Jacobi map in this way: $|\sum_{\Gamma=1}^3 \text{Dra}^\Gamma|^2 = I^2 = |\sum_{i=1}^2 I^i|^2$, $|\sum_{\Gamma=1}^3 \Pi_\Gamma^{\text{Dra}}|^2 = 2T = |\sum_{i=1}^2 \Pi_i|^2$, $\sum_{\Gamma=1}^3 \underline{\text{Dra}}^\Gamma \cdot \underline{\Pi}_\Gamma^{\text{Dra}} = 2\mathcal{D} = 2 \sum_{i=1}^2 \underline{\rho}^i \cdot \underline{\pi}_i$. However, $\sum_{\Gamma=1}^3 \underline{\text{Dra}}^\Gamma \times \underline{\Pi}_\Gamma^{\text{Dra}}$ is nothing like $\sum_{i=1}^2 \underline{\rho}^i \times \underline{\pi}_i$.

4.2.7 Triangleland in $(\mathbf{I}, \Theta, \Phi)$ spherical polar coordinates.

This is the $R, \alpha, \chi \rightarrow \mathbf{I}, \Theta, \Phi$, with $\{E - V\}/4\mathbf{I}$ in place of $E - V$ (giving an extra $\partial\{E/4\mathbf{I}\}/\partial\mathbf{I} = -E/4\mathbf{I}^2$ in the \mathbf{I} -evolution equation), $*$ \rightarrow $\bar{*}$ and $\mathcal{S}_A \rightarrow \mathcal{Dil}_\Gamma$ case of Sec 17.3.

4.2.8 Triangleland in $(\mathbf{I}_1, \mathbf{I}_2, \Phi)$ parabolic-type coordinates

These subsystem-split coordinates are useful in the context of a Φ -independent potential energy in the special and very special cases, for which the conjugate momenta are

$$P_i = \bar{\mathbf{I}}_i^*/4\mathbf{I}_i , \quad P_\Phi = \mathbf{I}_1 \mathbf{I}_2 \Phi^*/\mathbf{I} = \mathcal{J} , \quad \text{constant} , \quad \text{from the } \Phi \text{ evolution equation being } \{\mathbf{I}_1 \mathbf{I}_2 \Phi^*/\mathbf{I}\}^{\bar{*}} = -\partial V/\partial \Phi . \quad (341)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\bar{\mathcal{E}} := 2\mathbf{I}_1 P_1^2 + 2\mathbf{I}_2 P_2^2 + \frac{\mathcal{J}^2}{2} \left\{ \frac{1}{\mathbf{I}_1} + \frac{1}{\mathbf{I}_2} \right\} + \frac{K_1 \mathbf{I}_1 + K_2 \mathbf{I}_2}{8} = \frac{E}{4} , \quad (342)$$

the middle expression of which also serves as the Hamiltonian.

If V is independent of Φ , then Φ is a cyclic coordinate and the Φ evolution equation simplifies considerably: $I_1 I_2 \Phi^* / I = \mathcal{J}$, constant, so

$$\Phi^* = \mathcal{J} \{1/I_1 + 1/I_2\} . \quad (343)$$

This can be used to remove Φ^* from the other equations of motion,

$$\rho_i^{**} - \mathcal{J}^2 / \rho_i^3 = -\partial V / \partial \rho_i , i = 1, 2 . \quad (344)$$

4.2.9 General scale–shape split case

For $T = \{\dot{\sigma}^2 + \sigma^2 T_S\}/2$, the momenta are

$$p_\sigma = \sigma^* , \quad p_a^S = \sigma^2 M_{ab} S^{b*} . \quad (345)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} = \{P_\sigma^2 + N^{ab} P_a^S P_b^S / \sigma^2\} / 2 + V = E , \quad (346)$$

the middle expression of which also serves as the Hamiltonian. The evolution equations are

$$\sigma^{**} = \sigma M_{ab} S^a S^b - \partial V / \partial \sigma , \quad \{\sigma^2 S^{b*}\}^* + \Gamma_{ac}^b S^{a*} S^{c*} = -N^{ab} \partial V / \partial S^b , \quad (347)$$

(into which one can substitute Sec 3’s Christoffel symbol terms), one of which can be supplanted by the Lagrangian form of (346),

$$\{\sigma^{*2} + \sigma^2 M_{ab} S^{a*} S^{b*}\} / 2 + V = E \quad (348)$$

4.3 Physical interpretation of RPM momenta

The conjugates of the three kinds of quantity in Sec 3.18 could be termed distance or dilational momenta (one case of which is scale, and mathematically, radial), angular or rotational momenta and relative distance or relative dilational momenta. I choose to use the first option in each case. I use dilation, rather of the D_i quantities (partial dilations in parallel to the I_i being partial moments of inertia).

4.3.1 3- and 4-stop metroland cases

For 3-stop metroland in polar coordinates, the momenta are [dropping (a) labels and recycling the notation p_i to mean the conjugate of n^i],

$$\text{Dil} := p_\varphi = n_1 p_2 - n_2 p_1 = \mathcal{D}_2 n_1 / n_2 - \mathcal{D}_1 n_2 / n_1 \quad (349)$$

for \mathcal{D}_i the *partial dilations*. The second form of this is manifestly a shape-weighted *relative dilational quantity* corresponding to a particular exchange of dilational momentum between the {bc} and {a} clusters. It is indeed conceptually clear that the conjugate to the non-angular length ratio $\varphi^{(a)}$ will be a relative distance momentum. I generally use the notation \mathcal{Dil} for whichever type of relative distance momenta.

For 4-stop metroland in spherical coordinates, the momenta are [dropping (Hb) or (Ka) labels]

$$\mathcal{Dil}_\phi := p_\phi = n_1 p_2 - n_2 p_1 = \mathcal{D}_2 n_1 / n_2 - \mathcal{D}_1 n_2 / n_1 , \quad (350)$$

i.e. a a weighted relative dilational quantity corresponding to a particular exchange of dilational momentum between the {ab} and {cd} clusters in the H case or the {bc} and {Ta} clusters in the K case, and

$$\mathcal{Dil}_\theta := p_\theta . \quad (351)$$

A formula for the latter can readily be deduced from Sec 4.4.1.

4.3.2 Triangleland case

For triangleland in spherical coordinates, the momenta are [dropping (a) labels]

$$\mathcal{J} =: p_\Phi = \text{dra}_1 \Pi_2^{\text{dra}} - \text{dra}_1 \Pi_2^{\text{dra}} , \quad (352)$$

which, given that Φ is a relative angle in space, is a relative angular momentum in space (hence the notation \mathcal{J}), whilst

$$\mathcal{Dil}_\Delta := p_\Theta \quad (353)$$

is indeed a relative dilational quantity since Θ is a function of a relative distance (length ratio). A formula for the latter can readily be deduced from Sec 4.4.2.

As regards \mathcal{J} ’s interpretation as a relative angular momentum [30],

$$\mathcal{J} = I_1 I_2 \Phi^* / I = I_1 I_2 \{\theta_2^* - \theta_1^*\} / I = \{I_1 L_2 - I_2 L_1\} / I = L_2 = -L_1 = \{L_2 - L_1\} / 2 \quad (354)$$

(the fourth equality uses the zero angular momentum constraint). Thus it is interpretable as the angular momentum of one of the two constituent subsystems, minus the angular momentum of the other, or half of the difference between the two subsystems' angular momenta.

Franzen and I [50] termed the overall set of angular momenta and relative distance momenta (and mixtures of the two) *rational momenta* since they correspond to the general-ratio generalization of the angle-ratio's angular momenta. Rational momenta was previously called *generalized angular momenta* by Smith [580]; we jettisoned that name since it is not conceptually descriptive. Moreover, Serna and I found it to be conceptually cleaner to introduce this notion at the level of the momenta (as followed in this SSSec) rather than at the level of the isometries/conserved quantities. That makes it clear that these quantities' 'true name' [119, 533] ought to be **shape momenta**, since what are mathematically ratio variables can also be seen to be dimensionless shape variables, and the quantity in question is the momentum conjugate to such a quantity. Serna and I celebrate this by passing from the previous notation \mathcal{R} for 'rational' to \mathcal{S} for 'shape'.

This 'true naming' becomes clear in moving, away from the previous idea of interpreting in physical space the $SO(n)$ mathematics of the first few RPM models studied, to the following line of thought.

- 1) scale-shape splits are well defined, and then there are corresponding splits into conjugate scale momenta and shape momenta.
- 2) The shape momenta correspond to dimensionless variables, i.e. ratios (or functions of ratios), accounting for why the previously encountered objects were termed rational momenta.
- 3) Then in some cases, shape momentum mathematics coincides with (arbitrary-dimensional) angular momentum mathematics, and also some shapes/ratios happen to be physically angles in space, so the interpretation in space *indeed* is as angular momentum.
- 4) But in other cases, shapes can correspond physically to ratios other than those that go into angles in space, e.g. ratios of two lengths (then one's momentum is a pure relative distance momentum) or a mixture of angle and non-angle in space ratios (in which case one has a general shape momentum). Moreover, there is no a priori association between shape momenta and $SO(n)$ groups; this *happens* to be the case for the first few examples encountered (N -stop metroland, triangle land) but ceases to be the situation for quadrilateralland (and N -a-gonlands beyond that).

4.3.3 Quadrilateralland case

The Gibbons–Pope-type coordinates for quadrilateralland extend the above triangle land spherical polar coordinates in constituting a clean split into pure non-angle ratios and pure angle ratios (two of each). Thus their conjugates are again cleanly-split pure relative angular momenta and relative distance momenta as their conjugates (two of each):

$$\mathcal{J}_\psi := p_\psi, \quad \mathcal{J}_\phi := p_\phi, \quad \mathcal{D}_\beta := p_\beta, \quad \mathcal{D}_\chi := p_\chi. \quad (355)$$

The corresponding Hamiltonian is then

$$H = \frac{\mathcal{D}_\chi^2}{2} + \frac{2}{\sin^2 \chi} \left\{ \mathcal{D}_\beta^2 + \frac{1}{\sin^2 \beta} \{ \mathcal{J}_\phi^2 + \mathcal{J}_\psi^2 - 2 \mathcal{J}_\phi \mathcal{J}_\psi \cos \beta \} \right\} + \frac{2}{\cos^2 \chi} \mathcal{J}_\psi^2 + V. \quad (356)$$

See [53] for how various combinations of \mathcal{D}_β , \mathcal{J}_ϕ and \mathcal{J}_ψ can be constants of the motion for various types of potential (N -a-gonlands have a more intricate version of the very^k special tower) and for further study of quadrilateralland.

4.3.4 Scale momentum interpretation

In terms of the overall dilation object,

$$P_\rho = \mathcal{D}/\rho, \quad P_I = \mathcal{D}/2I. \quad (357)$$

4.4 Physical interpretation of RPM's relationalspace isometries/conserved quantities

4.4.1 N -stop metroland cases

The pure-shape case of 3-stop metroland is relationally trivial as per Sec 2.1.4, but it is part of dynamically nontrivial scaled 3-stop metroland problem. Here the generator of $\text{Isom}(\mathfrak{S}(3, 1)) = \text{Isom}(\mathbb{S}^1) = U(1) = SO(2)$ is just the above-described Dil . This is mathematically the 'component out of the plane' of 'angular momentum', albeit in *configuration space*, there clearly being no meaningful physical concept of angular momentum in 1- d space itself.

For 4-stop metroland, the three generators of $\text{Isom}(\mathfrak{S}(4, 1)) = \text{Isom}(\mathbb{S}^2) = SO(3)$ are

$$\text{Dil}_i = \epsilon_{ij}{}^k n^j p_k. \quad (358)$$

(358) are mathematically the three components of 'angular momentum' albeit again in configuration space. Their physical interpretation (for the moment in the setting of H -coordinates) in space is an immediate extension of that of the already-encountered 3-component of this object (350):

$$\text{Dil}_i = \mathcal{D}_k n^j / n^k - \mathcal{D}_j n^k / n^j. \quad (359)$$

Moreover, this example's interpretation relies, somewhat innocuously, on the three conserved quantities \mathcal{Dil}_i corresponding to three mutually perpendicular directions (the three DD axes picked out by using H-coordinates), as is brought out more clearly by the next example.

For 4-stop metroland in K-coordinates one has the above formulae again [dropping (Ka) labels instead of (Hb) ones]. They are clearly still all relative distance momenta, albeit corresponding to a different set of ratios. Then e.g. \mathcal{Dil}_3 is a (weighted) *relative dilational quantity* corresponding to a particular exchange of dilational momentum between the $\{12\}$ and $\{T3\}$ clusters. Here, one needs to use an axis system containing only one T-axis, e.g. a $\{T, M^*D, M^*D\}$ axis system.

Finally, for 4-stop metroland the *total shape momentum* counterpart of the total angular momentum is $\mathcal{T}_{\text{Tot}} = \sum_{i=1}^3 \mathcal{Dil}_i^2 = \mathcal{Dil}_\theta^2 + \sin^{-2}\theta \mathcal{Dil}_\phi^2 = 2\mathcal{T}$. [For 3-stop metroland, this is just $\mathcal{T}_{\text{Tot}} = \mathcal{Dil}^2$.]

The above pattern repeats itself, giving, for N -stop metroland, $n-1$ hyperspherical coordinates interpretable as a sequence of ratios of relative inter-particle cluster separations, shape space isometry group $SO(n)$ and a set of $n(n-1)/2$ isometry generators which are, mathematically, components of 'angular momentum' in configuration space.

4.4.2 Triangleland case

Here, the three Isom $(\mathbb{S}^2) = \text{Isom}\mathfrak{S}(3, 2) = SO(3) = SU(2)/\mathbb{Z}_2$ generators are given by

$$\mathcal{S}_i = \epsilon_{ij}{}^k \text{dra}^j \Pi_k^{\text{dra}}, \quad (360)$$

which are mathematically the three components of 'angular momentum' albeit yet again in configuration space rather than in space. Now on this occasion, there is a notion of relative angular momentum in space. There are even three natural such, one per clustering: \mathcal{J} Are these the three components of \mathcal{S}_Γ ? No! The three are coplanar and at 120 degrees to each other, so only can only pick one of these for any given orthogonal coordinate basis, much as in the above K-coordinate example. The other components point in an E and an S direction (c.f. Fig 27). E and D are then the two main useful choices of principal axes (I study QM with respect to both of these bases in Sec 8 and 9), furnishing the $\{E, D, S\}$ and $\{D, E, S\}$. Moreover the component pointing in the D direction has the form of a pure relative angular momenta, $\mathcal{S}_3 = \mathcal{J}$ of the $\{23\}$ subsystem relative to the 1 subsystem. The other two \mathcal{S}_Γ 's are mixed dilational and angular momenta with shape-valued coefficient [dropping (a) labels]:

$$\sin \Phi \mathcal{Dil}_\Delta + \cos \Phi \cot \Theta \mathcal{J} \quad \text{and} \quad -\cos \Phi \mathcal{Dil}_\Delta + \sin \Phi \cot \Theta \mathcal{J}. \quad (361)$$

Finally, for triangleland, the *total shape momentum* counterpart of the total angular momentum is $\mathcal{T}_{\text{Tot}} = \sum_{\Gamma=1}^3 \mathcal{S}_\Gamma^2 = \mathcal{J}^2 + \sin^{-2}\Theta \mathcal{Dil}_\Delta^2 = 2\mathcal{T}$.

4.4.3 Quadrilateralland case

$\text{Isom}(\mathfrak{S}(4, 2)) = \text{Isom}(\mathbb{CP}^2) = SU(3)$ [up to a usually not important quotienting by \mathbb{Z}_3] giving the same representation theory and mathematical form of conserved quantities as in the idealized flavour $SU(3)$ or the colour $SU(3)$ of Particle Physics [these *also* have this quotienting]. Our use of 1, 2, 3, +, and - is the standard one of $SU(2)$ mathematics. $SU(3)$ contains three overlapping such ladders (in fact three overlapping $SU(2) \times U(1)$'s, with the $SU(2)$'s being isospins I_+ , I_- , I_3 , V_+ , V_- , V_3 and U_+ , U_- , U_3 and the $U(1)$'s being hypercharges Y , Y_V and Y_U). The usual set of independent such objects, I_3 , I_+ , I_- , V_+ , V_- , U_+ , U_- and Y , are then represented by the Gell-Mann λ -matrices up to proportion. For quadrilateralland itself, I use the calligraphic-font analogues of these; Serna and I [53] relate some of these quantities to the Gibbons-Pope momenta that pick out one of the $SU(2) \times U(1)$'s

$$\mathcal{Y} = 2p_\psi = 2\mathcal{J}_\psi, \quad \mathcal{I}_3 = p_\phi = \mathcal{J}_\phi. \quad (362)$$

In terms of the quadrilateralland-significant inhomogeneous bipolar coordinates, these are

$$\mathcal{Y} = -2\{p_{\Phi_1} + p_{\Phi_2}\}, \quad \mathcal{I}_3 = p_{\Phi_2} - p_{\Phi_1}. \quad (363)$$

In H coordinates, the momentum associated with the ϕ coordinate represents a counter-rotation of the two constituent subsystems (\times relative to $\{12\}$ and $+$ relative to $\{34\}$). The momentum associated with the ψ coordinate represents a co-rotation of these two constituent subsystems (with counter-rotation in \times relative to $+$ so as to preserve the overall zero angular momentum condition). The momentum associated with the ϕ coordinate now represents a co-rotation of the two constituent subsystems (which are now $\{12\}$ relative to 4 and $\{+4\}$ relative to 3, and with counter-rotation in T relative to $+$ so as to preserve the overall zero angular momentum condition). The momentum associated with the ψ now represents a counter-rotation of these two constituent subsystems. Also,

$$\mathcal{I}_1 = -\sin \phi p_\beta + \frac{\cos \phi}{\sin \beta} \{p_\psi - \cos \beta p_\phi\} = -\sin \phi \mathcal{D}_\beta + \frac{\cos \phi}{\sin \beta} \{\mathcal{J}_\psi - \cos \beta \mathcal{J}_\phi\}, \quad (364)$$

$$\mathcal{I}_2 = \cos \phi p_\beta + \frac{\sin \phi}{\sin \beta} \{p_\psi - \cos \beta p_\phi\} = \cos \phi \mathcal{D}_\beta + \frac{\sin \phi}{\sin \beta} \{\mathcal{J}_\psi - \cos \beta \mathcal{J}_\phi\}. \quad (365)$$

Finally,

$$\mathcal{I}_{\text{Tot}} := \mathcal{I}^2 = p_\beta^2 + \frac{1}{\sin^2 \beta} \{p_\phi^2 - 2\cos \beta p_\psi p_\phi + p_\psi^2\} = \mathcal{D}_\beta^2 + \frac{1}{\sin^2 \beta} \{\mathcal{J}_\phi^2 - 2\cos \beta \mathcal{J}_\phi \mathcal{J}_\psi + \mathcal{J}_\psi^2\}. \quad (366)$$

Thus, whether for H's or for K's there is also a pair of coordinates β and χ : additionally dependent on only one corresponding ratio of relative separations, i.e. the \mathcal{I}_1 and \mathcal{I}_2 depend on β alone rather than on χ . These are conjugate to quantities that involve relative distance momenta in addition to relative angular momenta.

The other expressions ($\mathcal{U}_\pm, \mathcal{V}_\pm$) come out much more messily and less insightfully in these particular \mathcal{I} -adapted Gibbons–Pope type coordinates. Of course, \mathcal{U} and \mathcal{V} adapted Gibbons–Pope type coordinates exist as well, via omitting in each case a different choice of Jacobi vector. This is the end of this article's systematic coverage of quadrilateralland structure. Parallels of this article's subsequent classical and quantum solution work is in [53, 44].

4.4.4 Scaled counterparts

$\text{Isom}(\mathcal{R}(N, 1)) = \text{Eucl}(n)$ covering the preceding and the translations, and $\text{Isom}(\mathcal{R}(3, 2) = \text{Eucl}(3)$ likewise.

4.4.5 Internal ‘arrow’ degrees of freedom

As the word ‘spin’ itself has rotational and hence angular momentum connotations, Franzen and I called its generalization to RPM's that do not necessarily possess any notion angular momentum ‘arrow’, \mathcal{A} . Adding arrows to the present model could be nontrivial in the sense that the 1- d arrows need not just obey separate ‘tensored-on’ occupation rules. This is via addition of arrow and relative distance momentum quantities. By this means, spatially 1- d models can have arrow-relative distance momentum interactions that parallel the spin-orbital angular momentum couplings that occur in higher spatial dimensions.

4.5 Monopole issues at the classical level

Guichardet [288], Iwai [348], and Shapere and Wilczek [572] realized that certain gauge fields play an important role in the reduction process and in describing the reduced dynamics.

Monopole issues are known to affect classical, and in particular, quantum-mechanical, study of a system. E.g. consider a charged particle in 3- d . [Here, Cartesian coordinates are $\underline{x} = (x^1, x^2, x^3)$ and $(r, \theta_{\text{sp}}, \phi_{\text{sp}})$ in spherical polars, with corresponding mechanical momentum \underline{p} . m is the particle's mass and e is its charge.] Let this be in the presence of a Dirac monopole [207, 208, 647] of monopole strength g , corresponding to field strength

$$F_{\beta\gamma} = \epsilon_{\alpha\beta\gamma} g x^\alpha / r^3. \quad (367)$$

If one looks for a vector potential \underline{A} corresponding to this in just the one chart, one finds that it is singular somewhere – the monopole has a Dirac string emanating from it in some direction or other. However, there is no physical content in the direction in which it emanates, and one can avoid having such strings by using more than one chart (each chart's choice of Dirac string lying outside the chart). One such choice is to have an N-chart $\{\theta_{\text{sp}}, \phi_{\text{sp}} \mid 0 \leq \theta_{\text{sp}} \leq \pi/2 + \epsilon\}$, and an S-chart $\{\theta_{\text{sp}}, \phi_{\text{sp}} \mid \pi/2 - \epsilon \leq \theta_{\text{sp}} \leq \pi\}$, with the vector potential in each of these being given by

$$\underline{A}^{\text{N}} \cdot d\underline{x} = g\{x_1 dx_2 - x_2 dx_1\}/r\{r + x_3\}, \quad \underline{A}^{\text{S}} \cdot d\underline{x} = g\{x^1 dx_2 - x_2 dx_1\}/r\{r - x_3\}. \quad (368)$$

Then the classical Hamiltonian for the charged particle is built from the canonical momentum combination $\underline{p} - e\underline{A}$:

$$\mathbf{H} = \{\underline{p} - e\underline{A}\}^2/2m + V(\underline{x}), \quad (369)$$

with \underline{A} taking the above monopole form, i.e. \underline{A}^{N} in the N-chart and \underline{A}^{S} in the S-chart.

N.B. however how this working collapses in the case of an uncharged particle: the monopole is then not ‘felt’ so one has the mathematically-usual particle Hamiltonian (see Sec 6.9 for further QM consequences).

Now, monopole issues involving the triple collision are somewhat well-known to occur in 3-body problem configuration spaces. In the case of RPM's, is the preceding an indication of somewhat unusual mathematics arising analogously to in the above Dirac monopole considerations?

I begin by considering this for plain scaled triangleland in the Newtonian context of this sitting inside absolute space. Firstly, the connection involved in this case is Guichardet's [288] rotational connection (c.f. Appendix A.1). This is indeed a nontrivial connection as it has a nontrivial field strength (378). In this case, passing to e.g. Dra^Γ coordinates gives precisely the Dirac monopole (367) under the correspondence Dra^Γ for x^α , clustering (1)'s D in the role of N and its M in that of S.

Next, the Hamiltonian is (for relational space coordinates Q^A with corresponding mechanical momenta P_A with metric g_{AB} on configuration space)

$$\mathbf{H} = \underline{\underline{L}} \underline{\underline{L}}^{-1} \underline{\underline{L}} + g^{\text{AB}} \{P_A - \underline{\underline{L}} \cdot \underline{\underline{A}}_A\} \{P_B - \underline{\underline{L}} \cdot \underline{\underline{A}}_B\}/2 + V(Q^C). \quad (370)$$

From here, monopole effects would spread into the Schrödinger equation and its solutions.

However the key point is that the case of interest in RPM's is the one with *zero total angular momentum* $\underline{L} = \underline{\mathcal{L}} = 0$. For this, analogously to the special case of an uncharged particle in a Dirac monopole field, the Hamiltonian (370) collapses to a much simpler form,

$$H = g^{AB} P_A P_B / 2 + V(Q^C) \quad (371)$$

corresponding to ‘not feeling’ the monopole. Thus monopole effects do not enter the Schrödinger equation and its solutions in this way. (Nor does a distinction between mechanical and canonical momentum change the form of the relative rational momentum operator in this case of zero total angular momentum). Thus the associated $SO(3)$ mathematics is standard. And, in the case of a ‘central’ potential energy $V = V(I \text{ alone})$, the angular part of the Schrödinger equation does indeed give the spherical harmonics. (Determining this requires additional working due to operator ordering issues, which is provided in Sec 6). Thus the pure-shape RPM study of these in [31, 34] turns out to be reuseable in the study of the scaled triangle as the shape part of the scale-shape split.

Suppose that one chooses the case of O-shapes instead. (These are much more common in the literature due to the bias that the real world is 3- d .) Then the monopole is not now of Dirac-type in 3- d space but rather Iwai's monopole [348] in 3- d half-space. The mathematics remains similar to that of the Dirac monopole in any given chart and gauge. (Though now other choices of string are more convenient [348, 428] and the flux is halved due to involving half as much solid angle as before). In particular, the Hamiltonian remains of the form (370).

For more than three particles, the above simplifying effect of being in a $\mathcal{L} = 0$ theory. This is a useful result e.g. toward developing the quadrilateralland model. However, I have not as yet considered in these cases whether the g_{AB} coincides with the metric obtained from first principles/from reduction of BB82-type formulations.

We straightforwardly eliminated the translations early on in this article's treatment. I should next therefore reassure the reader that, if these had been left alone, they would not have given rise to a further significant connection. This is because, as a further manifestation of the mathematical simpleness of the translations, their analogue of the Guichardet connection form has trivial field strength (Appendix A.2). Thus one can be sure that for scaled N -stop metroland (involving at most translations), there are not any monopole effects to worry about. This protects my treatment of N -stop metroland as ‘ordinary Euclidean space physics’, among the coordinate systems for which the S^{n-1} polar coordinates have a particularly lucid scale-shape interpretation.

Next, in the case of pure-shape theories, can the dilations cause analogous monopole effects? I establish in Appendix A.2 that this is not the case because the dilational analogue of the Guichardet connection form also has trivial field strength. Thus combining this result and the above rotational results, pure-shape theories carry no vestiges of the excluded maximal collision through its action as a monopole. (Without establishing this, one could have feared e.g. that the shape spaces could contain a bad point of gauge-dependent position. Such could have arisen from the corresponding relational space's gauge and chart choice's Dirac string casting a ‘shadow’ on the shape space at the point intersection in relational space between the Dirac string and the surrounding shape space, c.f. [208]). [Finally, translations, rotations and dilations do not interfere with each other if treated together in this sense.]

4.6 Appendix A ‘Guichardet connection’ for various transformation groups

4.6.1 Appendix A.1 The Guichardet connection for rotations

Working in mass-weighted Jacobi coordinates,

$$\underline{\mathcal{L}} = \sum_{i=1}^n \underline{\rho}^i \times \underline{\pi}_i = \sum_{i=1}^n \underline{\rho}^i \times \{ \dot{\underline{\rho}}^i + \underline{\dot{B}} \times \underline{\rho}^i \} = \sum_{i=1}^n \underline{\rho}^i \times \{ \dot{Q}^\Gamma \partial \underline{\rho}^i / \partial Q^\Gamma + \underline{\dot{B}} \times \underline{\rho}^i \} \quad (372)$$

(for $\underline{\pi}_i$ the momentum conjugate to $\underline{\rho}^i$), then let

$$\underline{\mathcal{L}} = \underline{\mathbb{I}} \{ \underline{\dot{B}} + \underline{A}_\Gamma \dot{Q}^\Gamma \} \quad (373)$$

for $\underline{\mathbb{I}}$ the inertia tensor. The last term in this defines the Guichardet-type [288] gauge potential $\underline{A}_\Gamma = \underline{\mathbb{I}}^{-1} \underline{a}_\Gamma$ for $\underline{a}_\Gamma = \sum_{i=1}^n \underline{\rho}^i \times \partial \underline{\rho}^i / \partial Q^\Gamma$. For vanishing angular momentum, $\underline{\dot{B}} = -\underline{A}_\Gamma \dot{Q}^\Gamma$ i.e. the mapping between change of shape and corresponding infinitesimal rotation. In 2- d and for plain scaled triangleland, use 1) $Q^\Gamma = (\rho_1, \rho_2, \Phi)$ coordinates (closely related to parabolic coordinates [30, 37]). 2) Use what is referred to in the literature as the ‘xxy gauge’ [428]: $\underline{\rho}_1 = \rho_1(1, 0)$ and $\underline{\rho}_2 = \rho_2(\cos \Phi, \sin \Phi)$. Then the nonzero component of \underline{A}_Γ is

$$A_\Phi = \text{sharp} , \quad (374)$$

which, in passing to Dragt-type coordinates, gives

$$\underline{A}_\Gamma dQ^\Gamma = \{ \text{dra}_1 d\{\text{dra}_2\} - \text{dra}_2 d\{\text{dra}_1\} \} / 2\{1 - \text{dra}_3\} . \quad (375)$$

This is in direct correspondence with Wu–Yang's [647] A_μ^S (S for ‘South’) for the Dirac monopole if one applies the Dragt correspondence. It sets the monopole strength g to be $1/2$. The current triangleland context has several further lucid interpretative points. Firstly, in configuration space, its ‘South’ is the clustering in question's merger point, M. Secondly, in

space itself, the base of the triangle is aligned with the absolute x-axis, so I rename this gauge the *base = x gauge*. Thirdly, mathematically it is most simply and clearly presented as the gauge-fixing $\mathcal{F}_M := \theta_1 = 0$.

Likewise, if one inverts the roles of $\underline{\rho}_1$ and $\underline{\rho}_2$, the nonzero component of A_Γ in the resulting chart and gauge is

$$A_\Phi = \text{flat} , \quad (376)$$

which, in passing to Dragt coordinates, gives

$$A_\Gamma dQ^\Gamma = \frac{1}{2} \{ \text{dra}_1 d\{\text{dra}_2\} - \text{dra}_2 d\{\text{dra}_1\} \} / 1\{1 + \text{dra}_3\} . \quad (377)$$

This is in direct correspondence with Wu–Yang’s A_μ^N (N for ‘North’) for the Dirac monopole if one applies the Dragt correspondence. It likewise sets the monopole strength g to be $1/2$. In the current triangleland context, then, its ‘North’ in configuration space is the clustering in question’s double collision point, D. In space itself, it is the median of the triangle that is now aligned with the absolute x-axis, so I rename this gauge the *median = x gauge*. Finally, mathematically it is most simply and clearly presented as the gauge-fixing $\mathcal{F}_D := \theta_2 = 0$.

Then this D-chart and M-chart provide a full stringless description of this relational space monopole, just as the N-chart and S-chart do for the usual Dirac monopole in space. Given the precise nature of the correspondence between these, it is clear that the field strength is

$$F_{\Gamma\Sigma} = \epsilon_{\Lambda\Gamma\Sigma} \text{dra}^\Lambda / I^2 . \quad (378)$$

For scaled Otriangleland, these workings still hold except that 1) the $\text{dra}_3 = 4 \times \text{area} < 0$ half-plane has ceased to be part of the configuration space. 2) Other charts and gauges which position the Dirac string elsewhere are now more convenient. See e.g. [348, 428]. In this case one has an *Iwai monopole* on \mathbb{R}_+^3 (the Iwai string is restricted to the one hemisphere).

4.6.2 Appendix A.2 Translations and dilations give but trivial analogues

The below results hold for all particle numbers and spatial dimensions.

In the case of translations, the coordinates are mass-weighted particle positions, $\underline{X}^I = \sqrt{m_I} \underline{q}^I$ rather than mass-weighted Jacobi coordinates.

$$\mathcal{P}_\mu = \sum_{I=1}^N \pi_I = \sum_{I=1}^N m_I \{ \underline{q}_I + \underline{\dot{A}} \} = M \sum_{I=1}^N \{ \underline{\dot{A}} + M^{-1} \sum_{I=1}^N \sqrt{m_I} \dot{Q}^A \partial \underline{X}^I / \partial Q^A \} . \quad (379)$$

Here, the total mass is the analogue of inertia tensor, so $\underline{\dot{A}}_\text{trans} = \underline{\dot{a}}_\text{trans} = \partial_A \{ \sum_{I=1}^N \sqrt{m_I} \underline{X}^I / M \}$. As this is of gradient form $\partial_A \zeta$, the corresponding translational field strength $F_{AB} = 2\partial_{[A} \partial_{B]} \zeta = 0$ by symmetry–antisymmetry. Thus this connection is flat and so geometrically trivial.

In the case of dilations, taking the coordinates to be mass-weighted Jacobi coordinates,

$$\text{Dil} = \sum_{i=1}^n \underline{\rho}_i \cdot \underline{\pi}_i = \sum_{i=1}^n \underline{\rho}_i \cdot \{ \underline{\rho}_i + \dot{C} \underline{\rho}_i \} = \sum_{i=1}^n \underline{\rho}_i \cdot \{ \dot{Q}^A \partial \underline{\rho}_i / \partial Q^A + \dot{C} \underline{\rho}_i \} , \quad (380)$$

so

$$\mathcal{D} = I \{ \dot{C} + A_A \dot{Q}^A \} . \quad (381)$$

Here the scalar moment of inertia $I = \sum_{i=1}^n \{ \rho^i \}^2$ is the analogue of the inertia tensor, and $A_\text{A}^\text{dil} = I^{-1} a_\text{A}^\text{dil}$, $a_\text{A}^\text{dil} = \sum_{i=1}^n \underline{\rho}^i \cdot \partial \underline{\rho}^i / \partial Q^A$. Then for vanishing dilation, $\dot{C} = -A_\text{A}^\text{dil} \dot{Q}^A$ so it is a mapping between change of shape and corresponding infinitesimal size change. But again this can be cast in gradient form: $A_\text{A}^\text{dil} = \partial_A \{ \ln(\rho) \}$, so the corresponding field strength is also zero and so this connection is also flat/geometrically trivial.

Finally, composition of translational, rotational and dilational corrections is additive, so outcomes for each of these things do not affect each other. (To consider combinations involving the translations, note that the above presentations for mass-weighted relative Jacobi coordinates for rotations and dilations continue to hold identically under i to I , n to $N = n + 1$ and ρ^i to X^I . I.e., under the trivial position to relative Jacobi coordinates map, see e.g. [19, 20].)

4.6.3 Appendix A.3 Electromagnetism and GR counterparts

Question [Analogy 45]). The Guichardet connection [288] and configuration space monopole issues of RPM’s have at least formal counterparts for GR and electromagnetism. To what extent are these amenable to study?

4.7 Appendix B: some discussion of ‘q is primary’

This article is a fitting place in which to discuss the optional implications of taking this postulate further toward its logical conclusion.

Within the fixed vantage point and only seeking for the celestial sphere as per Appendix 3.C, the observer can obtain all measurements relevant within this scheme by multiple uses of a protractor localized at the vantage point.

However, if one acknowledges the universe to be along the lines of Fig 33b), one will require relative distance measurements and/or surveying with protractor from multiple vantage points. In this context, there is not a particular sense in which

multiple use of a protractor from a fixed vantage point is operationally privileged. In general, one must survey from multiple points, and angle-measuring and ratio-of-relative-distance measuring are to be taken to be on the same footing (and part interchangeable) in surveying conformal geometry. Using parallax, to estimate astronomical amounts to using multiple vantage points relative to the position of the Sun. Deducing distances to stars by this means is not a highly localized and instantaneous procedure, though it contains a key procedure that is much more so, namely the actual comparison of snapshots of the sky taken at different times of the year. Moreover, one could in principle obtain this info not from just relative angles but from direct measurements of multiple distances. These two sorts of geometrical measurements are *interchangeable* as regards building up one's picture of the world by moving around whilst performing geometrical surveying. The angles from a vantage point happens to be simple way to collect information, but it is more a matter of taste whether moving around with a ruler is more or less complicated than moving around with a protractor, and to measure the general angle one has to move around, whilst in actual rather than just celestial sphere models one does need to do at least some moving around. [At a simple level, this involves walking around with rulers. Though in astronomy one has to proceed instead by inferring distance from having a large population of astronomical object data and a mixture of applying laws known from elsewhere, noting correlations and deducing new laws based on those correlations.] Practical examples of indirect astronomical distance measurements include 1) that the stellar population form the Hertzsprung–Russell pattern; from that and assumptions about localized clusters, and the priorly known brightness-distance inverse square law, one can infer distance. 2) That galactic spectral shifted fits Hubble's law, from which redshift gives v and hence $v/H = \text{distance}$. The issue here whether the operational difference between configuration measuring and momentum measuring exceeds the differences among the ways of measuring just the configurations.

Note 1) It should then be clear that in discussing primality, I only wish to consider *direct* measurements rather than inferences (which could e.g. involve inferring distances from momentum-type measurements connected to photon brightness).

Note 2) Secondly, sight and spectra do involve photon momenta. One can dissociate from this problem by thinking in terms of simple 'blind groping' measurements on a scale small enough that one can reach out to grope (I agree that there will often be strong practical limitations on this). The virtue of this approach is that one's configuration measurements for particle mechanics are all *geometrical*: local angles, adjoining rulers to the separations between objects; the blind Geometer's work is free from the impact connotations that enter his sighted colleague's work (which will however often have greater practicality, e.g. if both work competitively as surveyors in a not particularly foggy landscape...)

Next, in GR, Riemannian 3-geometry measurements of space are again cleanly separated out from gravitational momentum/extrinsic curvature. Note how in the mechanics–geometrodynamics progression, one can pass from a simple notion of relational geometry measurements being operationally primary to a more complicated notion of relational geometry measurements being primary (argued for e.g. in [171, 173]. A theme that may then support part of the argument for the primality of configurational measurements is that geometrical measurements are primary.⁴⁷ It then appears somewhat incongruous for triad information to be considered as momentum-like within the Ashtekar variables type approaches given that this is another form of space-geometric information. It would likewise be somewhat incongruous if the Ashtekar \underline{A} were only to be indirectly inferable quantity.⁴⁸ The foundations of geometrodynamics are *not* incongruous in this way, though the above-linked footnote does place limitations on, and give alternatives to, this particular sense of incongruence fostered by this Appendix's tentative expansion of the point of view that \mathbf{q} is primary.

Note 3) The debate about whether lengths, impacts, times, frequencies... are the most primary things to measure is indeed an old and unsettled one (see also Sec 16.17). I wish to make a somewhat different point too: that, regardless of primality, one can tell apart whether what one is measuring is instantaneous-configuration information or impact information.

Note 4) QM is often taken to involve probability densities and operator expectation values over configuration space. However, this is not usually an argument for \mathbf{q} being primary since it is part of a wider position that QM concerns some suitable half-set of the degrees of freedom (the splitting involved being termed a *polarization* [645, 62]), \mathbf{q} then being just one such. This is closely tied to the common view that positions and momenta are on the same footing via being mappable into each

⁴⁷Moreover, this is *not* to be because of geometry constituting a layer of structure prior to the account of the physical objects themselves, since the context in which configurations are being considered as primary is a relational one and one which is specifically to encompass GR, for which there is indeed no prior fixed notion of Riemannian geometry.

⁴⁸Here one usually argues (see e.g. [246] instead for the progression from electromagnetism to Yang–Mills theory to GR in Ashtekar variables form as theories based on notions of connection. As regards the current Appendix's theme of operational distinction and most primary types of measurements, firstly I note that in classical electromagnetism it is of course \underline{B} that is measured, and that measuring \underline{B} and \underline{E} are operationally distinct. I concede that it may now be less clear-cut which of these is a more primary measurement; the \underline{B} -field does have more purely geometrical connotations by its being the curvature associated with that connection, but the physical realization of this geometry is more subtle than the mechanics and 3-metric geometries above. Secondly, mentioning Yang–Mills theory brings one face to face with problems with centring one's thinking in terms of classical measurements; of course, QM offers much wider challenges than this the idea of operational primality of measurements of configurations (this idea may only reflect a simple top-down thinking, a notion explained in Appendix 4.C). As regards measuring Ashtekar's \underline{A} , one has to make do with measuring curvature quantities and eventually inferring \underline{A} from them.

Moreover, the solidity of the above sequence as a first-principles scheme partly rests on clean conceptualization of the notion of 'connection' (to be more precise, the corresponding notion of holonomy is particularly important for LQG). For Ashtekar variables, this is itself a geometrical perspective, for a yet more complicated notion of geometry than GR's usual (semi)Riemannian geometry. (Both Ashtekar's \underline{E} and \underline{A} can be construed of as geometrical objects, though it is the \underline{E} whose sense of geometry ties more directly to the more straightforwardly realized spatial Riemannian geometry.) Moreover, the Ashtekar variables case does run into some issues (e.g. [555]) as regards which features are appropriate for a connection serving this purpose, though Thiemann [607] terms this 'aesthetic'. On the other hand, the solidity of this scheme as obtained from rearrangement of the usual geometrodynamical conceptualization of canonical GR rests on whether it is appropriate to extend the classical phase space to include degenerate configurations, as well as on canonical transformations. [The latter are, of course, usually accepted in Physics, though this Appendix and the next one specifically considers the possibility of doing at least some physics without accepting the canonical transformations.]

other using canonical transformations. Moreover, there are examples in which position and momentum are no longer distinct (albeit those of these that are realized in fundamental physics – spin, fermions – have quantum-mechanical character which may elsewhere take them outside of the intended arena of this Appendix). And yet, the ‘telling apart’ argument in the setting of classical measurements would appear to hold well enough: one can tell the difference between making measurements of an *physical* configuration as opposed to of some other half-set of degrees of freedom that is a *mathematical* a configuration in terms of canonically-related variables. [This is because, for mechanics, say the latter is partly measured using impact apparatus rather than just purely geometrical apparatus.] This suggests that, at least within a restricted setting, there are physically meaningful aspects of a system that are *not* taken into account by canonical transformations: namely what one actually needs to do in order to measure the system’s variables can discern whether the q ’s are physical q ’s or just mathematical Q ’s built from physical p ’s and q ’s via canonical transformation. This suggests that canonical transformations could be an over-mathematization: transformations that give an appearance of equivalence to situations which are in fact, at a more detailed operational level than is usually considered, distinguishable.

It is worth pointing out that this kind of suggestion (considering whether commonly-used structures are in fact unnecessary or even misleading) is very much part and parcel of relational thinking. That does not mean that the relationalist goes around throwing things out for no good reason, but rather that they are prepared to consider the outcome of doing so. This clearly plays dividends with absolute space and absolute time. See also Sec 16.6.1 on whether or not to discard scale from physics, Appendix 14.A on the conceptual reasons to favour compact without boundary space over asymptotically flat space for foundational studies, Sec 2 and 6 as regards relationalists’ questioning of Minkowski’s view on spacetime and also the idea in Sec 2 that space may be more primary than the greater amount of structure assumed spacetime. It then makes sense to question the ‘associated spaces’ of physics. There are very strong reasons why \mathbf{q} (or something with an equivalent amount of physical information in it) is indispensable, but the conventional structure of phase space may be questionable. Much as one can envisage preferred-foliation counterparts of GR spacetime, one can also construe of preferred-polarization versions of phase space. My argument then is that one can encode the operational distinction between configuration and impact measurements by using a preferred-polarization centred about the physical \mathbf{q} . See Appendix 4.C. The physicality of \mathbf{q} is *not* the centre of this argument; that is, rather, the operational distinguishability. If one includes fermions, one can operationally tell apart fermionic quantities that manage to be positions and momenta at once from quantities that solely manifest themselves as configurations and quantities that solely manifest themselves as momenta. In this way, the scheme does not die once the phase space is not just a simple tangent bundle over the configuration space. All that happens then is that operational distinguishability becomes more than bipartite.

As a general point, one should not necessarily postulate that entities that are operationally distinct *have* to be mixed just because transformations that mix them exist. One should not confuse ‘is often useful to transform’ with giving unwarrantedly unconditional significance to the transformations.

Another possible reason for doubting canonical transformations is their clash with the common notion of ‘adding in a potential’. If this is to be considered as a structurally minor change, as is often done, then one has a problem due to the asymmetry between the complication caused by adding a q^4 potential, say, as compared to the major structural upheaval of adding a p^4 kinetic term to one’s Hamiltonian. Of course, this can be dealt with by accepting that canonical transformations imply that ‘adding a potential’ is as great a structural upheaval (e.g. the above two additions to an HO are identical to each other under canonical transformation). This translates to changing the habitual simplicity requirements on Hamiltonians and Lagrangians due to their canonical disparity, and that doubtlessly cuts down on widely applicable theorems that depended on such canonically-disparate simplicity requirements. One might then hold all such results to have been empty anyway, but one might attempt to keep them by allotting particular significance to the physics in its presentation centred about \mathbf{q} .

A more worrying issue is that classical equivalence under canonical transformations is in general broken in the passage to QM by the Groenewold–van Hove theorem [276, 618], Sec 11. This may cast doubt on the deeper validity of canonical transformations, and so partly undermine the preceding widespread view. Though the Groenewold–van Hove theorem already affects configuration space itself (e.g. choice of q^3 versus that of q as an observable) so it should not be used as too hefty an argument against specifically the canonical transformations. One might however ask which weakening of the canonical transformations at the classical level it is that is preserved as unitary equivalence at the quantum level. I am not claiming that will be a neat or findable structure, but rather that looking for a such could be a way to question quantization programs that make use of canonical transformations as to whether they are equivalent to quantizations of the underlying formulations prior to applying those canonical transformations.

If canonical transformations are held in doubt, this affects internal time, Histories Theory, Ashtekar variables and the recent linking theory approach; the second and fourth of these make *more* than the usual amount of use of canonical transformations. This Appendix suggests the possibility of developing physics in the *opposite* direction.

4.8 Appendix C. (Rigged-)(differential-)(almost-)phase space and morphisms

I motivate this discussion by pointing out that quantization can be formally understood as a (bad) functor, linking the classical to the quantum-mechanical.⁴⁹

⁴⁹I also agree that one may well ultimately prefer to conceive of quantum theories, including QG, from first principles i.e. a down-up approach to quantization’s top-down approach, for all that the latter may have some early benefits due to the classical theory serving as a well understood

Categories (O, M) consist of objects O and *morphisms* M (the maps between the objects, $M : O \longrightarrow O$). *Functors* are then maps between categories.

It is then clear that so far we have only been studying objects [Relationalism 3), Appendix 4.B], but we should have also been studying morphisms. In particular the morphisms corresponding to \mathfrak{q} are the so-called *point transformations* (p. 15 of [415]) $\text{Point} : \mathfrak{q} \longrightarrow \mathfrak{q}$. Then the opening trident of Sec 3 thickens into Fig 34.

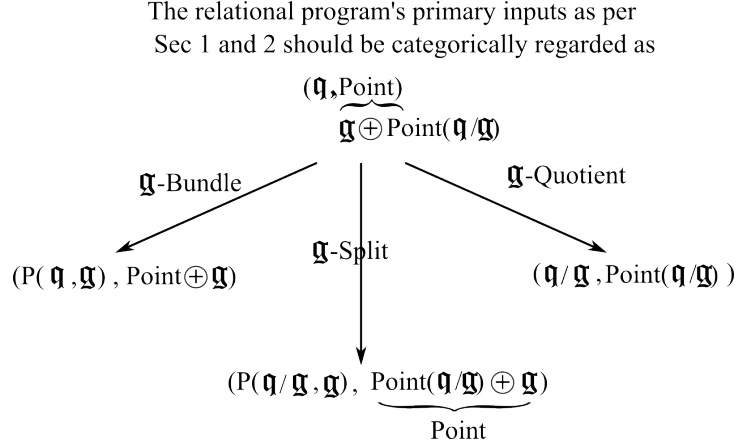


Figure 34:

Note 1) For 1- and 2-d RPM's, this makes $\text{Point}(\mathfrak{q}/\mathfrak{g})$ the coordinate transformations of known manifolds as listed in Sec 3, so this situation is well under control.

Note 2) \mathfrak{g} -quotient = \mathfrak{g} -constrain = \mathfrak{g} -reduce: $(\mathfrak{q}, \text{Point}) \longrightarrow (\mathfrak{q}/\mathfrak{g}, \text{Point}(\mathfrak{q}/\mathfrak{g}))$.

On the other hand, a distinct and far more common viewpoint is that the phase space is central.

$$\text{Phase} = (T^*(\mathfrak{q}), \{ , \}) \quad (382)$$

for $T^*(\mathfrak{q})$ the cotangent bundle of momenta over the configurations and $\{ , \}$ the Poisson bracket, so that this is a Poisson algebra. The canonical transformations $\text{Can} : \text{Phase} \longrightarrow \text{Phase}$ are the corresponding morphisms.

Once there is a \mathfrak{g} in play, APhase or DAPhase are relevant alternatives that are manifestly relational, as per Appendix 2.A.3. I note here that the bracket requires being on the cotangent space (matching a downstairs momentum index with an upstairs configuration index, and only concerns the physical part, so that the bracket part does not care how the \mathfrak{g} -auxiliaries are presented. I agree that momenta and Poisson brackets are part of the conception of this world. Thus I do not resist the passage to

$(T^*(\mathfrak{q}), \{ , \})$. However, I do at least *consider* resisting the suggestion that what mathematically preserves the Poisson bracket should be the associated morphisms of this. This is because of the previous SSec's argument of configuration variables and momenta being operationally distinguishable. Thus it is questionable to take bracket preserving morphisms that do not respect this physical insight. An alternative would be a rigged version RigiPhase of the phase space that preserves this distinction. Here, the \mathfrak{q} -first interpretation of this is that the q 's are fundamental and each p then follows by conjugation and only transforms on \mathfrak{q} are primarily meaningful (If the Q 's change, then the P 's follow suit by being the new conjugates, without any extra freedom in doing so.) Thus the morphisms of this are just Point again, the conjugate momenta being held to follow whatever the coordinates are rather than having their own morphisms (this approach's intent is to put configurations first, so one should not be surprised at these and not the momenta having the primary transformation properties. See Fig 35 for the ensuing options.

and reasonably well-motivated guide.

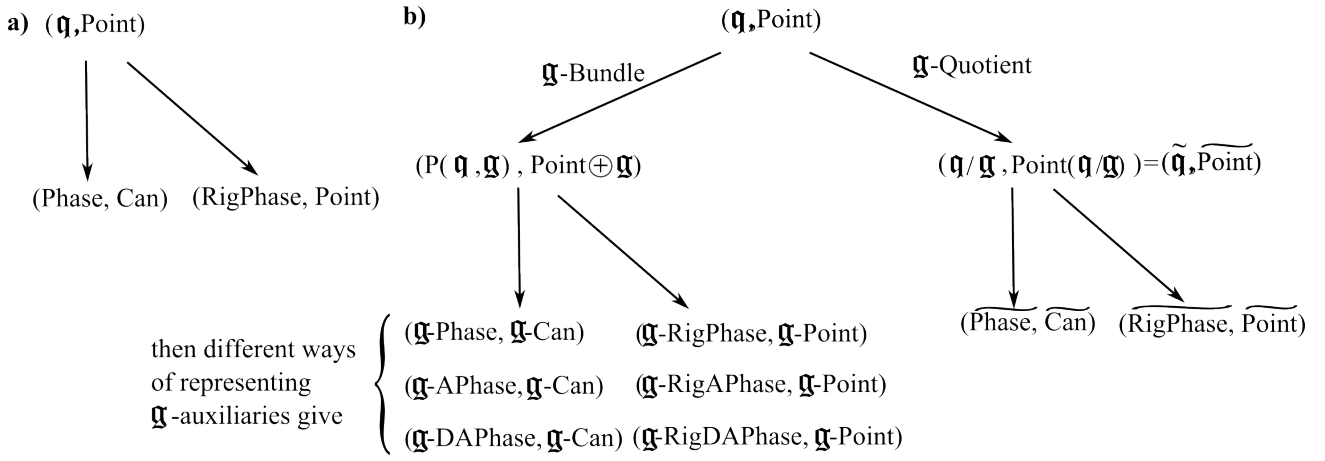


Figure 35: The options without and with the involvement of a \mathfrak{g} .

5 Classical RPM solutions

5.1 Some particular potentials for RPM's

Dynamics involves all of the kinetic metric, the potential and the value of the energy (often its sign in particular). Thus this section considers which specific potentials to use for RPM's. A priori, pure-shape RPM has a lot of potential freedom (any potential homogeneous of degree 0 is valid), while scaled RPM can have any potential at all. Thus I consider scaled RPM first. The no/constant potential case is often the easiest case, but tends to exhibit unbounded/non-normalizable behaviour,⁵⁰ so I tend to go beyond this case. (Moreover, the free problem does serve as an approximation is applicable to scattering problems, and, otherwise, being a thing to solve perturbatively about.) I look at HO-type potentials and then at particular cosmologically-inspired potentials.

5.1.1 Power-law potentials

Many potentials in physics are proportional to some power of the separation between two particles, $k_{IJ} \|\mathbf{q}^I - \mathbf{q}^J\|^n$, or are linear combinations of these. For scaled RPM then one can just consider the above power-law potentials (and sums of these). Then, in terms of relative Jacobi coordinates, power-law potentials take on forms built out of $\|\mathbf{R}^i\|$ or, more generally, $\underline{\mathbf{R}}^i \cdot \underline{\mathbf{R}}^j$.

5.1.2 Special potentials

Note that in some ways the special case is too much of a simplification: lack of interchange of rational momentum between constituent subsystems, also one does not want separability if aligned with the h-l split, as regards the Semiclassical Approach. In pure-shape RPM, however, power-law potentials need to occur multiplied by suitable powers of I to have the homogeneity of degree zero ($I^{-n/2}$); also terms like $\{\mathbf{q}^i - \mathbf{q}^j\} \cdot \{\mathbf{q}^j - \mathbf{q}^l\}$ that consistency requires. This is less of a big deal as it might be, because the powers of I are post-variationally constant in pure-shape RPM. What the above means is that while power-law potentials (other than the constant) are disallowed, they can nevertheless be mimicked by including powers of I. Examples of triangle and separable power laws are

$$V_{(n,m)} \propto N_1^{n-m} N_2^m \propto \{\mathcal{R}/\sqrt{1+\mathcal{R}^2}\}^{n-m} \{1/\sqrt{1+\mathcal{R}^2}\}^m \propto \sin^{n-m} \frac{\Theta}{2} \cos^m \frac{\Theta}{2}$$

so that

$$\bar{V}_{(n,m)} \propto \mathcal{R}^{n-m} / \{1+\mathcal{R}^2\}^{n/2+2} \quad \text{and} \quad \check{V}_{(n,m)} \propto \sin^{n-m} \frac{\Theta}{2} \cos^m \frac{\Theta}{2}. \quad (383)$$

The most physically relevant subcase therein are the power-law-mimicking potentials $m = 0$. These correspond to potential contributions solely between particles 2, 3. The case $m = n$ are potential contributions solely between particle 1 and the centre of mass of particles 2, 3 (which is less widely physically meaningful). It also turns out that the duality map sends $V_{(n,m)}$ to $V_{(n,m-n)}$. In the present case of 3 particles, I use k_1 as shorthand for k_{23} etc. I also replace some of the k 's with special labelling letters in the below examples.

5.1.3 HO-like potentials: motivation and limitations

I call these HO-like as pure-shape RPM only permits potentials that are homogeneous of degree 0. Thus these are not HO's but, rather, HO's divided by the total moment of inertia (which, given that this is constant) in many ways mimic actual HO's well. In the scaled case, one has bona fide HO potentials that are related to preceding by multiplication by I.

⁵⁰'Often' and 'tends' are needed here due to generally curved nature of configuration spaces and use of nontrivial PPST representations.

Harmonic oscillator-like potentials are motivated by

- 1) being quantum-mechanically well behaved [31, 50]/bounded
- 2) Being ubiquitous in theoretical physics or behaving similarly to systems common in theoretical physics.
- 3) Being reasonably analytically tractable
- 4) their scaled counterparts giving relevant terms from the perspective of analogue models of Cosmology [35].

However, such models are atypical in their simpleness and do not particularly parallel the dominant scale dynamics of commonly-used cosmological models. (Though some cosmological solutions do exhibit this sort of behaviour).

The meaning of multi-HO-like is as follows. To satisfy the pure-shape RPM homogeneity requirement, one is to use not scaled RPM's HO V but $V = V/I$; this amounts to using ρ^i in place of n^i .

5.2 HO(-like) potentials for RPM's

5.2.1 Analogies with ordinary physics

Some analogies for 4-stop metroland are as follows. The potential $V_0\{1 - \cos 2\theta\}$ [c.f. form 3 of (404)] occurs in modelling the rotation of a linear molecule in a crystal [505, 592, 506, 641]. Here, the further aspects of the analogy are that the axis and rotor in question are provided by the linear molecule itself, 'energy' \longleftrightarrow energy up to a constant,

$$K_1/2 \longleftrightarrow 2V_0 \text{ up to the same constant difference as in the energy analogy ,} \quad (384)$$

$$B \longleftrightarrow -2V_0 . \quad (385)$$

The potential $-\alpha_{||}\mathcal{E}^2 \cos^2\theta$, for $\alpha_{||}$ the polarizability along the axis occurs in the study [181, 614] of e.g. the CO_2 molecule in a background electric field \mathbf{E} is involved in the theoretical underpinning for Raman spectroscopy. The analogy here is (313),

$$B \longleftrightarrow -\alpha_{||}\mathbf{E}^2/2 . \quad (386)$$

Some useful mathematical analogies for pure-shape triangleland with multiple HO potentials are as follows.

$$\text{very special HO} \longleftrightarrow \text{linear rigid rotor} , \quad (387)$$

$$\text{special HO} \longleftrightarrow \text{linear rigid rotor in a background homogeneous electric field in the axial ('z')-direction} , \quad (388)$$

$$\text{general HO} \longleftrightarrow \text{linear rigid rotor in a background homogeneous electric field in an arbitrary direction} . \quad (389)$$

In particular, this classical problem has

$$T_{\text{rotor}} = I_{\text{rotor}}\{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\}/2 , \quad V_{\text{rotor}} = -\mathbf{M}\mathbf{E}\cos\theta , \quad (390)$$

where I_{rotor} is the single nontrivial value of the moment of inertia of the linear rigid rotor, and \mathbf{M} is the dipole moment component in that direction. Thus the correspondence is (318),

$$(\text{energy})/4 - (\text{sum of mass-weighted Jacobi-Hooke coefficients})/16 = \mathbf{E}/4 - A = \bar{\mathbf{E}} - A \longleftrightarrow E = (\text{energy}) , \quad (391)$$

$$(\text{difference of mass-weighted Jacobi-Hooke coefficients}) = B \longleftrightarrow -\mathbf{M}\mathbf{E} . \quad (392)$$

These all being well-studied at the quantum level [614, 458, 318], this identification is of considerable value in solving the relational problem in hand.

Next, note that $(N, 1)$ scaled RPM's are analogous to ordinary multi-HO's. Also, the very special scaled RPM HO PPSTC-maps to the Kepler problem with

$$(\text{radius}) = r \longleftrightarrow I \text{ (total moment of inertia)} , \quad (393)$$

$$(\text{test mass}) = m \longleftrightarrow 1 , \quad (394)$$

$$(\text{angular momentum}) = L \longleftrightarrow \mathcal{J} \text{ (relative angular momentum)} , \quad (395)$$

$$(\text{total energy}) = E \longleftrightarrow -A = -(\text{sum of mass-weighted Jacobi-Hooke coefficients})/16 , \quad (396)$$

$$\text{and } (\text{Newton's gravitational constant})(\text{massive mass})(\text{test mass}) = GMm \longleftrightarrow \bar{\mathbf{E}} \text{ (total energy)}/4 \quad (397)$$

[or to the 1-electron atom Coulomb problem with the last analogy replaced by

$$(\text{nuclear charge})(\text{test charge of electron})/4\pi(\text{permeability of free space}) = (Ze)e/4\pi\epsilon_0 \longleftrightarrow \bar{\mathbf{E}} \text{ (total energy)}/4] . \quad (398)$$

Note 1) The positivity of the Hooke's coefficients translates to the requirement that the gravitational or atomic energy be negative, i.e. to bound states. Next, the positivity of \mathbf{E} required for classical consistency corresponds to attractive problems like the Kepler problem or the atomic problem being picked out, as opposed to repulsive Coulomb problems.

Note 2) The special case corresponds to the same 'background electric field' that the rotor was subjected to above. Moreover, this is proportional to $\cos\Theta$, which is analogous to $\cos\theta_{\text{sp}}$, which is in the axial ('Z') direction. But it is *not* the well-known mathematics of the axial ('z') direction *Stark effect* for the atom, which involves, rather, $r \cos\theta_{\text{sp}}$. Nevertheless, it is both closely related to the rotor and to the mathematics of the atom in parabolic coordinates (see e.g. [416, 318]). The general case is then the same situation but with the 'electric field' pointing in an arbitrary direction.

5.2.2 3-stop metroland case

The 3-HO-like potential for scaled 3-stop metroland is $V = h_{23}||\underline{q}_2 - \underline{q}_3||^2/2I +$ cycles for h_{IJ} the Hooke's coefficients for the springs between the I th and the J th particles. This potential can be re-expressed as

$$V = K_1 n_1^2/2 + K_2 n_2^2/2 + L \rho_1 \rho_2 = A + B \cos 2\varphi + C \sin 2\varphi = A + B Y_{2c} + C Y_{2s} , \quad (399)$$

$$\text{from } K_1 = \{h_{23} + \{h_{13}m_2^2 + h_{12}m_3^2\}/\{m_2 + m_3\}^2\}/\mu_1, \quad K_2 = \{h_{12} + h_{13}\}/\mu_2, \quad L = 2\{h_{13}m_2 - h_{12}m_3\}/\{m_2 + m_3\}\sqrt{\mu_1\mu_2} , \quad (400)$$

$$A = \{K_1 + K_2\}/2, \quad B = \{K_2 - K_1\}/2, \quad C = L/2 . \quad (401)$$

I also use $\omega_r = \sqrt{K_r}$, which are frequencies, and likewise in terms of subsequent K 's below. This has a special case, for $C = 0$ corresponding to $m_2 h_{13} = m_3 h_{12}$:

$$V = K_1 n_1^2/2 + K_2 n_2^2/2 = A + B \cos 2\varphi . \quad (402)$$

Its physical meaning is that the resultant force of the second and third 'springs' points along the line joining the centre of mass of particles 2 and 3 to the position of particle 1. There is also a very special case, for $B = 0 = C$, corresponding to $K_1 = K_2$, which amounts to $m_1 h_{23} = m_2 h_{13} = m_3 h_{12}$, high-symmetry situation the various potential contributions balance out to produce the constant potential

$$V = A . \quad (403)$$

5.2.3 4-stop metroland case

Here, the HO-type potential is in general a linear combination of 6 inter-particle springs divided by I . This can be re-expressed as

$$V = \sum_{a=1}^3 \{K_a n^a/2 + L_a n^b n^c\} = A + B \cos 2\theta + C \sin^2 \theta \cos 2\phi + \sin^2 \theta \sin 2\phi + E \sin 2\theta \cos \phi + F \sin 2\theta \sin \phi$$

$$= a + b \mathcal{Y}_{2,0}(\Theta) + c \mathcal{Y}_{2,2c}(\Theta, \Phi) + d \mathcal{Y}_{2,2s}(\Theta, \Phi) + e \mathcal{Y}_{2,1c}(\Theta, \Phi) + f \mathcal{Y}_{2,1s}(\Theta, \Phi) \quad (404)$$

$$\text{for } A = \frac{1}{4} \left\{ K_3 + \frac{K_1 + K_2}{2} \right\}, \quad B = \frac{1}{4} \left\{ K_3 - \frac{K_1 + K_2}{2} \right\}, \quad C = \frac{K_1 - K_2}{4} . \quad (405)$$

For the moment, one can interpret this as corresponding to any H or K cluster. With eventual timeless records and structure formation goals in mind, I in particular follow a particular clustering – the $\{12,34\}$ one. This is via a particular permutation of h-coordinates being physically picked out by either ignoring D, E, F by following e.g. clustering $\{12,34\}$ with intra-cluster springs and a net inter-cluster spring between the centres of mass of the two clusters. Or, alternatively, via removing D, E, F by diagonalization [50]. This does however rotate the physical interpretation so that directions picked out by the potential no longer coincide with kinematically picked out directions [a 3 DD axis system or a $\{T, M^*D, M^*D\}$ axis system]. The \mathcal{Y} 's are spherical harmonics (c and s subscripts thereon standing for cosine and sine Φ -parts) and the precise form of the constants a, b, c is not required for this article. One can imagine whichever of these problems' potentials as a superposition of familiar 'orbital shaped' lumps. Though such a superposition will of course in general alter the number, size and position of peaks and valleys according to what coefficients each harmonic contribution has.

This has a special case, for $C = 0$ corresponding to $K_1 = K_2$, i.e. that cluster 1 and cluster 2 has the same 'constitution': the same Jacobi–Hooke coefficient per Jacobi cluster mass. This is a kind of 'homogeneity requirement' on the 'structure formation' in the cosmological analogy,

$$V = A + B \cos 2\theta . \quad (406)$$

There is also a very special case, for $B = 0 = C$, corresponding to $K_1 = K_2 = K_3$, for which high-symmetry situation the various potential contributions balance out to produce the constant,

$$V = A . \quad (407)$$

Additionally the $B \ll A$ perturbative regime about the very special case signifies $K_1 + K_2 \ll K_3$ so the inter-cluster spring is a lot stronger than the intra-cluster springs. This is in some ways is analogous to scalefactor dominance over inhomogeneous dynamics in cosmology. On the other hand, the $C \ll A$ regime corresponds to either or both of the conditions $K_1 + K_2 \ll K_3$, $K_1 \approx K_2$ the latter of which signifies high *contents homogeneity*. (This means that the particle clusters that make up the model universe are, among themselves, of similar constitution.) The multiplicity of forms of writing the potential above is useful to bear in mind in searching for mathematical analogues for the present problem in e.g. the Molecular Physics literature (c.f. Sec 8.1.6).

5.2.4 N-stop metroland case

For N -stop metroland pure-shape RPM, the multiple HO-type potential can be re-expressed as

$$V = \sum_{p=1}^n K_p n_p^2(\theta_{\bar{r}})/2 + \sum_{p>q} L_{pq} n_q(\theta_{\bar{r}}) n_q(\theta_{\bar{r}}) \quad (408)$$

which can likewise be re-expressed in terms of ultraspherical harmonics and admits a hierarchy of (very)^k special subcases whose significance extends the previous example's.

5.2.5 Triangleland case

The HO-type potential can here be re-expressed as

$$V = K_1 n_1^2/2 + K_2 n_2^2/2 + L \underline{n}^1 \cdot \underline{n}^2 = A + B \cos \Theta + C \sin \Theta \cos \Phi \quad (409)$$

with the A, B, C each being 1/4 of what they were for the 3-stop metroland example. The occurrence and significance of special and very special cases then follows suit (note now that one requires the special case in order to have separability).

Contrast with 4-stop metroland model is also interesting at this point – here $Y_{0,0}$ and just two of the first-order spherical harmonics arose. (As the also-quadratic $|\underline{R}_1 \times \underline{R}_2|_3$ is not a piece of the most general multi-harmonic oscillator potential between 3 particles in 2-d). The quadrilateralland counterpart of this is covered in [53].

5.2.6 Qualitative analysis for pure-shape 4-stop metroland

This is important in simplifying both this Sec’s solution study and Part II’s QM counterpart. 4-stop metroland’s special potential ($C = 0$) either having wells at both poles or an equatorial bulge that is shallower at the poles depending on the sign of B : here both of the near-polar regimes are simultaneously realisable.

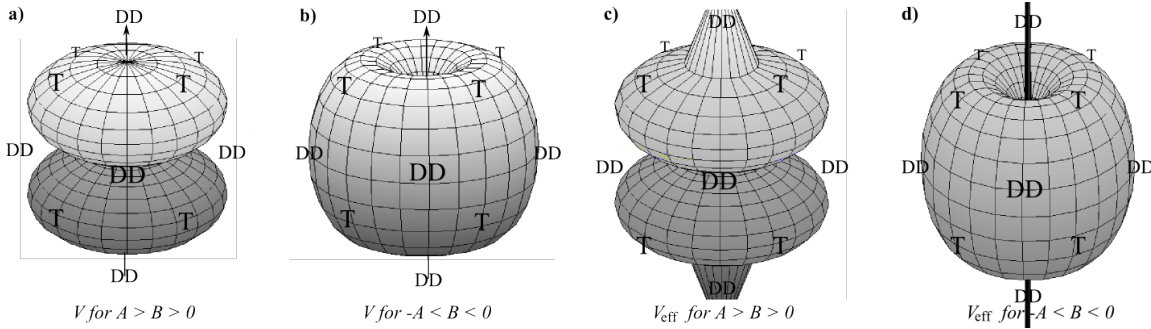


Figure 36: Sketches of V over the sphere for the mechanically significant cases a) the ‘peanut’ $A > B > 0$ and b) the ‘tyre’ $-A < B < 0$. The first has barriers at the poles and a well around the equator, while the second has wells at the pole and a barrier around the equator. In each case, considering V_{eff} for $Dil \neq 0$ adds a spike at each pole. (This now means that for $Dil \neq 0$ both islands cannot simultaneously collapse to their generally distinct centre of mass points). Finally, note that this potential is axisymmetric and reflectible about its equator so its symmetry group is $\mathbb{D}_\infty \times \mathbb{Z}_2$ (\mathbb{D} denotes dihaedral). If this is aligned with a DD axis of the physical interpretation, the overall problem retains a $\mathbb{D}_4 \times \mathbb{Z}_2$ symmetry group, of order 16.

Next, I consider small and large regimes for the special case. More precisely, these are near-North Pole and near-South Pole regimes in Θ but become large and small regimes in terms of $\mathcal{R} = \tan \frac{\Theta}{2}$. For this (including changing to the tilded PPSCT representation), and using ‘shifted energy’ $E' := E - A - B$

$$\overline{W} := \overline{E} - \overline{V} = 4E'/\{1 + \mathcal{R}^2\}^2 + 32B\mathcal{R}^2/\{1 + \mathcal{R}^2\}^4. \quad (410)$$

Then the near-North Pole regime ($\mathcal{R} \ll 1$) maps to the problem with flat polar kinetic term and

$$W = 4E' + 8\{4B - E'\}\mathcal{R}^2 \quad (411)$$

up to $O(\mathcal{R}^4)$. This has the mathematics of a 2-d isotropic harmonic oscillator,

$$W = \mathcal{E} - \Omega^2 \mathcal{R}^2/2, \quad (412)$$

provided that the ‘classical frequency’ (for us with units of I/time) $\Omega < 0$ (else it would be a constant potential problem or an upside-down harmonic oscillator problem), alongside $\mathcal{E} > 0$ to stand a chance of then meeting classical energy requirements. Writing \mathcal{E} and Ω^2 out by comparing the previous two equations, these inequalities signify that $2E > K_3$ and $2E > K_3 + 2\{K_3 - K_1\}$. The latter is more stringent if $K_3 > K_1$ (‘stronger inter-cluster binding’) and less stringent if $K_3 \leq K_1$ (‘weaker inter-cluster binding’). One can also deduce from the first of these and $K_3 \geq 0$ (spring) that $E > 0$.

Next, note that the near-South Pole regime ($\mathcal{R} \ll 1$) maps to the problem with flat polar kinetic term and

$$W = 4E'/\mathcal{R}^4 + 8\{4B - E'\}/\mathcal{R}^6 \quad (413)$$

up to $O(1/\mathcal{R}^8)$. Moreover, $\mathcal{U} = 1/\mathcal{R}$ maps the large case’s (410) to the small case’s (413), so this is also an isotropic harmonic oscillator – in (\mathcal{U}, Φ) coordinates and with the same \mathcal{E} and Ω as above. This is an exact ‘large–small’ or ‘antipodal’ duality. It halves the required solving to understand $\Theta \approx 0$ and $\approx \pi$.

Note 1) Another lesson learnt from [30, 31, 50] is that study of *second* approximations is considerably more profitable than that of first approximations.

Note 2) For the subsequent QM study, one wants the isotropic harmonic oscillator rather than cases corresponding to other values of the parameters \mathcal{E} and $-\Omega^2$. (E.g. the upside-down isotropic harmonic oscillator).

Note 3) The L_i terms [or, equivalently, D, E and F terms (404)] can be dropped in the sense that one can pass to normal coordinates for which the symmetric matrix of Jacobi–Hooke coefficients has been diagonalized [50]. Unlike in triangleland, however, this does not send one to the special case – the C -term survives and so requires addressing separately (e.g. perturbatively). The elimination thus of D, E and F terms is also subject to the mechanical interpretation of the normal-coordinate problem being more difficult algebraically than for $D = E = F = 0$. Thus it is conceivable that one might prefer to retain this simpler interpretation and treat D, E and F perturbatively.

5.2.7 Qualitative analysis for pure-shape triangleland

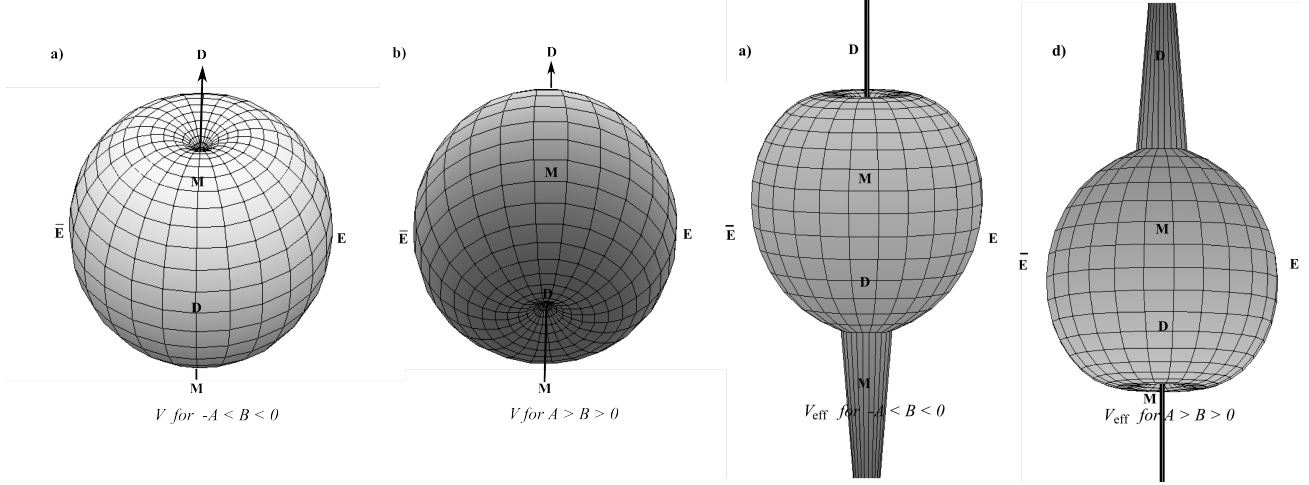


Figure 37: The heart shape of the potential for triangleland. The first has a well about the North ‘D’ pole and a barrier about the South ‘M’ pole, and the second has vice versa. In each case, considering V_{eff} for $\mathcal{S} \neq 0$ adds a spike at each pole. (This means that for $\mathcal{S} \neq 0$, the corresponding cluster cannot collapse to a point or have the third particle lie on its centre of mass.) Finally, note that this potential is axisymmetric so its symmetry group is \mathbb{D}_∞ . If this is aligned with a D–M axis of the physical interpretation as depicted, the overall problem retains a \mathbb{D}_3 symmetry group, of order 6.

For triangleland, the regime in which the (1)-clustering’s notion of special applies is $|B| \leq A$. In spherical polars, this corresponds to a heart or spheroidal shaped potential each with one end bulkier than the other. Thus one gets a well centred on either on North pole (D point) or the South pole (M point), depending on the sign of B (i.e. which of K_1 and K_2 is larger). [The $\mathcal{J} \neq 0$ case adds narrow infinite skewers to this that pass through the poles.] $K_1 > K_2$ has its well centred on the North pole, corresponding to sharp(1) triangles. The physics here is that the inter-cluster spring binding $\{23\}$ to the ‘external particle’ 1 is stronger than $\{23\}$ ’s intra-cluster spring. $K_1 < K_2$ has its well centred on the South pole, corresponding to flat(1) triangles. The physics here is that the intra-cluster spring of $\{23\}$ is more tightly binding than the inter-cluster spring between $\{23\}$ and 1. Thus only one of the near-polar regimes is actualized at once for a given problem, as, while small motions about the thick end of the potential are also admissible. These are unstable to escaping by rolling to where the potential is thinner.

In triangleland, the heart/spheroidal potential, even if inclined at an angle to the D–M axis (general case), continues to pick out a ‘small regime’ near its thin end. The physical meaning of this region does, however, vary with the angle. The potential confining $\Theta_{(\gamma)}$ to be small means that $\text{ellip}(\gamma) = \cos \gamma$ in this region, so that any value of sharpness or flatness can now be picked out. E.g. $\gamma = \pi/2$ (pure C term) picks out regular triangles (neither sharp nor flat).

Also note that the triangleland potential now furthermore breaks the tessellation group. For, the heart/spheroidal potential has symmetry group \mathbb{D}_∞ and involves an axis perpendicular to the $E\bar{E}$ axis. Thus the overall problem retains just a \mathbb{Z}_2 reflection symmetry about the plane of collinearity.

Working in spherical coordinates, set $0 = \partial V / \partial \Theta = -B \sin \Theta + C \cos \Theta \cos \Phi$, $0 = \partial V / \partial \Phi = -C \sin \Theta \sin \Phi$ to find the critical points. These are at $(\Theta, \Phi) = (\arctan(C/B), 0)$, $(-\arctan(C/B), \pi)$ which are antipodal (see Fig 38); in fact the potential is axisymmetric about the axis these lie on. The critical points are, respectively, a maximum and a minimum. [The very special case $B = C = 0$ is also critical, for all angles – this case ceases to have a preferred axis.]

A move useful for the study of the potential is to pass to (\mathcal{R}, Φ) coordinates; then the special potential is of form

$$2\bar{V} = \{K_1 \mathcal{R}^2 + K_2\} / \{1 + \mathcal{R}^2\}^3. \quad (414)$$

Now, \mathcal{R} small corresponds to Θ small, for which

$$\check{U} + \check{E} = \{\check{E} - A - B\} - C\Theta \cos \Phi + B\Theta^2/2 + O(\Theta^3), \text{ or} \quad (415)$$

$$2\{\check{U} + \check{E}\} = 2E - K_2 - L\mathcal{R} \cos \Phi - \{4E + K_1 - 3K_2\}\mathcal{R}^2 + O(\mathcal{R}^3) := u_0 - L\mathcal{R} \cos \Phi - \{u_2 \mathcal{R}^2\} + O(\mathcal{R}^3). \quad (416)$$

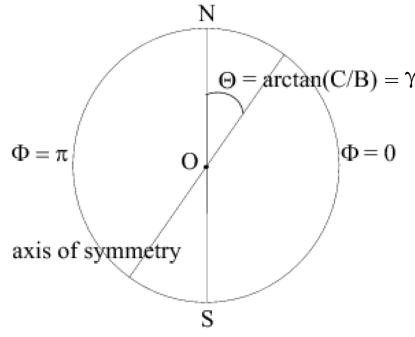


Figure 38: The preferred axis.

Thus the leading term is a constant, unless $u_0 = 2E - K_2$ ($\propto \bar{E} - A - B$) = 0. In this case, it is linear in Θ or \mathcal{R} and with a $\cos\Phi$ factor, unless also L ($\propto C$) = 0 (which is also the condition for the ‘special’ case). In this case, it is quadratic in Θ or \mathcal{R} , unless $B = 0$. (Given previous conditions, this is equivalent to $u_2 = 4E + K_1 - 3K_2 = 0$). This means that one is in the $K_2 = 2E$ subcase of the ‘very special’ case, for which $\mathbf{U} + \mathbf{E}$ has no terms at all.

\mathcal{R} large corresponds to the supplementary angle $\Xi := \pi - \Theta$ being small, so

$$\bar{\mathbf{U}} + \bar{\mathbf{E}} = \{\bar{E} - A - B\} + C\Xi \cos \Phi + B\Xi^2/2 + O(\Xi^3), \text{ or} \quad (417)$$

$$2\{\tilde{\mathbf{U}} + \tilde{\mathbf{E}}\} = \{2E - K_1\}\mathcal{R}^{-4} - L\mathcal{R}^{-5} \cos \Phi - \{4E + K_2 - 3K_1\}\mathcal{R}^{-6} + O(\mathcal{R}^{-7}) := w_0\mathcal{R}^{-4} - L\mathcal{R}^{-5} \cos \Phi - w_2\mathcal{R}^{-6} + O(\mathcal{R}^{-7}). \quad (418)$$

Thus the leading term goes as a constant in Ξ or as \mathcal{R}^{-4} , unless $w_0 = 2E - K_1$ ($\propto \bar{E} - A + B$) = 0. In this case, it goes linearly in Ξ or as \mathcal{R}^{-5} in each case also with a $\cos\Phi$ factor. This holds unless also L ($\propto C$) = 0 (‘special’ case), in which case it goes quadratically in Ξ or as \mathcal{R}^{-6} , unless $B = 0$. (Given previous conditions, this is equivalent to $w_2 = 4E + K_2 - 3K_1 = 0$). This means that one is in the $K_1 = 2E$ subcase of the ‘very special’ case, for which $\mathbf{E} + \mathbf{U}$ has no terms at all.)

Finally note that the large and small asymptotics are dual to each other. (The difference of 4 powers is accounted for by how the kinetic energy scales under the duality map). So that *one need only analyse the parameter space for one of the two regimes and then obtain everything about the other regime by simple transcription*. I.e., for triangle land, there is a duality under size inversion (swapping round what is large and what is small, or, alternatively, the antipodal map). Now, only the very special case is self-dual, the special case requiring additionally for K_1 and K_2 to be swapped around or alternatively the sign of B to be reversed.

5.2.8 Normal coordinates for triangle land

A rotation sending the general case to the special case in new coordinates is as follows. One can avoid having a C -term by using normal modes/adapted bases [30], again at the cost of making the physical interpretation more complicated. One can get to these by inserting such a rotation into the preceding solution process. I denote the new coordinates with N -subscripts (N for normal). This is done by inserting N -subscripts and, at the end of the calculation, rotating back from normal Jacobi coordinates to a more complicated form in terms of the original Jacobi coordinates.

What is the requisite rotation angle? From the matrix equation $\underline{\mathbf{R}}(\alpha, y)\underline{n} = \underline{n}_N$ (rotation through angle α about y -axis),

$$\begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} C \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ B_N \end{pmatrix}, \quad (419)$$

where \underline{n} , \underline{n}_N are unit vectors considered to be in unit-radius spherical coordinate form about the original and normal-coordinate axes, the requisite rotation is through an angle of

$$\gamma = \arctan(C/B). \quad (420)$$

This sends $B \cos \Theta + C \sin \Theta \cos \Phi$ to $B_N \cos \Theta_N$.

In addition to the axis systems described in Sec 3.11.3, one can also take $\mathbf{E}\bar{\mathbf{E}}$ as principal axis while placing the 2-axis at a general angle γ within the collinearity plane (measured without loss of generality from the (1)-axis. I denote these bases by $[\gamma]$ superscripts (the axis systems in Sec 3.11.3 are the $[\gamma = 0]$ case of these. These bases serve to align the axis system with the general HO problem’s symmetry axis (potential-adapted bases). For general B and C the harmonic oscillator-like potential acts as a well centred on some point of the equator of collinearity. Clearly the azimuthal angle is unaffected by passing to this set-up, so $\cos \Theta_{[\gamma]}$ is also $4 \times \text{area}$, while $\Phi_{[\gamma]} = \Phi_{[1]} - \gamma$. [This is measured without loss of generality from the (1)-axis.] Thus $\cos \Phi_{[\gamma]}$ and $\sin \Phi_{[\gamma]}$ are obtainable by 2-angle formulae. Finally, one could take the γ -axis as principal axis and $\mathbf{E}\bar{\mathbf{E}}$ as second axis. I denote these basis by (γ) ; this is in various ways even better-adapted to the general problem of Sec 5.2.8. Unfortunately, formulae for $\Theta_{(\gamma)}$ and $\Phi_{(\gamma)}$ are somewhat more complicated; they are derivable e.g. by spherical

trigonometry similar to that in [30]. Note that $[\gamma]$ and (γ) include the preceding three bases under the identifications $\gamma = 0 \mapsto 1$, $\gamma = 2\pi/3 \mapsto 2$ and $\gamma = 4\pi/3 \mapsto 3$.

Despite the new axis being confined to the plane of collinearity, it can nevertheless still pass through/near a range of qualitatively different points. Thus $C \neq 0$ is an even richer dynamics than $B \neq 0$ $C = 0$ was found to be in [30]. Whether this richness translates to interestingness from the perspective of dynamical systems remains to be seen. (This did turn out to some extent to be the case [621] in the minisuperspace counterpart.)

Also,

$$\sin \gamma = C/\sqrt{B^2 + C^2}, \quad \cos \gamma = B/\sqrt{B^2 + C^2}, \quad B_N = \sqrt{B^2 + C^2}. \quad (421)$$

$$\cos \Theta_N = \{\cos \gamma \cos \Theta + \sin \gamma \sin \Theta \cos \Phi\} \quad \text{and} \quad \Phi_N = \arctan(\sin \Theta \sin \Phi / \{\cos \gamma \sin \Theta \cos \Phi - \sin \gamma \cos \Theta\}). \quad (422)$$

The HO-like potential is now

$$\{K_1^N R_{1N}^2 + K_2^N R_{2N}^2\}/8I = A_N + B_N \cos \Theta_N. \quad (423)$$

It is also useful to note for later use the following coefficient interconversions:

$$A_N = A, \quad B_N = \sqrt{B^2 + C^2}, \quad (424)$$

$$K_1^N = 8\{A - \sqrt{B^2 + C^2}\}, \quad K_2^N = 8\{A + \sqrt{B^2 + C^2}\}, \quad (425)$$

$$K_1^N = K_1 + K_2 - \sqrt{\{K_1 - K_2\}^2 + 4Q^2}, \quad K_2^N = K_1 + K_2 + \sqrt{\{K_1 - K_2\}^2 + 4Q^2}. \quad (426)$$

From the spherical perspective, the normal coordinates solution has the same form as the special solution in the original coordinates. However, now one is to project onto the general tangent plane rather than the tangent plane at the North Pole. One is interpreting the general stereographic coordinate to now be the ratio of the square roots of the barycentric partial moments of inertia. This permits use of tessellation to interpret the ‘shifted’ pattern in N -variables.

5.3 Mechanics–Cosmology analogy as chooser of potentials

As regards (sums of) power-law potentials; these are motivated by being common in mechanics and by their mapping to the commonly-studied terms in the Friedmann equation of cosmology also. E.g. for scaled 3-stop metroland,

$$V = C_1|q_2 - q_3|^n + C_2|q_3 - q_1|^n + C_3|q_1 - q_2|^n \quad (427)$$

becomes

$$V = C_1\rho^n|\cos \varphi|^n + C_2\rho^n|\cos \varphi + \sqrt{3}\sin \varphi|^n + C_3\rho^n|\cos \varphi - \sqrt{3}\sin \varphi|^n. \quad (428)$$

On the other hand, for scaled triangleland, (427) becomes

$$V = C_1\rho^n|\sin \frac{\Theta}{2}|^n + C_2\rho^n|\cos \frac{\Theta}{2} - \frac{1}{2}\sin \frac{\Theta}{2}|^n + C_3\rho^n|\cos \frac{\Theta}{2} + \frac{1}{2}\sin \frac{\Theta}{2}|^n. \quad (429)$$

HO-type potential RPM models are then exactly soluble, which motivates them as highly tractable. More widely, these analogies with cosmology require that any shape factors present being slowly-varying so that one can carry out the following scale-dominates-shape approximation holding at least in some region of interest.

5.3.1 The ‘scale dominates shape’ approximation

I originally considered an action-level scale-dominates shape approximation [35], most clearly formulable as

$$||d_B \mathbf{S}||_{\mathbf{M}}/d\{\ln \rho\} \ll 1, \quad (430)$$

However, further consideration (Sec 12) reveals that this assumption is better justified if first made at the level of the equations of motion/forces, so that it involves

$$\rho S^{*2}/\rho^{**} = \epsilon_{\text{sds1}} \ll 1 \quad (431)$$

(which is encompassed under ‘S is fast’ compared to ρ), whilst $\partial V_\rho/\partial S^a$ does not in any case contribute to the shape evolution equations). Also,

$$|\partial V_{\rho S}/\partial S|/|\partial V_S/\partial S| = \epsilon_{\text{sds2}} \ll 1 \quad (432)$$

[Without the ‘scale dominates shape’ approximation, one cannot separate out the heavy (here scale) part so that it can provide the approximate timefunction with respect to which the light (here shape) part’s dynamics runs. Moreover, N.B. that isotropic cosmology itself similarly suppresses small anisotropies and inhomogeneities, so that exact solutions of this are really approximate solutions for more realistic universes too.] For the spherical presentation of triangleland, the barred, $\rho \longrightarrow \text{I}$ counterpart of this holds.

Analogy 46) GR counterparts of the scale-dominates-shape analogy are the leading-order neglect of scalar field terms in e.g. [252], of anisotropy in e.g. [9] and of inhomogeneity in e.g. [296].

Note: isotropic cosmology itself similarly suppresses small anisotropies and inhomogeneities, so that exact solutions thereof are really approximate solutions for more realistic universes too.] Take the barred, $\rho \longrightarrow \text{I}$ counterpart of all this for the \mathbb{S}^2 presentation of triangleland. Throughout, these analogies are subject to any shape factors present being slowly-varying so that one can carry out the approximation (431–432), at least in some region of interest.

5.3.2 Ordinary 1-particle Mechanics–Cosmology analogy

Next, I use the analogy between Mechanics and Cosmology to broaden the range of potentials under consideration and pinpoint ones which parallel classical and Quantum Cosmology well. This would seem to be making more profitable use of the potential freedom than previous mere use of simplicity.

Isotropic cosmology (in $c = 1$ units) has the Friedmann equation

$$H^2 := \left\{ \frac{a^*}{a} \right\}^2 = -\frac{k}{a^2} + \frac{8\pi G\epsilon}{3} + \frac{\Lambda}{3} = -\frac{k}{a^2} + \frac{2GM_{\text{dust}}}{a^3} + \frac{2GM_{\text{rad}}}{a^4} + \frac{\Lambda}{3}, \quad (433)$$

the second equality coming after use of energy–momentum conservation and assuming non-interacting matter components. Here, a is the scalefactor of the universe, $*$ = d/dt^{cosmic} (aligned with d/dt^{Newton} here). H is the Hubble quantity. k is the spatial curvature which is without loss of generality normalizable to 1, 0 or -1 . G is the gravitational constant. ϵ is matter energy density. Λ is the cosmological constant. M is the mass of that matter type that is enclosed up to the radius $a(t)$.

A fairly common analogy is then between this and

$$\left\{ \frac{r^*}{r} \right\}^2 = \frac{2E}{r^2} + \frac{K_{\text{Newton}}}{r^3} + \frac{K_{\text{Conformal}}}{r^4} + \frac{K_{\text{Hooke}}}{3} \quad (434)$$

i.e. the unit-mass ordinary mechanics energy equation)/ r^2 . Here and elsewhere in this article the various K 's are constant coefficients, and $*$ is here d/dt^{Newton} . A particularly well-known subcase of this is that 1- d mechanics with a $1/r$ Newtonian Gravity type potential is analogous to isotropic GR cosmology of dust. This extends to an analogy between the Newtonian dynamics of a large dust cloud and the GR isotropic dust cosmology [460, 451, 461, 105]. (Here, shape is least approximately negligible through its overall averaging out to approximately separated out shape and cosmology-like scale problems.) For [35] and the present article's purposes, enough of these parallels ([304, 465, 507, 529]) survive the introduction of a pressure term in the cosmological part, and the using of an N -stop metroland RPM in place of ordinary mechanics.

In considering a large number of particles, another way in which shape could be at least approximately negligible arises [105]. This is through its overall averaging out to produce a highly radial problem (in a factorization into a cosmology-like scale problem and a shape problem). In the case of dust in 3- d , the many Newtonian gravity potential terms average out to produce the effective dust, and one's equations are split into 1) the standard dust cosmological scale equation. 2) The also well-known central configuration problem for shape.⁵¹ The Newton–Hooke problem, amounting to cosmology with dust and cosmological constant, has also been studied in a somewhat similar context (see e.g. [253]). It is an interesting question to me whether the averaging out to produce a radial equation and a shape equation occurs for other power-law potentials and their superpositions. Furthermore, are any of the resulting shape problems of an already mathematically-known form?

5.3.3 N -stop metroland–Cosmology analogy

Analogy 47) The N -stop metroland–Cosmology analogy itself (all of the rest of this SSsec above applies also to the \mathbb{CP}^{N-2} presentation of N -a-gonland) is between the above Friedmann equation and (RPM energy relation)/ ρ^2 ,

$$\left\{ \frac{\dot{\rho}}{\rho} \right\}^2 = \frac{2E}{\rho^2} - \frac{\mathcal{T}_{\text{ot}}}{\rho^4} - \frac{2V(\rho, S^u)}{\rho^2} = \frac{2E}{\rho^2} + \frac{2K}{\rho^3} + \frac{2R - \mathcal{T}_{\text{ot}}}{\rho^4} - 2A, \quad (435)$$

where $*$ is now this article's usual $d/dt^{\text{em(JBB)}}$. Then,

Analogy 48) the spatial curvature term k becomes -2 times the energy E .

Analogy 49) The cosmological constant term $\Lambda/3$ becomes -2 times the net A from the (upside down) HO potentials.

Analogy 50) The dust term $2GM/a^3$'s coefficient $2GM$ becomes -2 times the net coefficient K from the Newtonian Gravity potential terms. N.B. the K of interest has a particular sign shared by dust and Newtonian Gravity (and the attractive subcase of Coulomb).

Analogy 51) The radiation term coefficient $2GM/a^4$'s coefficient $2GM$ becomes $-\mathcal{T}_{\text{ot}} + 2R$ for $2R$ the coefficient of the $V_{(0)}$ contribution from the $1/|r^{IJ}|^2$ terms. It corresponds to the conformally invariant potential term, which is quite well-studied in Classical and Quantum Mechanics.

Difference 18) Note that \mathcal{T}_{ot} itself is of the wrong sign to match up with the ordinary radiation term of cosmology. In the GR cosmology context, 'wrong sign' radiation fluid means that it still has $p = \epsilon/3$ equation of state (for p the pressure), but its density ϵ is negative. Thus it violates all energy conditions, making it unphysical in a straightforward GR cosmology context, and also having the effect of singularity theorem evasion by 'bouncing'. From the ordinary mechanics perspective, however, this is just the well-known repulsion of the centrifugal barrier preventing collapse to zero size.

Note 1) This difference in sign is underlied by an important limitation in the Mechanics–Cosmology analogy. For, in mechanics the kinetic energy is positive-definite, while in GR the kinetic energy is indefinite, with the scale part contributing negatively.

⁵¹This being well-known is, however, in a somewhat different context, i.e. the Celestial Mechanics literature considers the few-particle case (see e.g. [471] for an introduction and further references). On the other hand, the late universe cosmological interest is in the large-particle limit [105].

This does not affect most of the analogy, as follows. The energy and potential coefficients can be considered to come with the opposite sign. On the other hand, the relative sign of the shape and scale kinetic terms cannot be changed thus, and it is from this that what is cosmologically the ‘wrong sign’ arises.

Note 2) Two distinct attitudes to wrong-sign term are as follows. Firstly, one can suppress this \mathcal{T}_{Tot} term by taking toy models in which this term. (The significance of this is the relative distance momentum of two constituent subsystems is zero or small, or swamped by ‘right-sign’ $1/|r^{IJ}|^2$ contributions to the potential.) Secondly, it should also be said that more exotic geometrically complicated scenarios such as brane cosmology can possess ‘dark radiation’ including of the ‘wrong sign’ [603]. (Indeed, this can possess what appears to be energy condition violation from the 4- d spacetime perspective due to projections of higher-dimensional objects [54]). Thus such a term is not necessarily unphysical.

Difference 19) The analogy does not however have a metric interpretation or a meaningful interpretation in terms of an energy density ϵ , it is after all just a particle mechanics model.

5.3.4 Spherical presentation of triangleland – Cosmology analogy

Analogy 47') It is between the above Friedmann equation and (434)/ I^2 ,

$$\left\{ \frac{I^*}{I} \right\}^2 = \frac{2E}{I^3} - \frac{\mathcal{T}_{\text{Tot}}}{I^4} - \frac{2V(I, S^u)}{I^3} = \frac{-2A}{I^2} + \frac{2E}{I^3} + \frac{2R - \mathcal{T}_{\text{Tot}}}{I^4} - 2S. \quad (436)$$

Thus now

Analogy 48') the spatial curvature term k becomes the net HO terms' $2A$.

Analogy 49') The cosmological constant term $\Lambda/3$ becomes -2 times the surviving lead term from the $|r^{IJ}|^6$ potentials' coefficient S .

Analogy 50') The dust term $2GM/a^3$'s coefficient $2GM$ becomes $2E$ so to have an analogy with a physical kind of dust it is $E > 0$ that is of interest.

Analogy 51') just coincides with analogy 49) again.

In the spherical triangleland analogy, the Newtonian $1/|r^{IJ}|$ type potentials that one might consider to be mechanically desirable to include produce $1/I^{7/2}$ terms. These are analogous to an effective fluid with equation of state $P = \epsilon/6$ i.e. an interpolation ‘halfway between’ radiation fluid and dust.

5.3.5 The Hamiltonian Mechanics–Cosmology analogy

Analogy 47'') Because of the 3-metric nature of the variables, for e.g. isotropic minisuperspace with q scalar fields ς_A , $p_a \propto -aa^*$ and $p_{\varsigma_A} \propto a^3\varsigma^*$, giving, in Hamiltonian form,

$$\frac{1}{a^4} \left\{ p_a^2 - \frac{p_{\varsigma}^2}{a^2} \right\} \propto -\frac{k}{a^2} + \frac{2GM_{\text{dust}}}{a^3} + \frac{2GM_{\text{rad}}}{a^4} + \frac{\Lambda}{3} \quad (437)$$

which has a different correspondence from the Lagrangian one above. In this formulation,

Analogy 48'') spatial curvature corresponds to HO's,

Analogy 49'') cosmological constant to $|r^{IJ}|^4$ potentials,

Analogy 50'') dust to linear potentials and

Analogy 51'') radiation to constant potential.

Moreover, note that the case of constant plus HO potential that is straightforward to study in mechanics continues to be relevant in this analogy. This analogy is useful for quantum scale equations, though if these are treated semiclassically as is my main intention, the analogy reverts to that in the main text.

Analogy 47''' One is now to compare (436) and the momentum-containing version of (437).

Analogy 48'') spatial curvature corresponds to $|r^{IJ}|^6$ potentials,

Analogy 49'') cosmological constant to $|r^{IJ}|^4$ potentials,

Analogy 50'') dust to $|r^{IJ}|^4$ potentials and

Analogy 51'') radiation to HO's.

Overall, the pattern is that an a^c term in Friedmann's equation becomes a $c + 2$ power in the Lagrangian ρ form, a $c + 4$ power in the Hamiltonian ρ form, a $2c + 6$ power in the Lagrangian I form and a $2c + 10$ power in the Hamiltonian ρ form,

If one makes this analogy, one gets the direct analogy if $p(q - 1) = 2(k - 1)$. (This ignores disparity in sign between shape and scalar field kinetic terms. Also, p the spatial dimension of GR (usually 3), k the dimension of the RPM configuration

space and q is the number of minimally-coupled scalar fields (mcsf's). Thus e.g. 3-stop metroland is analogous to 1 mcsf in any dimension, 4-stop metroland has no such analogies for nontrivial geometrodynamics. (In the sense of $p \geq 3$ being required to have local degrees of freedom.) Whether there are further such analogies with nonminimally coupled scalar fields, anisotropies and inhomogeneities remains work in progress.

5.3.6 The invariant analogy implies another timefunction

The scale $= \rho$ Lagrangian analogy makes substantially more physical sense (especially the dust to Newtonian Gravity potential correspondence). This is to the extent that I shall suggest one reinterprets the analogy in such a way that this specific copy of it remains invariant, if one wishes to maximally parallel quantum cosmology.

Analogy 52) One can redefine the time such that the Lagrangian Mechanics-Cosmology and Hamiltonian Mechanics-Cosmology analogies coincide. This involves other than the $t^{\text{em(JBB)}} \longleftrightarrow t^{\text{cosmic}}$ identification.

5.3.7 Stability of the approximation and sketches of unapproximated potentials

$HO/\Lambda < 0$ problems include as a subcase finite-minimum wells about the poles of the 4-stop metroland configuration space sphere ($C = 0$ case) [30, 50, 34]. Adding a \mathcal{T}_{ot} effective potential term adds spokes to the wells. These models are exactly soluble. Positive power potentials are still finite-minimum wells, but cease to be exactly soluble. For Newtonian Gravity/dust models (or negative power potentials more generally), near the corresponding lines of double collision, the potential has abysses or infinite peaks. Thus the small-shape approximation is definitely not valid there, and so some assumptions behind the Semiclassical Approach fail in the region around these lines. Thus for negative powers of relative separations the heavy approximation only makes sense in certain wedges of angle. There is then the possibility that dynamics set up to originally run in such regions falls out from them: a stability analysis is required to ascertain whether semiclassicality is stable. Fig 39 illustrates this for single and triple potential terms. This can be interpreted as a tension between the procedure used in the Semiclassical Approach and the example of trying to approximate a 3-body problem by a 2-body one.

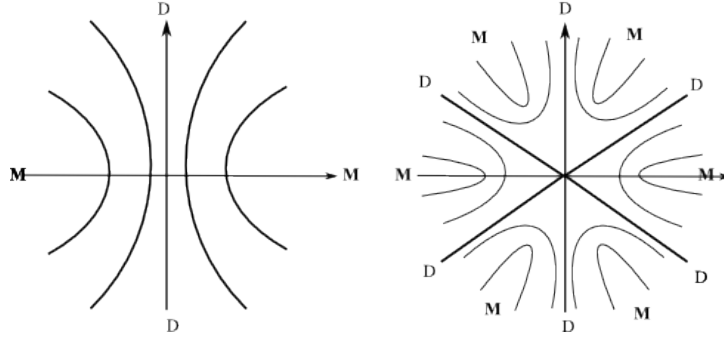


Figure 39: Contours for single and triple negative power potentials. These have abysses along the corresponding double collision lines and high ground in between these (for negative-power coefficients such as for the attractive Newtonian Gravity potential).

Moreover, at least in the simple minisuperspace quantum cosmologies I have looked at so far, it is potential powers that are common, and these then do not possess the abysses and instabilities of the $1/|r^{IJ}|$ potentials.) This furnishes is one further reason why I mostly concentrate on positive power models in the current article. E.g. the Cosmology-spherical presentation of triangle-land analogy cause this article's cosmologically-inspired set of potentials to all be such positive powers.

5.4 Classical solutions for pure-shape 4-stop metroland

Classical study of solution includes finding the shapes of the orbits (which is a relational and timeless concern) as well as consideration of traversal times (which I treat partly qualitatively and partly in analogy with cosmology).

5.4.1 Classical solutions for $\mathcal{D}il = 0$

Here, $0 = \mathcal{D}il = \sin^2 \theta \phi^*$. Thus, either $\sin \theta = 0$, so one is stuck on a pole (the DD's corresponding to the H-coordinates with respect to which θ is defined, taken to be $\{12, 34\}$ and $\{34, 12\}$), or, ϕ is constant, so that one is stuck on a meridian. This situation corresponds to losing a dilational momentum exchange freedom, which renders this SSSec's solutions of somewhat limited interest. How far one goes in the θ direction, and whether the motion turns around, is determined by considering the other equation of motion (see below for examples).

Some significant examples of such meridians include 1) the one in the direction of the 1-axis that corresponds to cluster $\{34\}$ always being collapsed while cluster $\{12\}$ varies in size. As well as the two DD's, this passes through the $\{134\}$ and $\{234\}$ T's. 2) The $\{12\} \longleftrightarrow \{34\}$ of this going in the direction of the 2-axis. 3, 4) Going in the direction of a $n_y = \pm n_x$ lines, that correspond to the clusters always being of the same size (contents homogeneity) but with that size varying from

zero (at the $\{12,34\}$ and $\{34,12\}$ DD collisions) to maximal. [Here, the two clusters are superposed into the $\{13,24\}$ or $\{14,23\}$ DD collisions]. [This SSsec's results, and those of SSsec 5.5.1, clearly hold for any special potential rather than just a multi-HO-type one. There being no 'centrifugal' barrier term, it has the simpler mathematics of motion in a 1- d potential.]

5.4.2 Classical solution in the very special case: the geodesic solution

For $\mathcal{D}il \neq 0$ in general, the very special case is solved by the geodesics on the shape space sphere,

$$\cos(\phi - \phi_0) = \kappa \cot \theta \quad (438)$$

for $\kappa = \mathcal{D}il / \sqrt{2\{E - A\} - \mathcal{D}il^2}$ and ϕ_0 constants. Then in terms of the n^i (or RelSize variables),

$$\kappa n_z = n_x \cos \phi_0 + n_y \sin \phi_0 , \quad (439)$$

i.e. restriction to a plane through the origin, with arbitrary normal $(\cos \phi_0, \sin \phi_0, -\kappa)$. Since also $\sum_{i=1}^3 n^{i2} = 1$, one is restricted to the intersection of the sphere and the arbitrary plane through its centre, i.e. another well-known presentation of the great circles as circles within \mathbb{R}^3 .

The disc in the equatorial plane is useful in considering the mechanics of the problem with clusters $\{12\}$ and $\{34\}$ picked out by the choice of Jacobi H-coordinates. Eliminating n_z projects an ellipse onto this disc,

$$\kappa^2 = \{\kappa^2 + \cos^2 \phi_0\} n_x^2 + 2 \cos \phi_0 \sin \phi_0 n_x n_y + \{\kappa^2 + \sin^2 \phi_0\} n_y^2 , \quad (440)$$

centred on the origin with its principal axes in general not aligned with the coordinates. E.g. for $\phi_0 = 0$, the ellipse is

$$1 = \left\{ \text{RelSize}(1,2) / \{1 + \kappa^{-2}\}^{-1/2} \right\}^2 + \text{RelSize}(3,4)^2 , \quad (441)$$

which has major axis in the $\text{RelSize}(3,4) = n_y$ direction and minor axis in the $\text{RelSize}(1,2) = n_x$ direction, while the value of $\text{RelSize}(12,34) = n_z$ around the actual curve can then be read off (439) to be $n_z = n_x / \kappa$. With reference to the tessellation of 4-stop metroland, as $\mathcal{D}il \rightarrow \infty$, the dynamical trajectory is the equator, corresponding to maximally-merged configurations including four DD collisions. For $\mathcal{D}il$ small, the motion approximately goes up and down a meridian. E.g. this forms a basic unit of a narrow cycle from the polar DD to slightly around the T on the Greenwich meridian (reflections of) which is repeated various times to form the whole trajectory. [The actual limiting on-axis motion $\mathcal{D}il = 0$ gives back the meridians considered in the preceding section; the motion now goes round and round the entirety of whichever of these meridians.]

Other ϕ_0 straightforwardly correspond to rotated ellipses. However the mechanical meaning of these differs. E.g. about $\pi/2$ clusters $\{12\}$ and $\{34\}$ are interchanged, while about $\pm\pi/4$ also has distinct sharp physical significance. Throughout, note the periodicity of the motion (already clear in the spherical model as the great circles are closed curves). The tessellation lines are great circles, projecting to the disc rim, the axes, the lines at $\pi/4$ to the axes and ellipses with principal directions aligned with the preceding.

5.4.3 Approximate solutions for the special case of HO-like problem

In the first approximation for stereographic radius \mathcal{R} small⁵² gives, by integrating the \mathcal{R} quadrature (1034) the orbits

$$\pm \sqrt{2u} \sec(\phi - \bar{\phi}) = \mathcal{R} = \tan \frac{\theta}{2} = n_1 / n_2 . \quad (442)$$

I cast the answer in terms of straightforward relational variables, and using the $\mathcal{D}il$ -absorbing constant $f_0 = 2\mathcal{E}/\mathcal{D}il^2$, $f_2 = \Omega^2/\mathcal{D}il^2$ and $g = \sqrt{f_0^2 - f_2}$. However, the \mathcal{R} form of the orbits are parallel straight lines (vertical for $\phi_0 = 0, \pi$). Such are known not to be very good approximands in that they totally neglect the non-constant part of the potential (i.e. deflections due to forces) and thus are precisely rectilinear motions. This argument for the use of the second approximation pervades this article.

In the second approximation,

$$\sqrt{f_0 + g \cos(2\{\phi - \phi_0\})} = 1/\mathcal{R} = \cot \frac{\theta}{2} . \quad (443)$$

In terms of \mathcal{R} , this is straightforwardly rearrangeable into quite a standard form (e.g [482, 636])

$$\mathcal{R}^2 = 1 / \{u_0 + \sqrt{u_0 - v} \cos(2\{\Phi - \Phi_0\})\} , \quad (444)$$

the case-by-case analysis of which is provided in Fig 40. Casting into a well-known form (e.g. [482]):

$$\mathcal{R}^2 = 1 / \{\alpha + \beta \cos^2(\phi - \phi_0)\} , \quad (445)$$

where α and β are also constants. These describe closed curves around the DD collision the potential is centred about. The cases of interest are then ellipses (including the bounding case of circles but excluding the other bounded case of pairs of

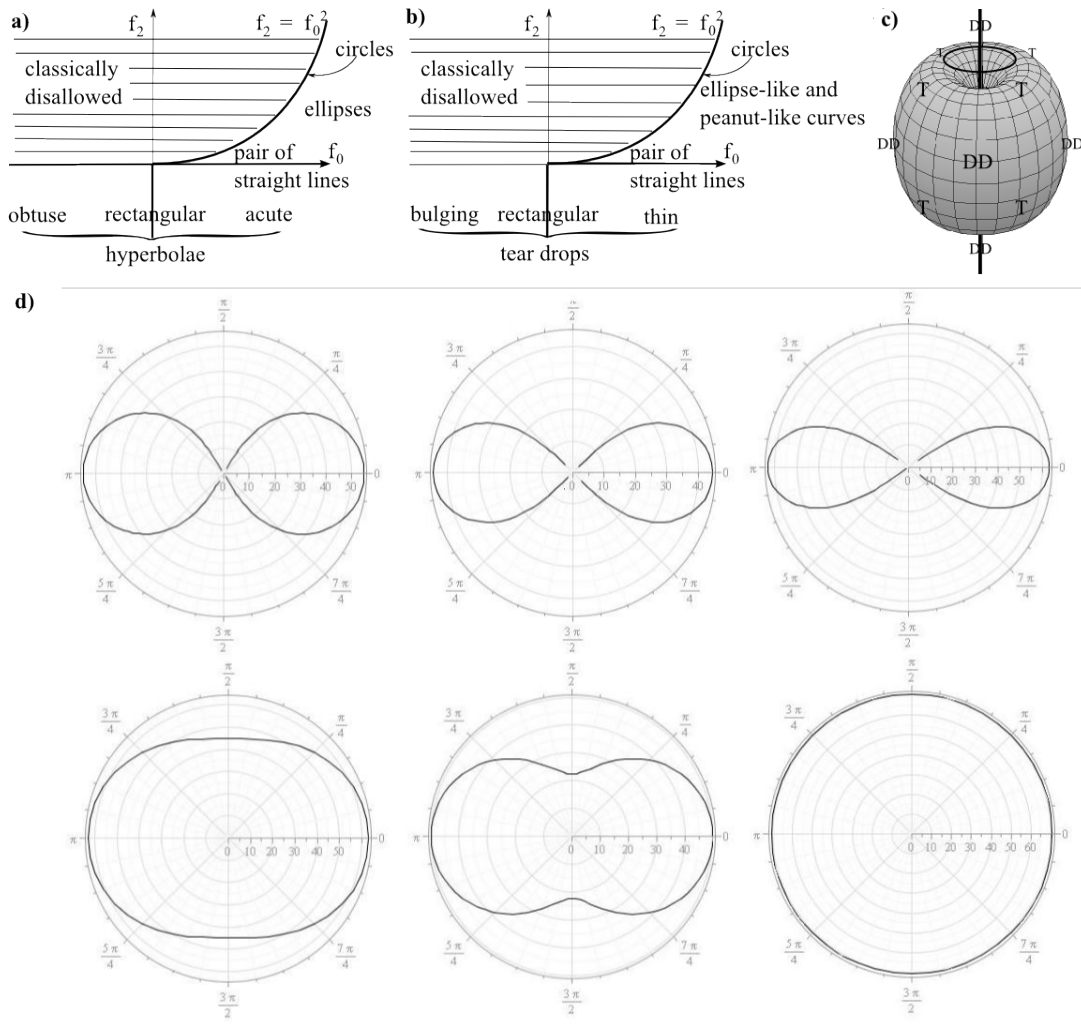


Figure 40: a) For the small approximation in (\mathcal{R}, ϕ) coordinates, the desired parameter space is the indicated wedge populated by ellipses. On the bounding parabola, I get circles, coinciding with the edge of the n_x, n_y disc for $f_0 = 1 = f_2$ and becoming smaller in either direction. This is standard isotropic HO mathematics. b) The large approximation in (\mathcal{R}, ϕ) coordinates involves somewhat more unusual curves, and yet same breakdown of parameter space as for a). This is explained by it taking the same form as a) in (\mathcal{U}, ϕ) coordinates by the size duality/antipodal map. c) The relevant small behaviour are ellipses (or pieces of these or other of the curves that then exit the region where the approximation is appropriate). I illustrate how this is qualitatively clear by providing the form of the potential as a backdrop. As regards RPM interpretation, these are curves for which DD is small. d) I provide the forms of the second large approximation's curves, since these often illustrate the large behaviour of pure-shape RPM solutions (thus ignore inner parts of curves where this does not apply well). The scale of the figure is too small to distinguish between all the DD's and T's (bar the South Pole DD, which is at infinity), these all lying piled up on/very near the origin in the figures.

straight lines, as in). [N.B. this is isotropic HO mathematics rather than Kepler problem mathematics, so that the ellipses are centred on the origin rather than having the origin at one focus.]

N.B. there is now a third inequality on \mathcal{E}, Ω :

$$4\mathcal{E}^2 \geq \mathcal{D}\text{il}^2\Omega^2, \quad (446)$$

that replaces $\mathcal{E} > 0$ as it is more stringent. Thus in terms of \mathbf{E} and the K_i I get the allowed wedge of parameter space to be

$$2\{2\mathbf{E} - K_3\}^2/\mathcal{D}\text{il}^2 \geq \{2\mathbf{E} - K_3\} + 2\{K_1 - K_3\} > 0. \quad (447)$$

Saturation of this corresponds to circular trajectories. For such circles to exist, the discriminant gives the condition $\{\mathcal{D}\text{il}/4\}^2 \geq K_3 - K_1$, so that the relative dilational quantity is bounded from below by the amount by which the inter-cluster spring dominates.

Then by elementary trigonometry and (259), (443) becomes

$$\sqrt{f_0 + g \frac{\{n_x^2 - n_y^2\} \cos 2\phi_0 + 2n_x n_y \sin 2\phi_0}{n_x^2 + n_y^2}} = \sqrt{\frac{1 + n_z}{1 - n_z}}. \quad (448)$$

⁵²Not all dynamical orbits enter such a regime – sometimes the quadrature's integral goes complex before the small \mathcal{R} regime is attained (\mathcal{R} large 'classically forbidden'), for $\tilde{\mathcal{U}}(\mathcal{R})\mathcal{R}^2 - \mathcal{D}\text{il}^2 < 0$.

So, solving for n_z and applying the on-sphere condition for $\phi_0 = 0$, say, gives

$$\sqrt{1 - n_x^2 - n_y^2} = n_z = \frac{\{f_0 + g - 1\}n_x^2 + \{f_0 - g - 1\}n_y^2}{\{f_0 + g + 1\}n_x^2 + \{f_0 - g + 1\}n_y^2}. \quad (449)$$

Then one can write down a curve in terms of two independent variables such as RelSize(1,2) and RelSize(3,4). Either RelSize(1,2) + RelSize(3,4) = 0 (so both are 0 because they are positive and so both clusters have collapsed) or

$$\{\{f_0 + g + 1\}\text{RelSize}(1,2)^2 + \{f_0 - g + 1\}\text{RelSize}(3,4)^2\}^2 = 4\{\{f_0 + g\}\text{RelSize}(1,2)^2 + \{f_0 - g\}\text{RelSize}(3,4)^2\}. \quad (450)$$

The large regime then has

$$\sqrt{f_0 + g \cos(2\{\phi - \phi_0\})} = 1/\mathcal{W} = \mathcal{R} = \tan \frac{\theta}{2}, \quad (451)$$

which is, for the cases of interest, an ellipse-like or peanut-like curve in the $\{\mathcal{R}, \Phi\}$ plane [30]. Applying the interpretation in terms of n^i or RelSize variables, the *same* answer as for the small regime arises again. This conclusion just reflects the potential imposed possessing the large–small duality/antipodal symmetry, which physically translates to shapes and their mirror images behaving in the same fashion.

5.4.4 Discussion of more general HO-like problems

The L_i terms (or, equivalently, D, E and F terms) can be dropped in the sense that one can pass to normal coordinates for which the symmetric matrix of Jacobi–Hooke coefficients has been diagonalized. Note that, unlike in the below triangleland counterpart, this does not send one to the special case – the C-term survives and so requires addressing separately (e.g. perturbatively). The elimination thus of D, E and F terms is also subject to the mechanical interpretation of the normal-coordinate problem being more difficult algebraically than for D = E = F = 0. Thus it is conceivable that one might prefer to retain this simpler interpretation and treat D, E and F perturbatively.

5.5 Pure-shape triangleland

5.5.1 Classical solutions for $\mathcal{J} = 0$

$0 = \mathcal{J} = \sin^2 \Theta \Phi^*$. Thus, either $\sin \Theta = 0$, so one is stuck on a pole (the D or M corresponding to the clustering with respect to which Θ is defined, taken to be $\{1, 23\}$), or, Φ is constant, so that one is stuck on one of the corresponding meridians. These correspond to losing an angular momentum exchange freedom, which renders this subsection’s solutions to be of somewhat limited interest. Again, how far one goes in the Θ direction, and whether the motion turns around, is determined by considering the other equation of motion for Θ (in analogy to the above examples).

These meridians include 1) the equator of collinearity of Fig 19 that goes through all 3 D’s and all 3 M’s. 2) The one in the direction of the area axis, that goes through the equilateral configurations E, \bar{E} as well as the $\{1,23\}$ cluster’s M and D.

5.5.2 Exact solution for the ‘very special case’ of HO

For $\mathcal{J} \neq 0$ in general, the very special case is solved by the geodesics on the shape space,

$$\kappa^{-1} \cos(\Phi - \Phi_0) = 2 \cot \Theta = \{1 - \mathcal{R}^2\}/2\mathcal{R} = \{n_2^2 - n_1^2\}/n_1 n_2 \quad (452)$$

(following through [30] with variable transformations so as to cast it in terms of straightforward relational variables, and using $\mathcal{J}\kappa^{-1} = 2\sqrt{2\{\bar{E} - A\} - \mathcal{J}^2} = \sqrt{2E - K_2 - 4\mathcal{J}^2} = \sqrt{u_0 - 4\mathcal{J}^2}$). κ and Φ_0 are constants. Further note that the \mathcal{R} form above can be rearranged to the equation of a generally off-centre circle in the stereographic plane. Also, in terms of shape quantities, this solution can be written as

$$\{\text{aniso} \cos \Phi_0 + 4 \times \text{area} \sin \Phi_0\} = 2\kappa \text{ellip} \quad (453)$$

with the $\cos \Phi_0 = 0$ or $\sin \Phi_0 = 0$ cases obviously being simpler.

Of particular significance among these are the various great circles visible in the tessellation of triangleland figure. As well as the abovementioned equator of collinearity, these now include A) all three of the meridians of isoscelesness that go through a particular cluster’s D and M as well as the two equilateral configurations, E and \bar{E} . B) All three of the the meridians of regularity that go through the two E’s and are constituted of configurations with $I_1 = I_2$ that cycle round all possible values of the relative angle Φ .

5.5.3 Small relative scale asymptotic behaviour

This is the $\theta \longrightarrow \Theta$, $\phi \longrightarrow \Phi$, $\text{Dil} \longrightarrow \mathcal{S}$ of Sec 5.4.3. Thus the first approximation is

$$\sec(\Phi - \Phi_c) = n_1/n_2, \quad (454)$$

but, likewise, this is not a sufficiently good approximation for almost any purposes). Then, for the second approximation,

$$\pm 1/\sqrt{u_0 + v \cos(2\{\Phi - \Phi_c\})} = \mathcal{R} = \tan \frac{\Theta}{2} = n_1/n_2 \quad (455)$$

(following through with variable transformations so as to cast it in terms of straightforward relational variables, and using the \mathcal{J} -absorbing constant $u_2 = u_2/\mathcal{J}^2$) and $v = \sqrt{u_0^2 - u_2}$. Thus they are ellipses centred about the equilateral configuration at the pole, E.

In shape quantities, this gives

$$\text{ellip} \pm \sqrt{4 - 3 \text{ellip}^2} = \pm \sqrt{u + v \left\{ \frac{\text{aniso}^2 - 4 \times \text{area}^2}{4} \cos(2\Phi_c) + \frac{\text{aniso} \ 4 \times \text{area}}{2} \sin(2\Phi_c) \right\}}. \quad (456)$$

In the special case, now there is a well trapping paths around e.g. the (1)-clustering's D or M – a (1)-sharp triangle region or a (1)-flat triangle region. Then [dropping (1) labels] for $\mathcal{J} \neq 0$, these have Φ go round and round, so all regular and all isosceles configurations are periodically attained for small trajectories. In the small approximation, to second order, one gets, as the case most relevant to subsequent QM work, the isotropic harmonic oscillator in (\mathcal{R}, Φ) variables. This is solved by ellipses centred about the origin [30]. Now one only has duality up to $K_1 \longleftrightarrow K_2$, meaning that the coefficients are different in each case.

5.5.4 Large relative scale asymptotic behaviour

For analogous notions of first and second *large* approximations, now the quadrature in $\mathcal{U} = 1/\mathcal{R}$ takes the same form as the \mathcal{R} -quadrature in the above workings with

$$u_0 \longrightarrow w_0, \quad v \longrightarrow x \quad (457)$$

(w_0 , w_2 and x below are then the obvious analogues of μ_0 , μ_2 and v). Hence the solutions are dual to those of Sec 5.5.3. Thus all of Sec 5.5.3's results apply again under the duality substitutions. (The subsequently-induced language changes are as follows. 'small' \longrightarrow 'large'. "2, 1, 3 collinearity with particle 1 at the centre of mass of particles 2, 3" \longrightarrow "collision between particles 2, 3, which is also interpretable as particle 1 escaping to infinity").

The first approximation is then

$$\pm \sqrt{2w} \cos(\Phi - \Phi_0) = \mathcal{R} = \tan \frac{\Theta}{2} = n_1/n_2. \quad (458)$$

In the (\mathcal{R}, Φ) plane and for $\Phi_0 = 0$, this takes the form of a family of circles of radius $\sqrt{w/2}$ and centre $(\sqrt{w/2}, 0)$, so that they are all tangent to the vertical axis through the origin [24]. 2) The second approximation gives

$$\pm \sqrt{w + \sqrt{w^2 - x} \cos(2\{\Phi - \bar{\Phi}\})} = \mathcal{R} = \tan \frac{\Theta}{2} = n_1/n_2. \quad (459)$$

In the \mathcal{R}, Φ plane and for $\Phi_0 = 0$, this takes the forms in Fig 40d) in the parameter regions of Fig 40b).

Note: for general $\tilde{w}_{(n,0)}$ with constant of proportionality Λ_n , one gets exactly the same large-asymptotics analysis as here, with $u_0 = 2E - \Lambda_n$, $u_2 = 4E - \{4 + n\}\Lambda_n$. Thus generally this duality to the isotropic HO of the universal large-scale asymptotics of pure-shape triangle land is a useful and important result for this Machian mechanics. This usefulness is due to the classical and quantum mechanics of isotropic HO's being rather well-studied and thus a ready source of classical and quantum methods, results, and insights.

The second large approximation's solutions are likewise [by the duality] ellipses but in the corresponding (\mathcal{W}, Φ) -chart. These ellipses map to more unusual closed curves (given in [30]) in the (\mathcal{R}, Φ) -chart, and correspond to a sequence of very flat triangles.

5.5.5 Comments on second small and large asymptotics

1) The second small approximation allows for turn-around behaviour rather than having to complete the first small approximation's circles: approximate circular arc, turn-around, approximate circular arc in opposite direction to the original. This is similar to the $C = 0$ approximation having turn-around ellipses rather than having to follow straight lines,

2) One can now have now asymmetric bulges where one had symmetric ones before. (E.g. in the first large approximation or in the peanut case of the second large approximation.) Indeed one bulge can non-generically become infinitely big, like the straight line in the first large approximation). It can even form another contribution 'beyond infinity' which shows up on the other side of the opposite bulge.

5.6 Scaled 3-stop metroland with free and HO potentials

The $\mathcal{D}il = 0$ orbit shape is always a radial half-line. Some of these are physically/shape-theoretically special by being along an M or D direction. The general such line corresponds to ρ_1 and ρ_2 being in a fixed proportion, so that the shape is fixed and one has a dynamics of pure scale. The traversal time is

$$\rho = \sqrt{2E} \{t^{\text{em(JBB)}} - t_c^{\text{em(JBB)}}\} . \quad (460)$$

With $\mathcal{D}il \neq 0$, the traversal time is

$$\rho = \sqrt{2E} \sqrt{\{t^{\text{em(JBB)}} - t_c^{\text{em(JBB)}}\}^2 + \mathcal{D}il^2} , \quad (461)$$

and the shapes of the orbits are, for φ_c constant,

$$\rho = \mathcal{D}il \sqrt{4E^2 \tan^2(2E\{\varphi - \varphi_c\}) + 1/\sqrt{E}} . \quad (462)$$

In the special HO case,

$$t^{\text{em(JBB)}} - t_{c(\bar{r})}^{\text{em(JBB)}} = \frac{1}{\sqrt{K_{\bar{r}}}} \arcsin \left(\sqrt{\frac{K_{\bar{r}}}{2E_{\bar{r}}}} \rho_{\bar{r}} \right) , \quad (463)$$

and the shapes of the orbits are

$$\frac{1}{K_1} \arcsin \left(\sqrt{\frac{K_1}{2E_1}} \rho_1 \right) = \frac{1}{K_2} \arcsin \left(\sqrt{\frac{K_2}{2E_2}} \rho_2 \right) + \text{const} , \quad (464)$$

the constant arising from the two $c(\bar{t})$'s not being the same in general.

In the isotropic HO case, one can also straightforwardly solve in scale-shape coordinates. If $\mathcal{D}il = 0$,

$$\rho = \sqrt{E/A} \sin \left(A \{t^{\text{em(JBB)}} - t_c^{\text{em(JBB)}}\}/2 \right) . \quad (465)$$

If $\mathcal{D}il \neq 0$,

$$t^{\text{em(JBB)}} - t_c^{\text{em(JBB)}} = \frac{1}{\sqrt{2A}} \arcsin \left(\frac{A}{E^2/4A - \mathcal{D}il^2} \{\rho^2 - E/2A\} \right) , \quad (466)$$

and then the shapes of the orbits are

$$\varphi - \varphi_0 = \text{artanh} \left(\frac{k - \sqrt{l - m\rho^2}}{m\rho^2 - 1} + \bar{k} \right) \quad (467)$$

for some constants k, \bar{k}, l, m .

5.7 Exact solutions for scaled triangleland

The free problem in Dragt coordinates gives

$$\text{Dra}^\Gamma = A^\Gamma t^{\text{em(JBB)}} + B^\Gamma \quad (468)$$

for A^Γ, B^Γ constants. Then

$$\{\text{Dra}_1 - B_1\}/A_1 = \{\text{Dra}_2 - B_2\}/A_2 = \{\text{Dra}_3 - B_3\}/A_3 . \quad (469)$$

This method is of limited use upon the introduction of potential terms however, since the usual potentials tend to become complicated nonseparable combinations of the Dragt coordinates.

5.7.1 Simple exact solutions for triangleland with Φ -free potentials

Each of the free-free, attractive Newton-Coulomb-free, HO-free and the aforementioned special multiple HO setting problems separate into single-variable problems.

In the case of zero relative angular momentum, the motion is linear (and indeed equivalent to the 1- d problem at the classical level. The free-free problem's solution is

$$\rho_2 = \text{const } \rho_1 + \text{Const} , \quad \Phi \text{ fixed} . \quad (470)$$

The HO-free problem's solution is

$$\rho_1 = \sqrt{\frac{E_1}{K_1}} \sinh \left(\sqrt{\frac{K_1}{E_2 + K_2}} \sqrt{\mu_2} \{\rho_2 - \text{const}\} \right) , \quad \Phi \text{ fixed} . \quad (471)$$

5.7.2 Very special case

Here,

$$I^{*2}/2 + \mathcal{J}^2/2I^2 \sin^2 \Theta_0 + A = E/I \quad (472)$$

[usually one would set $\Theta_0 = \pi/2$ without loss of generality, however the present physical interpretation has the value of Θ_0 be meaningful, as $\Theta_0 = 2\arctan(\rho_1/\rho_2)$]. Thus the solutions are conic sections

$$I = l/\{1 + e \cos(\Phi - \Phi_0)\} \quad (473)$$

where the semi-latus rectum and the eccentricity are given by

$$l = \mathcal{J}^2/\check{E} \sin^2 \Theta_0, \quad e = \sqrt{1 - 2A \mathcal{J}^2/\check{E}^2 \sin^2 \Theta_0}. \quad (474)$$

So, in terms of straightforward relational variables,

$$I_1 + I_2 = l/\{1 + e \cos(\Phi - \Phi_0)\}. \quad (475)$$

In moment of inertia–relative angle space, for $2A = \{\check{E} \sin \Theta_0/\mathcal{J}\}^2$ one has circles. For $0 < 2A < \{\check{E} \sin \Theta_0/\mathcal{J}\}^2$ one has ellipses. For $A = 0$ one has parabolae (corresponding to the case with no springs). The hyperbolic solutions ($A < 0$) are not physically relevant here because this could only be attained with negative Hooke's coefficient springs. The circle's radius is $I = l = \check{E}/2A$ while for the ellipses I is bounded to lie between $\mathcal{J}/\sqrt{2A} \sin \Theta_0$ and $\check{E}/2A$. The smallest I attained in the parabolic case is $\mathcal{J}^2/2\check{E} \sin^2 \Theta_0$. The period of motion for the circular and elliptic cases is $\pi\check{E}/\sqrt{2A^3}$.

As regards the individual subsystems, combining the fixed plane equation and the $I = I(\Phi)$ relation,

$$I_1 = l \sin^2 \frac{\Theta}{2} / \{1 + e \cos(\Phi - \Phi_0)\}, \quad I_2 = l \cos^2 \frac{\Theta}{2} / \{1 + e \cos(\Phi - \Phi_0)\} \quad (476)$$

so each of these behave individually similarly to the total I . In the $\Theta_0 = \pi/2$ plane, they are both equal (and so equal to $I/2$). The circle and ellipse cases have I_1 and I_2 as closed bounded curves which sit inside the curve that I traces. The parabolic case has I_1, I_2 curves to the 'inside' of the parabola that I traces.

5.7.3 Special case

This is solved by

$$\bar{t}^{\text{em(JBB)}} - \bar{t}_c^{\text{em(JBB)}} = \{2/\sqrt{K_i}\} \arccos \left(\{I_i K_i - E_i\} / \sqrt{E_i^2 - K_i \mathcal{J}^2} \right) \quad (477)$$

(in agreement with [24], once differences in convention are taken into account). Thus, synchronizing, one part of the equation for the orbits is

$$\sqrt{K_2} \arccos \left(\{I_1 K_1 - E_1\} / \sqrt{E_1^2 - K_1 \mathcal{J}^2} \right) = \sqrt{K_1} \arccos \left(\{I_2^2 K_2 - E_2\} / \sqrt{E_2^2 - K_2 \mathcal{J}^2} \right). \quad (478)$$

[One can see how the arccosines cancel in the very special case... Then $E_1 = E_2 = E/2$ gives $\rho_1 = \rho_2$ i.e. $\Theta = 2\arctan(\rho_1/\rho_2) = 2$, $\arctan(1) = \pi/2$, so are confined to the plane perpendicular to the chosen Z-axis.] Then the Φ evolution equation implies

$$\Phi - \Phi_c = \mathcal{J} \int d\bar{t} \left\{ \frac{1}{I_1} + \frac{1}{I_2} \right\} = \frac{\mathcal{J}}{2} \sum_{i=1}^2 \sqrt{K_i} \int \frac{d\tau_i}{F_i \cos \tau_i + E_i} = \sum_{i=1}^2 \arctan \left(\sqrt{\frac{\{E_i - F_i\}\{F_i - A_i\}}{\{E_i + F_i\}\{F_i + A_i\}}} \right)$$

(for $\tau_i = 2\sqrt{K_i}\{t - t_0\}$ and $F_i := \sqrt{E_i^2 - K_i \mathcal{J}^2}$, $A_i = K_i I_i - E_i$), which simplifies to

$$\Phi - \Phi_c = \sum_{i=1}^2 \arctan \left(\sqrt{\frac{\left\{ \left\{ \sqrt{E_i^2 - K_i \mathcal{J}^2} - E_i \right\} I_i + \mathcal{J}^2 \right\}}{\left\{ \left\{ \sqrt{E_i^2 - K_i \mathcal{J}^2} + E_i \right\} I_i - \mathcal{J}^2 \right\}}} \right) \quad (479)$$

in the straightforward relational variables.

5.7.4 Separate working required for single HO case

For $K_1 = K_2 = 0$, the trajectories are given by, after some manipulation,

$$\sqrt{E_2/E_1} I_2 = I_1 = \sec(E_1\{\Phi - \Phi_0\}/E)/\sqrt{2E_1}, \quad (480)$$

which is obviously the expected straight line in the absence of forces.

For $K_2 = 0$, $K_1 \neq 0$, the trajectories are given by, in straightforward relational variables,

$$\{I_1 K_1 - E_1\} / \sqrt{E_1^2 - K_1 \mathcal{J}^2} = \cos \left(\sqrt{2K_1/E_2} \sqrt{2E_2 I_2 - \mathcal{J}^2} \right) \quad (481)$$

and

$$\Phi - \Phi_0 = \arctan \left(\sqrt{\left\{ \left\{ \sqrt{E_1^2 - K_1 \mathcal{J}^2} - E_1 \right\} I_1 + \mathcal{J}^2 \right\} / \left\{ \left\{ \sqrt{E_1^2 - K_1 \mathcal{J}^2} + E_1 \right\} I_1 - \mathcal{J}^2 \right\}} \right) + \arctan \left(\sqrt{\frac{2E_2 I_2^2}{\mathcal{J}^2} - 1} \right). \quad (482)$$

5.7.5 A brief interpretation of the previous two SSsecs' examples

In Sec 5.7.4's example, ρ_2 (or I_2) makes a good time-standard as the absolute space intuition of it 'moving in a straight line' survives well enough to confer monotonicity. It is convenient then to rewrite (481, 482) as a curve in parametric form with I_2 playing the role of parameter.

In Sec 5.7.3's example, ρ_1 and ρ_2 oscillate boundedly, so neither of these (or I_1 or I_2) is globally a good clock parameter from the point of view of monotonicity. There is again some scope for variation in relative angle Φ , including 'sporadic' amplitude variations.

5.7.6 General HO counterpart for triangleland

The working of Sec 5.2.8 holds again (using now x , x_N in place of n and n_N) but radii are unaffected by rotations and so cancel out giving the same rotation and Θ , Φ to Θ_N , Φ_N coordinate change as before. For scaled RPM, one can uplift from the preceding parts of Sec 6 by inserting the above extra steps into the triangleland case of Sec 5.2.8.

The evolution equations and energy integral here are, respectively and after discovering the conserved quantity J and eliminating it, as for the special case but with N -subscripts appended. The momenta, Hamiltonian and the energy constraint are as before with N -subscripts appended to the various quantities. One can then rotate the above two exact solutions for the special case to obtain solutions to the general case, but these are too lengthy to include in this article.

5.8 Types of behaviour of cosmologically-inspired (approximately) classical solutions

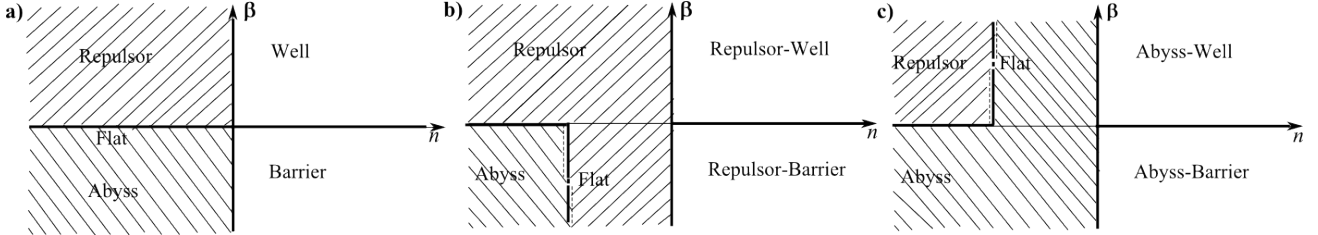


Figure 41: a) 5 classical behaviours for $V = \beta \rho^n$ potentials (these could just as well be scaled triangleland βI^n potentials). Solid lines denote included edges and dashed lines denote excluded edges. In the top-right quadrant, a representative is the HO. In the bottom-right quadrant, a representative is the upside-down HO. In the bottom-left quadrant, a representative 'abyss' is the bound $l = 0$ analogue of the hydrogen atom model. In the top-left quadrant, a representative 'repulsor' is the $L = 0$ analogue of the electron-electron model. The fifth region is the axes, for which the potential is constant.

b) Next consider the also commonly occurring (under the $\rho \rightarrow r$, $\mathcal{N}et = \mathcal{T}ot - 2R \rightarrow L_{Total}$ analogy) case of $V = \beta \rho^n + \mathcal{N}et/\rho^2$, i.e. with a $\mathcal{N}et > 0$ 'centrifugal barrier' added. This sends the HO to the repulsor HO, the upside-down HO to the repulsor upside-down HO. Also now the repulsor behaviour takes over the axes and a strip plus partial boundary of the bottom-left quadrant, pushing out the 'abyss' behaviour to $n < -2$ and the lower part of $n = 2$. The only case exhibiting the flat behaviour now is the critical value of β for $n = -2$.

Note: one of the representatives of repulsor, however, ($L \neq 0$ bound Hydrogen analogue) has a well next to the repulsor.

c) Finally, consider the $\mathcal{N}et < 0$ case c), which is not mechanically standard but is cosmologically standard (radiation term). This sends the HO to the abyss-HO, the upside-down HO to the abyss-upside-down-HO. Also now it is the abyss behaviour that takes over the axes and a strip plus partial boundary of the top-left quadrant, pushing out the 'repulsor' behaviour to $n < -2$ and the upper part of $n = 2$.

As there are many, one needs a more efficient method of qualitatively understanding these than the previous SSsecs on HO-type potentials. The shape of the potentials can be classified into the regions of Fig 41. The corresponding qualitative classical behaviours of these are in Fig 4; therein I make use of Robertson's kind of [530]'s notation for types of solution, also used in e.g. [304, 529]. This involves using O for oscillatory models of the types O_1 ($0 \leq \rho \leq \rho_{\max}$), O_2 ($\rho_{\min} \leq \rho \leq \rho_{\max}$) and O_3 ($\rho_{\min} \leq \rho \leq \infty$). M for monotonic solutions of type M_1 ($0 \leq \rho \leq \infty$), and M_2 ($\rho_{\min} \leq \rho \leq \infty$). S for static solutions of types S_1 unstable, S_2 stable and S_3 stable for all ρ . A for solutions asymptotic to static solutions at a finite ρ_A , of types A_1 coming in from 0 and A_2 coming in from ∞ .

Some solutions corresponding to QM models studied in this article are as follows. $\rho = \sin(\sqrt{2At}^{em})/\sqrt{2A}$ is analogous to the Milne in anti de Sitter solution. $\rho = \cosh(\sqrt{-2At}^{em})/\sqrt{-2A}$ is analogous to the positively-curved de Sitter model. [Both are models with just analogues of k , Λ of various signs.] $\rho = \{-9K/2\}^{1/3} t^{em 2/3}$ is analogous to the flat dust model, with well

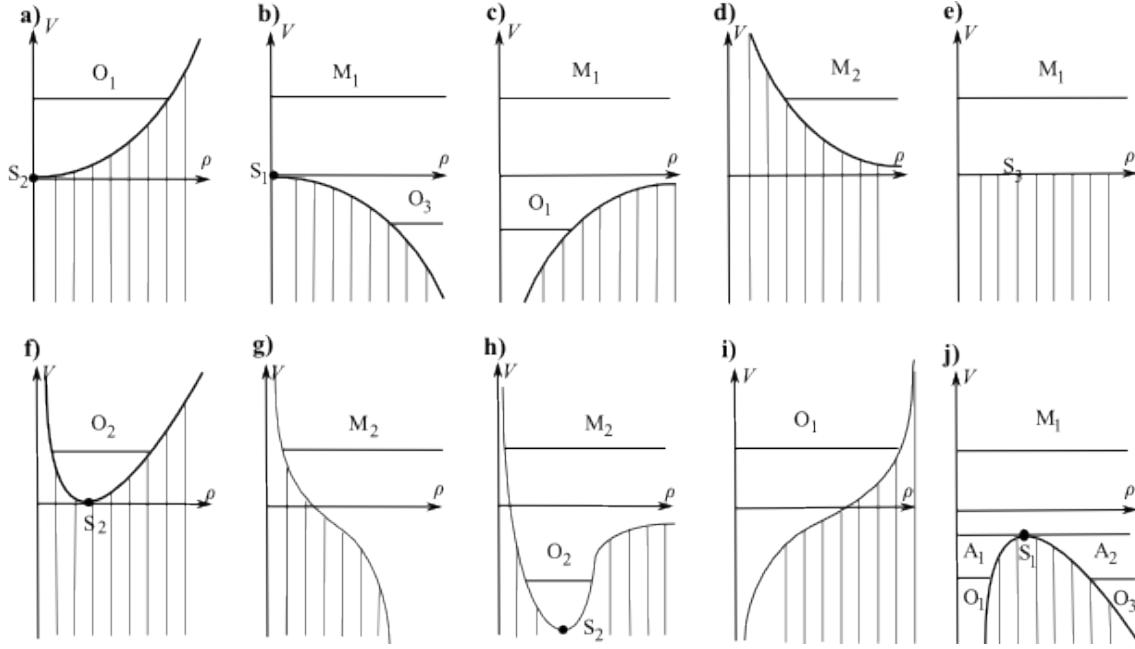


Figure 42: Qualitative classical behaviour and identification of corresponding cosmological solutions. Shaded regions are classically forbidden.

a) The HO model corresponds to a Milne in AdS cosmology (\mathcal{A} vii) in [35]; the potential is nonsingular at 0.

b) For zero and positive energy, the upside-down HO model corresponds to the ($k = 0$, ordinary) de Sitter and $k = -1$ de Sitter sinh cosmologies, with instability causes separation to keep on increasing and faster, which motion corresponds to a significant expansion phase in cosmology. For negative energy, it can be taken to bounce (O_3) at a finite ρ_{\min} . This corresponds to the $k = -1$ de Sitter cosh solution.

c) This behavior covers pure dust and corresponds to Newtonian Gravity mechanics models) and pure radiation and corresponding to conformal potential mechanics models). The flat and negatively curved (open in cosmology) models therein involve expanding forever (M_2). On the other hand, the positively curved (closed in cosmology) models involve recollapse (O_3) from and to a point at which the potential is singular.

d) This behaviour covers pure wrong-sign radiation, corresponding to mechanics with centrifugal term and/or positive conformal potential.

e) This constant-potential case involves expansion forever (M_2) or everywhere static stable behaviour.

f) This is wrong-sign radiation (centrifugal term and/or positive conformal potential) alongside negative cosmological constant (alias HO).

g) This is wrong-sign radiation corresponding to conformal potential alongside positive cosmological constant/HO.

h) This is the $L \neq 0$ Hydrogen analogue, corresponding to Newton-conformal potential problem and has oscillatory ('bound') and monotonic ('ionized') regions.

i) This is radiation (sufficiently negative conformal potential) alongside negative cosmological constant/HO and involves recollapse from and to a point at which the potential is singular.

j) This is radiation (sufficiently negative conformal potential) alongside positive cosmological constant/upside-down HO, covering many different possible behaviours.

[Models with all of dust, radiation and cosmological constant [304, 617, 183, 189] exhibit behaviours f), g), i) and j), with scope for turning points in the intermediate region.]

known cycloid and hyperbolic counterpart solutions in the positively and negatively-curved cases. $\rho = \{4\{2R - \mathcal{T}_{\text{ot}}\}\}^{1/4} t^{\text{em} 1/2}$ is analogous to the flat radiation model, with well-known curved counterparts (one analogous to the Tolman model). The present problem also requires cosmologically less familiar wrong-sign radiation models familiar (but these are familiar in ordinary mechanics as solutions with central term), such as $\rho = \sqrt{t^{\text{em} 2} + \mathcal{T}_{\text{ot}}}$.

5.9 Scaled RPMs' approximate scale solutions (semiclassically relevant)

I build these by the mathematical analogy with mostly Appendix 5.A and occasionally Appendix 5.B.

5.9.1 N -stop metroland analogue cosmology approximate scale equation solutions

I argue that E -normalizing suffices for the current qualitative treatment. I also point out that this work also applies to the \mathbb{CP}^{N-1} presentation of N -a-gonland.

\mathcal{A}) Models with energy and (upside-down) HO potentials' scale contribution are as follows.

- i) $E = 0$, $A = 0$ is ρ constant: a static model universe.
- ii) $E = 1/2$, $A = 0$ is $\rho = t^{\text{em}(\text{JBB})}$.
- iii) $E = -1/2$ and $A \geq 0$ is impossible.
- iv) $E = 0$ and $A < 0$ is $\rho = \exp(\sqrt{-2A} t^{\text{em}(\text{JBB})})$.
- v) $E = 1/2$ and $A < 0$ is $\rho = \{1/\sqrt{-2A}\} \sinh(\sqrt{-2A} t^{\text{em}(\text{JBB})})$.
- vi) $E = -1/2$ and $A < 0$ is $\rho = \{1/\sqrt{-2A}\} \cosh(\sqrt{-2A} t^{\text{em}(\text{JBB})})$.
- vii) $E = 1/2$ and $A > 0$ is $\rho = \{1/\sqrt{2A}\} \sin(\sqrt{2A} t^{\text{em}(\text{JBB})})$.

\mathcal{B}) Models with energy and Newtonian potentials' scale contribution are as follows.

- i) $E = -1/2$ is a cycloid solution.

ii) $E = 0$ is $\rho = \{-9K/2\}^{1/3} t^{\text{em(JBB)} 2/3}$.

iii) $E = 1/2$ is a hyperbolic analogue of the cycloid.

C) Models with energy and scale-invariant potential terms have the following approximate heavy-scale solutions (the R is but the constant lead term of an expansion in the shape variables). For $2R - \mathcal{T}_{\text{ot}} > 0$ (corresponding to right-sign radiation in Cosmology),

i) $E = -1/2$: $\rho = \sqrt{2R - \mathcal{T}_{\text{ot}}} \{1 - \{1 - t^{\text{em(JBB)}}/\sqrt{2R - \mathcal{T}_{\text{ot}}}\}^2\}^{1/2}$.

ii) $E = 0$: $\rho = \{4\{2R - \mathcal{T}_{\text{ot}}\}\}^{1/4} t^{\text{em(JBB)} 1/2}$.

iii) $E = 1/2$: $\rho = \sqrt{2R - \mathcal{T}_{\text{ot}}} \{1 + t^{\text{em(JBB)}}/\sqrt{2R - \mathcal{T}_{\text{ot}}}\}^2 - 1\}^{1/2}$

For $2R - \mathcal{T}_{\text{ot}} < 0$, including $\mathcal{T}_{\text{ot}} \neq 0$, $R = 0$, and corresponding to wrong-sign radiation in Cosmology, so that the direct analogy is with the mechanics of Appendix B,

iv) $E = -1/2$ is impossible,

v) $E = 0$ is also impossible, and

vi) $E = 1/2$ gives $\rho = \sqrt{t^{\text{em(JBB)} 2} + \mathcal{T}_{\text{ot}} - 2R}$.

E) Some further approximate (the K is but the constant lead term of an expansion in the shape variables) heavy-scale solutions are as follows.

i) $E = 0$, with upside-down HO $A < 0$ and Newtonian potential terms, solved by $\rho = \{\{K/2A\}\{\cosh(3\sqrt{-2A}t^{\text{em(JBB)}}) - 1\}\}^{1/3}$.

ii) $E = 0$, with HO $A > 0$ and Newtonian potential terms, solved by $\rho = \{\{-K/2A\}\{1 - \cos(3\sqrt{2A}t^{\text{em(JBB)}})\}\}^{1/3}$.

The forms with the E 's not scaled out of the following out of the above models are subsequently needed in this article.

For model $\mathcal{A}v$), $t^{\text{em(JBB)}} = \{1/\sqrt{-2A}\}\text{arsinh}(\sqrt{-A/E}\rho) + \text{const}$. This model's full name is 'positive energy upside-down HO RPM analogous to negative-curvature dS' (de Sitter).

For model $\mathcal{A}vi$), $t^{\text{em(JBB)}} = \{1/\sqrt{-2A}\}\text{arcosh}(\sqrt{A/E}\rho) + \text{const}$. This model's full name is 'negative energy upside-down HO RPM analogous to positive-curvature dS'.

For model $\mathcal{A}vii$), $t^{\text{em(JBB)}} = \{1/\sqrt{2A}\}\text{arcsin}(\sqrt{A/E}\rho) + \text{const}$. This model's full name is 'HO RPM analogous to Milne-in-AdS'. (HO necessarily implies positive energy; all the positives and negatives in this SSsec are strict).

For model $\mathcal{B}ii$), $t^{\text{em(JBB)}} = \sqrt{-2K/9\rho^{3/2}} + \text{const}$. This model's full name is 'zero-energy Newtonian potential RPM analogous to zero-curvature dust cosmology'.

I note that all of the above approximate heavy-scale timefunctions are monotonic apart from model $\mathcal{A}vii$), which nevertheless has a reasonably long era of monotonicity as regards modelling early-universe Quantum Cosmology. Model $\mathcal{A}vii$) has periods proportional to $\sqrt{E/A}$ for N -stop metroland, which plays a role proportional to that of $1/\sqrt{\Lambda}$ in GR Cosmology. Finally, for model $\mathcal{A}vi$) there is a nonzero minimum size.

5.9.2 Triangleland analogue cosmology approximate scale equation solutions

A) Models with (upside down) HO and $|r^{IJ}|^6$ potential terms are as follows (these are just approximate heavy solutions in cases with $S \neq 0$).

i) $A = 0$, $S = 0$ is I constant: static universe.

noindent ii) $A = -1/2$, $S = 0$ is $I = t^{\text{em(JBB)}}$.

iii) $A = 1/2$ and $S \leq 0$ is impossible.

iv) $A = 0$ and $S > 0$ is $I = \exp(\sqrt{2S}t^{\text{em(JBB)}})$.

v) $A = -1/2$ and $S > 0$ is $I = \{1/\sqrt{2S}\}\sinh(\sqrt{2S}t^{\text{em(JBB)}})$.

vi) $A = 1/2$ and $S > 0$ is $I = \{1/\sqrt{2S}\}\cosh(\sqrt{2S}t^{\text{em(JBB)}})$.

vii) $A = -1/2$ and $S < 0$ is $I = \{1/\sqrt{-2S}\}\sin(\sqrt{-2S}t^{\text{em(JBB)}})$.

B) Models with (upside down) HO and energy have the following solutions.

i) $A = 1/2$ is a cycloid.

ii) $A = 0$ $I = \{9E/2\}^{1/3} t^{\text{em(JBB)} 2/3}$.

iii) $A = -1/2$ is the hyperbolic analogue of the cycloid.

C) Models with conformally invariant potential and (upside down) HO include the following solutions. For $2R - \mathcal{T}_{\text{ot}} > 0$ (corresponding to right-sign radiation in Cosmology),

i) $A = 1/2$: $I = \sqrt{2R - \mathcal{T}_{\text{ot}}} \{1 - \{1 - t^{\text{em(JBB)}}/\sqrt{2R - \mathcal{T}_{\text{ot}}}\}^2\}^{1/2}$.

ii) $A = 0$: $I = \{4\{2R - \mathcal{T}_{\text{ot}}\}\}^{1/4} t^{\text{em(JBB)} 1/2}$.

iii) $A = -1/2$: $I = \sqrt{2R - \mathcal{T}_{\text{ot}}} \{1 + t^{\text{em(JBB)}}/\sqrt{2R - \mathcal{T}_{\text{ot}}}\}^2 - 1\}^{1/2}$.

For $2R - \mathcal{T}_{\text{ot}} < 0$, including $\mathcal{T}_{\text{ot}} \neq 0$, $R = 0$, and corresponding to wrong-sign radiation in Cosmology,

iv) $A = 1/2$ is impossible,

v) $A = 0$ is also impossible, and

vi) $A = -1/2$ gives $I = \sqrt{t^{\text{em(JBB)} 2} + \mathcal{T}_{\text{ot}} - 2R}$.

D) The model with Newtonian potentials and $E = 0$ has the approximate heavy solution $I = \{2E\}^{13/21} \{4t^{\text{em(JBB)}}/3\}^{4/7}$ which parallels the flat cosmology with $P = \rho/6$.

E) Some further approximate heavy-scale solutions are as follows.

i) $A = 0$, $S > 0$ and E-term of suitable sign: $I = \{\{E/2S\}\{\cosh(3\sqrt{2S}t^{\text{em(JBB)}}) - 1\}\}^{1/3}$.

ii) $A = 0$, $S < 0$ and E-term of suitable sign: $I = \{\{-E/2S\}\{1 - \cos(3\sqrt{-2S}t^{\text{em(JBB)}})\}\}^{1/3}$.

With the A 's not scaled out, the particular model among these which are used later on in the present article is as follows (presented in the subsequently-used inverted form for $t^{\text{em(JBB)}}$).

A slight generalization of model \mathcal{A}_{ii} with $\bar{t}^{\text{em(JBB)}} = \sqrt{-2A}I + \text{const}$. This model's full name is 'zero-energy upside-down HO spherical triangleland analogous to negative-curvature vacuum cosmology'.

5.9.3 The preceding in terms of t^{geom}

$t^{\text{geom}} = \int dt^{\text{em}}/\rho^2$ is in a particular sense more geometrically natural for shape space (the ρ^2 factor is that from ds^2 versus $\rho^2 ds^2$) and will later be found to be particularly useful. In all of the cases considered (both here and more extensively in [36, 37]) this is analytically possible, and invertible so that one can make the scale a function of t^{geom} analytically, by composition of two analytical inversions. For model \mathcal{A}_{v} $t^{\text{geo}} = -\sqrt{-A/2} \coth(\sqrt{-2A}t^{\text{em}})/E = -\sqrt{1-2A\rho^2}/\sqrt{2E\rho}$, for model \mathcal{A}_{vi} $t^{\text{geo}} = \sqrt{A/2} \tanh(\sqrt{-2A}t^{\text{em}})/E = \sqrt{-2A\rho^2-1}/\sqrt{2E\rho}$ and for model \mathcal{A}_{vii} , $t^{\text{geo}} = -\sqrt{A/2} \cot(\sqrt{2A}t^{\text{em}})/E = -\sqrt{1-2A\rho^2}/\sqrt{2E\rho}$. For the spherical presentation of triangleland model \mathcal{A}_{ii} using I^2 rather than ρ^2 ,

$$t^{\text{geo}} = \sqrt{\frac{2}{-A}} \left\{ \frac{1}{I_0} - \frac{1}{I} \right\}. \quad (483)$$

Thus, inverting, $I = 1/\{1/I_0 - \sqrt{-A/2}t^{\text{geom}}\} = \sqrt{2/-A}/\{t^{\text{geo}}_c - t^{\text{geom}}\}$ for $t^{\text{geo}}_c := \sqrt{2/-A}/I_0$.

5.9.4 Comments on these RPM solutions

Note in particular that the cyclic trial models with HO mathematics \mathcal{A}_{vii} of [50, 30, 31, 34] do correspond to a known cosmology (Milne in anti de Sitter). Also, having some upside-down HO's, rather (also readily tractable), is de Sitter/inflationary in character [\mathcal{A}_{iv} , v) and vi)]. Other models parallel the dynamics of fairly realistic simple models of the early universe involving radiation, spatial curvature and cosmological constant type terms.

5.10 Appendix A: a range of standard GR isotropic cosmology solutions

\mathcal{A}) Models with spatial curvature and cosmological constant are as follows (see e.g. [529]).

- i) $k = 0$, $\Lambda = 0$ is a constant: a static universe.
- ii) $k = -1$, $\Lambda = 0$ is $a = t^{\cos}$.
- iii) $k = 1$ and $\Lambda \leq 0$ is impossible.
- iv) $k = 0$ and $\Lambda > 0$ is $a = \exp(\sqrt{\Lambda/3}t^{\cos})$: a deSitter/inflationary model with zero curvature.
- v) $k = -1$ and $\Lambda > 0$ is $a = \sqrt{3/\Lambda} \sinh(\sqrt{\Lambda/3}t^{\cos})$: a de Sitter/inflationary type model with negative curvature.
- vi) $k = 1$ and $\Lambda > 0$ is $a = \sqrt{3/\Lambda} \cosh(\sqrt{\Lambda/3}t^{\cos})$: a de Sitter/inflationary type model with positive curvature.
- vii) $k = -1$ and $\Lambda < 0$ is $a = \sqrt{-3/\Lambda} \sin(\sqrt{-\Lambda/3}t^{\cos})$ – a 'Milne in anti de Sitter' [529] oscillating solution.

\mathcal{B}) Models with spatial curvature and dust are as follows.

- i) $k = 1$ is the well-known cycloid solution.
- ii) $k = 0$ $a = \{9GM/2\}^{1/3} t^{\cos 2/3}$.
- iii) $k = -1$ is the also well-known hyperbolic analogue of the cycloid.

\mathcal{C}) Models with radiation and spatial curvature include the following solutions [622].

- i) $k = 1$: $a = \sqrt{2GM}\{1 - \{1 - t^{\cos}/\sqrt{2GM}\}^2\}^{1/2}$ (the Tolman model).
- ii) $k = 0$: $a = \{8GM\}^{1/4} t^{\cos 1/2}$.
- iii) $k = -1$: $a = \sqrt{2GM}\{1 + t^{\cos}/\sqrt{2GM}\}^2 - 1\}^{1/2}$.

\mathcal{D}) the case of $P = \epsilon/6$ is not exactly integrable except for $k = 0$, in which case the solution is $a = \{2GM\}^{13/21}\{4t/3\}^{4/7}$.

Finally, I consider further combinations of the well-motivated potential terms. E.g.,

\mathcal{E}) the cosmologically standard model comprising dust, spatial curvature and cosmological constant is covered e.g. in [529, 218]. Solutions of this include the Lemaître model, a model in which the Big Bang tends to the Einstein static model, the Eddington–Lemaître model, and various oscillatory models including a bounce. Solutions for k and Λ both nonzero involve in general elliptic integrals. Some subcases that are solvable in basic functions are

- i) $k = 0$, $\Lambda > 0$, solved by $a = \{\{3GM/\Lambda\}\{\cosh(\sqrt{3\Lambda}t^{\cos}) - 1\}\}^{1/3}$, and
- ii) $k = 0$, $\Lambda < 0$, solved by $a = \{\{-3GM/\Lambda\}\{1 - \cos(\sqrt{-3\Lambda}t^{\cos})\}\}^{1/3}$.

$|x^{IJ}|^6$ potential terms for triangleland, can readily be obtained by Models with radiation and spatial curvature and cosmological constant include a subcase of what is covered by Harrison [304] and Vajk [617]. Models with all of radiation, dust, spatial curvature and cosmological constant are also a subcase of Harrison's work [304]. They are also treated more explicitly by Coqueraux and Grossmann [183] and by Dabrowski and Stelmach [189]. [Note that I am considering models in which

dust and radiation contributions do not interact with each other.] On the other hand, the analogy with ordinary mechanics covers combining a ‘wrong sign radiation’ ‘central term’ with these other terms (e.g. in the Newton–Hooke problem).

5.11 Appendix B: further support from ordinary Mechanics

‘Wrong sign radiation’ in Cosmology clearly corresponds via the Cosmology–Mechanics analogy to just the kind of effective potential term that one has for a centrifugal barrier, which is often present and well-studied in ordinary mechanics. Thus this case, while cosmologically unusual, does not lead to any unusual calculations either. In any case, such sign changes usually do not change exact integrability, but can change the qualitative behaviour in at least some regimes. (Consider e.g. trigonometric functions becoming hyperbolic functions under a sign change in some elementary integrals). With E and overall angular momentum term (= wrong-sign radiation), C_{iv},v are impossible cases, while C_{vi} gives $r = \sqrt{t^2 + L^2}$. Some cases of this remain readily tractable if a Kepler–Coulomb potential term is added to this.

5.12 Appendix C: can RPM’s model the real world at the classical level?

5.12.1 C.1 Could pure-shape RPM describe the real world?

Pure-shape RPM serves as an intriguing suggestion of how a symmetry principle is capable of locally reproducing standard physics at smaller scales while significantly diverging nonlocally. In this model, this can be interpreted as an effect of the ‘wider matter distribution in the universe’ (e.g. triangleland’s apex particle) affecting the physics of other subsystems (e.g. triangleland’s base pair). This is interestingly ‘Machian’ (albeit in a distinct sense of the word from that used in the Introduction [81]). This may have some capacity to account for deviations from standard physics at larger scales without having to invoke (as many instances of) dark matter. E.g. better explaining (at least at a nonrelativistic level) the rotation curves of galaxies without incurring unacceptably large deviations in solar system physics. Some problems with modelling the world using SRPM [84, 85] are as follows.

1) Technically it is a lot less straightforward to study in 3- d than in this article’s 1- and 2- d cases.

However, scale and rotational relationalism do not interfere with each other, and it is only the latter that is harder in 3- d , so that the nonrotational RPM can serve as a toy model for the investigation of the specific effects of regarding scale as unphysical in a 3- d world.

2) The observed universe’s angular momentum does look to balance out to zero (see below). However, the observed universe’s contents look to be expanding far more than contracting, and we do not know of a satisfactory model to fit the details of cosmological redshift without invoking scale. Thus zero dilational momentum prediction, unlike zero angular momentum one, would seem to be incompatible with observation, at least within the framework of all hitherto-conceived theories that explain the observations.

5.12.2 C.2 Is scaled RPM restrictive at the classical level?

1) Scaled RPM is a Leibnizian/Machian mechanics, and yet is in agreement with a subset of Newtonian Mechanics – the zero total angular momentum universes.⁵³

2) Subsystems need not have zero angular momentum or energy conservation. Thus models with such arise in relational setting too, in particular nonzero angular momentum island subuniverses [92, 433]. Because of this, the restriction to $\mathcal{L} = 0$ universes is by no means as severe as might be naïvely suggested.

3) One issue that I have not seen resolved is that the viability of this rests on the N -body problem’s theoretical framework being such that initially distant subsystems do not fall together below any desired finite timescale internal to the subsystem under study. This falling together can involve unboundedness of some velocities and hence of T and hence of V , which is termed a *singularity* in the N -body problem literature. While it used to be thought that this could only occur in situations involving collisions, it has been shown that for 5 bodies it is possible to attain this by merely coming arbitrarily close to collisions.⁵⁴ Moreover, it remains mere conjecture that singular solutions are of measure zero in the set of all solutions. In any case (and unsurprisingly), sufficiently accurate physical modelling of the universe acts to prevent arbitrarily distant subsystems from falling together in finite time. On the other hand, at least in astrophysics with realistic matter, there will be short-range forces that interfere with the potential becoming arbitrarily negative and/or GR (which respects a positive energy theorem) will take over. Also, SR will bound infall velocities.

4) Within the context of mechanical modelling, one can carry out checks on whether \mathcal{L} is zero or not in some patch of the universe around us. Angular momentum can be estimated object-by-object from short-term observations. Additionally, \mathcal{L} zero and nonzero (sub)systems are capable of evolving qualitatively differently (see e.g. [469] for an account of Hill’s work in that direction).

5) There is astrophysical and cosmological evidence for zero total angular momentum of the universe. Though it does concern a *null outcome*, so it is a relevant further issue as to how accurately is this statement known to hold?

⁵³That many of the results and mathematical structures (at least in the simpler cases) previously obtained by standard absolutist physics are recovered from relationalism is an interesting reconciliation as regards the extent of prejudice incurred by studying absolutist physics. This article does however get far enough and detailed enough to list a number of differences (see Secs 6, 10 and 16).

⁵⁴This was conjectured by Painlevé [503]; Xia [648] eventually constructed an example of it, which is incidentally, an $\mathcal{L} = 0$ example and thus carries over to the scaled RPM arena. For further examples and a simple introduction to this subject, see [206].

Then note that scaled RPM *predicts* this (to the extent that it is a satisfactory cosmological model, see below) whereas Newtonian mechanics merely allows for it as a subset of its solutions (which are qualitatively different). [Is it a retrodiction since it was known observationally before RPM explained it? However, as it relates to a null result, it remains a prediction that this result will *stay null* as our detailed knowledge of the universe's contents increases with the scientific progress.]

There is however a loophole in that we do not know whether there is some more complicated way of relationally formulating $\mathcal{L} \neq 0$ mechanics. What is known is that $\mathcal{L} \neq 0$ Newtonian mechanics is a whole lot more complicated at the level of configuration space bundle geometry [428], so that, if it were relationally formulable, the complexity of its formulation would likely considerably exceed that of all hitherto-studied RPM's.

6) Another loophole is that, logically, RPM applies to the whole universe, whereas the observations apply to the observable universe i.e. within the cosmological horizon, and the latter is very often only a portion of the former within the set of cosmological models that are posited to match reality.

7) RPM's lack a practical robustness to modelling that Newtonian mechanics possesses well. This is in the sense that particular distinctive features of RPM's can go away, or substantially change, if one improves the detail of one's model. E.g. if the deeper truth about the 3-particle universe is that those particles are planets, then the 2- d to 3- d indiscernibility is absent because the detailed structure of the objects themselves distinguishes these. Likewise, the 1 and 2 body problems being meaningless/relationally trivial can cease to make sense if they have internal structure. In other words, indiscernibility and relational triviality in fact depend on where one sits in the hierarchy of increasingly detailed models of nature. This is a case of objects with internal structure allowing for self-relations rather than just mutual relations.

Note: overall, the practical advantages of working with an RPM are the modelling of *whole-universe* issues, and the provision of an emergent timestandard.

5.12.3 C.3 Is scaled RPM a realistic model of Cosmology?

That scaled RPM itself is a fairly good cosmological model as claimed in Sec 5.3.2 needs some qualification. Cosmology is conventionally modelled in terms of the Einstein–Euler system of GR alongside phenomenological fluid matter. In this picture, our universe is currently deep into a matter era in which the fluid is dust. This extends back to the early universe, at which point, somewhat after the onset of nucleosynthesis, a radiation fluid is additionally required.

On the other hand, Newtonian Mechanics of many particles suffices to model the large-scale matter era of Cosmology [460, 451, 461, 253, 105]. This makes good physical sense: dust, having no pressure, should be modellable more simply than by a fluid, and not require any GR input at least so as to explain many of its basic and long-standingly established features. That *Newtonian* cosmology itself suffices for this follows from how the scale–shape split of the equations of motion yields the ‘dust case of the Raychaudhuri equation’ for the scale of the universe, alongside an equation for the shape part (see e.g. [105]). The latter is well-known in the Celestial Mechanics literature, albeit in a slightly different context. (I.e. for a few particles, whereas cosmology has the large particle number case; these could be galaxy clusters, galaxies, stars or even dark matter particles, gravitation being universal and none of these things exerting pressure on each other.) Also, the above analogy continues to make sense in models with pressure terms [304, 465, 507, 529], allowing for radiation models, and with cosmological constant terms. The last form of matter model which Sec 5.3.4 parallels is sensible as a rough model of a mixture of dust and radiation corresponding to the conventional universe model around 60000 years after the Big Bang.

A limitation: Microlensing [562] is a salient example of a fairly large-scale phenomenon that requires additional GR input (or perhaps an equivalent but rather probably less insightful input of the universe being an unusual optical medium). I see this as a reasonable indication of the extent to which the Newtonian Cosmology paradigm can be pushed.

Moreover, the above Newtonian Cosmology paradigm is a zero total angular momentum universe, and therefore can be conceived of as scaled RPM. Admittedly, viewing it as scaled RPM has no immediate *technical* advantages that I can think of, though it does attribute somewhat better explanatory power for features such as the zero total angular momentum and emergent rather than assumed timestandard.

On the other hand, this Sec's specific models are obviously related to cosmology by more tenuous analogies than the above (parallel form of the scale dynamics but in a lower dimension, with the benefit of then being reduced/relationalspace formulated). They have the limitation of not having an underlying concept of energy density. [The analogies are at the level of the cosmological equations after making the $\epsilon = \epsilon(\text{scale})$ substitution. This is wholly unsurprising given that they are only few-particle models rather than the averaged approximations to GR of the most common cosmological paradigm. Newtonian/RPM models would aspire to that, but only as the particle number becomes very large, which lies outside of the scope of the present article's contents and interests.]

Both these models and the 3- d version itself have a **further analogy with structure formation** in classical and quantum cosmology: in each case, one has a simple scale problem coupled to a complicated structure formation shape problem. For these papers' models, these shape problems are a lot simpler than GR's, making them further investigable. On the other hand, the 3- d version's late-time physics should in many ways coincide with the standard cosmology, though in this setting one has to face that the quantum-cosmological origin of the perturbations themselves clearly do not belong to the classical matter era setting for which 3- d Newtonian Mechanics/RPM is a good model of GR. That is not an issue for this article's models since their relation to cosmology is not meant to exceed a good qualitative analogy.

PART II: QUANTUM THEORY OF RPM's

6 Introduction to Quantum RPM's

6.1 Minor motivation: do absolute and relational mechanics lead to different QM?

This motivation follows on from the historical interest in freeing mechanics from absolute structure. Would a different sort of QM or generalization thereof have arisen if the conception of mechanics itself had been relational [98, 539, 583, 19, 25, 35] rather than absolute?⁵⁵ I.e., has the traditional absolutist approach been misleading us at the quantum level? Or does an essentially identical theoretical framework arise in the relational case too? (Perhaps due to some deeper-seated underlying foundation to both approaches? Or perhaps due to curious parallels/technical coincidences?) N.B. what is new in the relational approach is certainly not the use of relative/relational quantities in calculations – that is already widespread in the standard approach to standard QM.

6.1.1 A start on quantum relationalism

In this respect, I find a *lack* of non-standard QM at least in the simpler examples (c.f. the mathematical analogies listed in Sec 8 and 9). By this means earlier work with absolutist QM gives us plenty of machinery to ease the solving of the simpler examples of relational QM too. Note however that the latter but not the former are appropriate as whole-universe models and for detailed analysis of the Problem of Time. Thus this coincidence is a triumph as regards finding tractable toy models (to the extent that these are useful toy models of generally-relativistic whole-universe models).

Moreover, there are differences between absolutist and relational QM.

Analogy 53) There are differences of the subsystem versus the closed whole-system sort between absolutist QM and relational QM Subsystem physics in a relational universe would be expected to be linked by global restrictions due to its constraint equations. Such differences were already found to be present in the GR case by DeWitt [201].

There is no a priori time, but many parts of QM do not depend on time, and we will also see how the remaining parts can be accessed using emergent time dependent wave equations in Part III.

My principal motivation remains the study the quantization of RPM's is that it is likely to be valuable, via the RPM-geometrodynamics analogy for the following Quantum Gravity and Quantum Cosmology investigations. Thus what particular features quantum GR has is a bigger thrust here than 'does relationalism paint a different picture of QM?' We build general enough to be able to (at least formally) discuss GR.

Note: RPM's also make for interesting places to test out techniques for ([98, 539, 583, 283, 284, 93, 32, 287] and the present article) and possibly foundations of quantization [19].

6.2 On the nature of quantization: the case with no \mathfrak{g} /no \mathcal{L}_{in}

Quantization is a bad functor, conventionally taken to be

$$\text{Quant: (Phase, Can)} \longrightarrow (\text{Hilb}, \text{Uni}) . \quad (484)$$

The Hilbert space $\text{Hilb} = (\text{Vec}, \text{IP})$ for Vec a vector space (of wavefunctions) and IP an inner product, and such that completeness is obeyed. The corresponding morphisms are the unitary transformations $\text{Uni: Hilb} \longrightarrow \text{Hilb}$.

I note that conventionally (and already in good keeping with relationalism), the objects containing tangible physics are (using Dirac notation for the IP)

$$\langle \Psi_1 | \hat{O} | \Psi_2 \rangle \quad (485)$$

for Ψ_1, Ψ_2 wavefunctions and \hat{O} an operator. This includes expectations of operators ($\Psi_1 = \Psi_2$), overlap integrals ($\hat{O} = \text{id}$) and probabilities for regions (taking both of these conditions but restricting the integration implicit in the inner product to apply only to a region of configuration space; in fact it can be necessary to only consider ratios of these). These objects are invariant under unitary transformations U due to cancellation of the straightforward $U\Psi$ action and the adjoint $U\hat{O}U^\dagger$ action.

⁵⁵This issue does not go away in considering relativistic QM or QFT on flat spacetime as these have a privileged Killing vector that effectively re-assumes Newtonian time's absolutist role (see also the Preface and Sec 2.6.4). One can extend this to relativistic particle dynamics in other stationary spacetimes, but, unfortunately, that is as far as it will go [398].

Reasons why Quant is a bad functor are legion. In discussing these, it is helpful to 1) take (Phase, Can) to be augmented by the Hamiltonian for the system, H . 2) For now, take there to be no \mathfrak{g} first. 3) ‘factorize’ this into a string of (bad) functors as follows.

$$\text{Quant} = \text{DynQuant} \circ \text{H-Rep} \circ \text{Assoc} \circ \text{KinQuant} \quad (486)$$

I explain what each of these are below.

6.2.1 KinQuant: kinematical quantization

This step is very obvious in the simplest example $q^A \rightarrow \hat{q}^A$, $p_A \rightarrow \hat{p}_A$ and the fundamental Poisson brackets going over (‘correspondence principle’) to the fundamental equal-time commutation relations,

$$[\hat{q}^A, \hat{p}_B] = i\hbar\delta_A^B. \quad (487)$$

And yet it is often ‘forgotten about’ in more complicated cases, for which it in fact comes to exhibit a number of subtleties [332] (take these as some reasons for ‘bad’).

More generally,

$$\text{KinQuant}: (\text{Phase}, \text{Can}) \rightarrow (\text{ComAl}, M) \quad (488)$$

for $\text{ComAl} = (\text{a set of } f(\hat{q}, \hat{p}, i\hbar[\ , \]))$ [M are the corresponding commutator-preserving morphisms.]

In doing so more generally than for (487), one has to make a *choice* of a preferred subalgebra of Phase objects [332] to promote to QM operators. Some context for this is that nonlinear systems exhibit the **Groenewold–van Hove phenomenon** [276, 618].⁵⁶ By this, classical equivalence can be lost in passage to QM.

$$\begin{array}{ccc} (\mathcal{Q}, \mathcal{P}, H) & \xrightarrow{\text{KinQuant, H-Rep}} & (\mathcal{T}(\hat{\mathcal{Q}}, \hat{\mathcal{P}}), \hat{H}) \\ \text{Can} \downarrow & & \downarrow \text{Uni} \\ (\mathcal{Q}', \mathcal{P}', H) & \xrightarrow{\text{KinQuant, H-Rep}} & (\mathcal{T}(\hat{\mathcal{Q}}', \hat{\mathcal{P}}'), \hat{H}) \end{array}$$

Figure 43: A major problem in quantization.

There are also global considerations by which the quantum commutator algebra will not in general be isomorphic to the classical Poisson bracket algebra. Quantization on the half-line already demonstrates this [332]. Here,

$$q > 0 \quad (489)$$

impedes \hat{p} as represented by $i\hbar\partial/\partial q$ from being properly self-adjoint. On the other hand, \hat{p} represented by $-i\hbar q\partial/\partial q$ does not have this problem, corresponding to the affine commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\hat{q}. \quad (490)$$

Note 1) Preferred subalgebra choice and global considerations are two reasons for the KinQuant functor to be ‘bad’ (or, at least, ambiguous).

Some further developments in the context of Isham’s $\mathfrak{q}/\mathfrak{g}$ example for \mathfrak{g} a subscript of \mathfrak{q} all specific RPM’s in this article are nested within this. For this, the relevant spaces involved in kin quant can be decomposed as semisimple products $\mathfrak{v}^*(\mathfrak{q}) \otimes \mathfrak{g}_{\text{can}}(\mathfrak{q})$. Thus here $\text{ComAl} = (\mathfrak{v}^* \otimes \mathfrak{g}_{\text{can}}(\mathfrak{q}), i\hbar[\ , \])$. Here, $\mathfrak{g}_{\text{can}}(\mathfrak{q})$ is the canonical group and \mathfrak{v}^* is the dual of a linear space \mathfrak{v} that is natural via carrying a linear representation of \mathfrak{q} with the property that there is a \mathfrak{q} -orbit in \mathfrak{v} that is diffeomorphic to $\mathfrak{q}/\mathfrak{g}$ [332]. A useful note here is that Mackey theory is a powerful tool for finding the representations of such semidirect products.

Note 2) A further reason for badness is that the commutator algebra’s closure depends on operator orderings that were trivially equivalent for the brackets’ prior closure, which can be algebra-altering including in ‘harsh’ ways that go by the name of **anomalies**.

Note 3) This construct of Isham’s can be interpreted to be in line with Relationalism 3)’s \mathfrak{q} primality as $\mathfrak{g}_{\text{can}}$ comes from the form of \mathfrak{q} and then \mathfrak{v}^* is chosen to be compatible with \mathfrak{q} and $\mathfrak{g}_{\text{can}}(\mathfrak{q})$.

⁵⁶In fact, this result is only established for a number of simple phase spaces (see e.g. [268]).

6.2.2 Simple \mathfrak{g} -less RPM example of KinQuant

Here, the canonical group $\mathfrak{g}_{\text{can}} = \text{Isom}(\mathcal{R}(N, 1)) = \text{Eucl}(n) = \text{Tr}(n) \mathbb{S} \text{Rot}(n) = \mathbb{R}^n \mathbb{S} SO(n)$ and an appropriate finite subalgebra is $\mathbb{R}^n \mathbb{S} \text{Eucl}(n)$.

Note 1) In greater generality, from [332], $\mathfrak{g}_{\text{can}}(\mathfrak{q}) = \text{Isom}(\mathfrak{q})$ for the span of all of this article's specific RPM's (in fact all the 1 and 2 d RPM's).

Note 2) Also in greater generality, $\mathfrak{v}^* = \mathfrak{v}$ for finite examples.

For scaled 4-stop metroland, the nontrivial commutation relations are then the $U^A \rightarrow \mathfrak{n}^i$, $P_A \rightarrow \pi_i$ and $\mathcal{S}_A \rightarrow \mathcal{D}\text{il}_i$ of Sec 18.2. These can be recast in a dual form, with $\rho_i = \delta_{ij} \rho^j$ and $\mathcal{D}\text{il}_{ij} = \rho_i \pi_j - \rho_j \pi_i$: (1037) alongside

$$[\widehat{\mathcal{D}\text{il}}_{ij}, \widehat{\mathcal{D}\text{il}}_{kl}] = i\hbar \{ \delta_{ik} \widehat{\mathcal{D}\text{il}}_{jl} + \delta_{jl} \widehat{\mathcal{D}\text{il}}_{ik} - \delta_{il} \widehat{\mathcal{D}\text{il}}_{jk} - \delta_{jk} \widehat{\mathcal{D}\text{il}}_{kl} \} , \quad (491)$$

$$[\widehat{\rho}^i, \widehat{\mathcal{D}\text{il}}_{kl}] = i\hbar \{ \delta_{lj} \delta_{ik} - \delta_{lk} \delta_{ij} \} \widehat{\rho}^k , \quad [\widehat{\pi}_i, \widehat{\mathcal{D}\text{il}}_{kl}] = i\hbar \{ \delta_{lj} \delta_{ik} - \delta_{ij} \delta_{lk} \} \widehat{\pi}_k , \quad (492)$$

which has the benefit of extending to all of the higher- N pure-shape N -stop metrolands. 3-stop metroland, however, has to be treated differently: the $U^A \rightarrow \mathfrak{n}^i$, $P_A \rightarrow \pi_i$ and $\mathcal{S} \rightarrow \mathcal{D}$ of Sec 18.3.

6.2.3 Assoc, H-rep and DynQuant

Assoc associates a pre-Hilbert space PreHilb of pre-wavefunctions for these operators to act on. I.e.

$$\text{Assoc}: (\text{ComAl}, M) \rightarrow (\text{PreHilb}, M') . \quad (493)$$

H-rep is how to promote the Hamiltonian \mathbf{H} to a function of KinQuant's operators $\widehat{\mathbf{H}}$. This procedure yields a wave equation. In the present article's context, it is a timeless wave equation rather than such as a time-dependent Schrödinger equation, Klein–Gordon equation or Dirac equation, each of which is backed by absolutist time notions. Note furthermore that there are operator-ordering issues at stake, as well as well-definedness issues (functional derivatives, combinations with no functional-analytic reasons to be well-behaved) and regularization issues. Take these as more reasons for 'bad'.

DynQuant is then solving this: passing from PreHilb to Hilb: the Hilbert space within that is annihilated by $\widehat{\mathbf{H}}$. I.e.

$$\text{DynQuant}: (\text{PreHilb}, M') \rightarrow (\text{Hilb}, \text{Uni}) . \quad (494)$$

Note: the IP within Hilb restricts what are valid operators by self-adjointness.

6.3 Constrained Quantization: Overview and KinQuant

The general formula is

$$\text{ConstrainedQuant} = \text{DynQuant} \circ \mathfrak{g}_2\text{-Constrain} \circ \text{Assoc} \circ \text{C-rep} \circ \text{KinQuant} \circ \mathfrak{g}_1\text{-Constrain} , \quad (495)$$

the first move being a purely classical reduction, and with \mathfrak{g}_1 -or- \mathfrak{g}_2 -Constrain = id if \mathfrak{g}_1 or $\mathfrak{g}_2 = \text{id}$. If there are linear constraints, putting all the constraining in the first slot is reduced quantization and all in the second is Dirac quantization. Splitting the constraining between the two is a more general possibility. Constrain is not a commuting move, though on occasion versions of the other moves exist so that for that example it does commute.

It thus makes sense for \mathfrak{g} -constrain to be presented as a functor. We have

$$\mathfrak{g}\text{-constrain}: (\mathfrak{g}\text{-Phase}, \text{Can}, \mathfrak{g}\text{-Lin}, \text{Quad}) \rightarrow (\widetilde{\text{Phase}}, \widetilde{\text{Can}}, \widetilde{\text{Quad}}) \quad (496)$$

in the first position, and

$$\mathfrak{g}\text{-constrain}: (\text{PreHilb}, M, \mathfrak{g}\text{-Lin}, \text{Quad}) \rightarrow (\text{MidHilb}, \text{Uni}, \text{Quad}^r) . \quad (497)$$

The relevant part of Quant, PreDynQuant, is defined as mapping to $(\text{MidHilb}, \text{Uni}, \widetilde{\text{Lin}}, \widetilde{\text{Quad}})$, whether or not \mathfrak{g} and thus $\mathfrak{g}\text{-Lin}$ are trivial, and whether or not this is a tilded category. [MidHilb is those PreHilb states that are annihilated by $\widehat{\mathcal{L}\text{in}}$]. As a non-constraining move, this is defined for a constraint set. The logic is that Phase, Can supports \mathcal{C} that close under Phase's bracket with Can leaving the total(DA)Hamiltonian invariant. But certainly allotting a \mathcal{C} is an element of choice of system (equivalent to action prescribe or, + cleanly naive H prescribe to be met with the Dirac procedure. Thus

$$\text{Quant} = \text{Assoc} \circ \mathcal{C}\text{-Rep} \circ \text{KinQuant} , \quad (498)$$

giving a chosen kinematical quantization subalgebra F , a commutator, a space these operators act on and a representation of the constraints (which entails an operator-ordering choice and that the constraints close under the commutator, possibly with a quantum supplement or some strong restriction on what was a free parameter classically – the dimension in string theory being a good example). \mathcal{C} corresponds to $\mathcal{C}_W = (\mathcal{L}_{\text{in}} \text{ and } \text{Quad})$.

Once one considers $(\mathfrak{g}\text{-Phase, Can, } \mathfrak{g}\text{-Lin, Quad})$, KinQuant is only part of a functor, with $\mathcal{C}\text{-Rep}$ complementing it.

$$\text{RedPreDynQuant} = \text{Assoc} \circ \text{C-Rep} \circ \text{KinQuant} \circ \text{Constrain} , \quad (499)$$

$$\text{DirPreDynQuant} = \text{Constrain} \circ \text{Assoc} \circ \text{C-Rep} \circ \text{KinQuant} , \quad (500)$$

but the two constrains are very different functors in terms of what spaces they are between (they are similar by what is cut out rather than by between what they are).

The issue then is that $\tilde{}$ and $\hat{}$ as acting on \mathbf{H} are not expected to commute, so the final DynQuant step's outcome for each of the above could well be different:

$$\text{DirQuant} = \text{DynQuant} \circ \text{DirPreDynQuant} \neq \text{RedQuant} = \text{DynQuant} \circ \text{RedPreDynQuant} . \quad (501)$$

Note 1) the above can be written out for both, using \mathbf{r} labels for reduced with a smaller Solve step, and using \mathbf{D} labels at the Hilbert space level for the versions obtained via the Dirac approach.

Analogy 54) In each case, the presence of additional linear constraints permits one to choose to attempt the Dirac quantization approach. Geometrodynamics, not being reducible in general, that is one of few schemes that can be attempted in that case. There one does not have in general the luxury of reduced/relationspace quantization.

Difference 20) As consequence of Sec 3, one has the good fortune of being able to perform reduce/relationspace quantization for RPM's; such are not open in general for midisuperspace GR and upwards.

Note 2) The a posteriori $\mathfrak{q}\text{--}\mathfrak{g}$ incompatibility of Relationalism 11) can strike again now, via brackets not closing including anomalies.

6.3.1 RPM example of KinQuant down the Dirac route

Take $\mathfrak{q} = \mathfrak{r}(N, 2)$ and $\mathfrak{g} = \text{Rot}(2)$. One is on \mathbb{R}^{nd} , so one has the canonical group $\mathfrak{g}_{\text{can}}(\mathfrak{r}(N, 1)) = \text{Isom}(\mathbb{R}^{nd})\text{Eucl}(nd)$ and so $\mathfrak{v} \otimes \mathfrak{g}_{\text{can}} = \mathbb{R}^{nd} \otimes \text{Eucl}(n, d)$ (the Heisenberg group). Thus the usual canonical commutation relations apply:

$$[\hat{\rho}^{i\alpha}, \hat{\pi}_{j\beta}] = i\hbar \delta^i_j \delta^\alpha_\beta \quad (502)$$

alongside the ρ 's and π 's being $SO(nd)$ -vectors and $SO(nd)$ commutation relations among the \mathcal{S} 's.

Note: As regards using APhase or DAPhase in place of Phase here, in coming from the Poisson bracket, the commutator is not affected by the \mathfrak{g} -sector, and so using \mathbf{A} or \mathbf{dA} does not amount to promoting anything other than configurations and momenta to operators. The QM concerns only the part-tangible \mathbf{p} 's and \mathbf{q} 's. This is a 'Hamiltonian collapse' issue (c.f. Appendix 2.A).

6.3.2 RPM examples of KinQuant down the \mathbf{r} route

For N -stop metroland's shape space $\mathfrak{S}(N, 1) = \mathbb{S}^{n-1}$, the canonical group $\mathfrak{g}_{\text{can}}(\mathfrak{S}(N, 1)) = \text{Isom}(\mathbb{S}^{n-1}) = SO(n)$ and an appropriate finite subalgebra is $SO(n) \otimes \mathbb{R}^n$ (p 1269-70 of [332]). In the present case, this can be taken to be generated by $n\{n-1\}/2$ $SO(n)$ objects that are interpreted as relative distance momenta $\text{Dil}_{\tilde{\mathbf{r}}}$,⁵⁷ alongside n coordinates u^Γ such that $\sum_\Gamma \{u^\Gamma\}^2 = 1$, that are interpreted as the unit Cartesian directions in the surrounding relational space \mathbb{R}^n .

For pure-shape N -a-gonland's $\mathfrak{S}(N, 2) = \mathbb{CP}^{n-1}$ shape space, the canonical group is $\mathfrak{g}_{\text{can}}(\mathfrak{S}(N, 2)) = \text{Isom}(\mathbb{CP}^{n-1}) = SU(n)/\mathbb{Z}_n$ (perhaps one needs to take care here with including the further quotienting out by \mathbb{Z}_n). Moreover, this shape space can also be written as $SU(n)/U(n-1)$ (see Sec so it is a subcase of the general form $\mathfrak{q}^\Gamma = \mathfrak{q}/\mathfrak{g}$ for \mathfrak{g} a subgroup of \mathfrak{q} that is considered in [332]). Thus a suitable finite algebra is $\mathfrak{v} \otimes \mathfrak{g}_{\text{can}} = SU(n)/\mathbb{Z}_n \otimes \mathbb{C}^n = SU(n)/\mathbb{Z}_n \otimes \mathbb{R}^{2n}$. The triangleland case of this is $\mathfrak{v} \otimes \mathfrak{g}_{\text{can}} = SO(3) \otimes \mathbb{R}^3$ as per Sec 18.2 with $U^A \longrightarrow \text{Dra}^\Gamma$. The quadrilateralland subcase of this has 8 $SU(3)$ objects [Secs 4.1.6, 4.3] and 6 $\mathbb{R}^6 = \mathbb{C}^3$ objects.

For the scaled case, a possibly useful Lemma it as follows.

Lemma. Given a kinematical quantization algebra \mathfrak{C} for a shape space, then if the corresponding relational space has no 'extra' symmetries, the kinematical quantization algebra of the corresponding relational space is $\mathfrak{C} \otimes \mathfrak{Aff}$ for \mathfrak{Aff} the 'radial'/ \mathbb{R}_+ problem's affine algebra.

⁵⁷For 4-stop metroland, this is the case in \mathbf{H} -coordinates, while in \mathbf{K} -coordinates these are linear combinations of the Dil_i .

However, this fails for N -stop metroland's relational space $\mathcal{R}(N, 1) = \mathbb{R}^n$ as this has a number of extra symmetries. Moreover, this particular case's mathematics is, of course well-known (Sec 6.2.2).

It also likewise fails for triangleland's relationalspace \mathbb{R}^3 , which works like the $n = 3$ case of the preceding except that the ρ^i and π_i are now, rather Dra^Γ and Π_Γ^{Dra} . Note that using e.g. parabolic coordinates instead does not change the underlying canonical group. This should be related to coordinate changes merely leading to new bases of wavefunctions that are linear combinations of other coordinatizations'.

An appropriate kinematical quantization [332] for scaled triangleland involves $\text{Eucl}(3) \otimes \mathbb{R}^3$. as per Sec 18.2 with $U^A \rightarrow \text{Dra}^\Gamma$, $\underline{P}_A \rightarrow \Pi_\Gamma^{\text{Dra}}$. A choice of objects is then Dra^Γ for the first \mathbb{R}^3 , the translational generators Π_Γ^{Dra} and the $\text{SO}(3)$ generators \mathcal{S}_Γ . The above Lemma could still be of use as is not yet known to me whether $C(\mathbb{CP}^2)$ and $C(\mathbb{CP}^n)$ have extra symmetries.

For pure-shape 4-stop metroland, the nontrivial commutation relations are then the $U^A \rightarrow n^i$ and $\mathcal{S}_A \rightarrow \text{Dil}_i$ of Sec 18.1. These can also be written in the following dual form, with $n_i = \delta_{ij} n^j$ and $\text{Dil}_{ij} = n_i p_j - n_j p_i$,

$$[\widehat{\text{Dil}}_{ij}, \widehat{\text{Dil}}_{kl}] = i\hbar\{\delta_{ik}\widehat{\text{Dil}}_{jl} + \delta_{jl}\widehat{\text{Dil}}_{ik} - \delta_{il}\widehat{\text{Dil}}_{jk} - \delta_{jk}\widehat{\text{Dil}}_{il}\} , \quad (503)$$

$$[\widehat{n}^i, \widehat{\text{Dil}}_{kl}] = i\hbar\{\delta_{lj}\delta^i_k - \delta_{lk}\delta^i_j\}\widehat{n}^k , \quad (504)$$

which has the benefit of extending to all of the higher- N pure-shape N -stop metrolands. For pure-shape triangleland, the spherical presentation's nontrivial commutation relations are the $U^A \rightarrow \text{dra}^\Gamma$ and $\mathcal{S}_A \rightarrow \mathcal{S}_\Gamma$ of Sec 18.1. For scaled triangleland, the spherical presentation's nontrivial commutation relations are then the $U^A \rightarrow \text{Dra}^\Gamma$, $\underline{P}_A \rightarrow \Pi_\Gamma^{\text{Dra}}$ and $\mathcal{S}_A \rightarrow \mathcal{S}_\Gamma$ of Sec 18.3.

6.3.3 KinQuant for GR

The geometrodynamics case of kinematical quantization involves $\mathcal{C}^\infty(\Sigma, M(3, \mathbb{R})) \otimes \mathcal{C}^\infty(\Sigma, \text{GenLin}^+(3, \mathbb{R}))$ where the latter factor is closely associated with the mathematical identity of $\text{Riem}(\Sigma)$ GenLin stands for 'general linear' and $M(3, \mathbb{R})$ are real 3×3 matrices.

Analogy 55) At the level of kinematical quantization, 1 and 2- d RPM's are contained within Isham's work [332]; so is GR.

Difference 21) However, the ensuing representation theory in the GR case is considerably harder and to date largely impassible in any substantial detail. (The 1- and 2- d RPM examples involve $\text{SO}(n)$ and $\text{SU}(n)/\mathbb{Z}_n$ as canonical groups, and the representation theory of these is, of course, elementary and well known from particle physics.)

As regards commutation relations, naïvely geometrodynamics' $h_{\mu\nu}$ and $\pi^{\mu\nu}$ might follow the simplest ('plain') case 487: promote to $\widehat{h}_{\mu\nu}$ and $\widehat{\pi}^{\mu\nu}$ obeying

$$[\widehat{h}_{\mu\nu}(x), \widehat{\pi}^{\rho\sigma}(x')] = 2i\hbar\delta_{(\mu}{}^\rho\delta_{\nu)}{}^\sigma\delta^3(x, x') . \quad (505)$$

However, classically, there is an inequality on the determinant

$$\det h > 0 \quad (506)$$

(the nondegeneracy condition). This looks more like quantizing \mathbb{R}_+ than \mathbb{R} , and already that has a distinct kinematical quantization from the naïve one. The affine geometrodynamics [340, 341, 381] commutation relations that take this into account are

$$[\widehat{h}_{\mu\nu}(x), \widehat{\pi}_\rho{}^\sigma(x')] = 2i\hbar\widehat{h}_{\rho(\nu}\delta_{\mu)}{}^\sigma(x)\delta^3(x, x') . \quad (507)$$

I mention that Otriangleland has a similar inequality $\text{Area} \geq 0$, whilst 3-cornerland has an even more similar one: $\text{Area} > 0$.

6.3.4 Further global issues

1) There are global issues [332] that stem from $\pi_1(\mathbf{q}) \neq 0$. (However, this is 0 for all complex projective spaces and for spheres other than the circle, and for \mathbb{R}^k though not for this with points deleted as the potential may require.)

2) There are also global issues that stem from Chern class nontriviality. The first Chern class comes into the classification of the twisted representations. The second Chern class is related to the instanton number Here, for pure-shape RPM's second Chern classes are trivial for triangleland's $\mathbb{S}^2 = \mathbb{CP}^1$ (and 4-stop metroland) but are nontrivial for quadrilateralland's \mathbb{CP}^2 (from Sec 3.6.5).

3) For $\mathbf{q} = \mathbf{r}/\mathbf{g}$, the cocycles of \mathbf{r} lead to a further range of global effects; I have not as yet thought about these for RPM's. See [332] for the counterparts of the above global issues in GR. Note that for a theory like GR, the involvement of $\pi_1(\mathbf{q})$ is unlikely to bear much connection to one's final quantum theory.

That the configuration spaces in question are homogeneous spaces carries the further implication of there being a unique orbit, which allows for group quantization techniques to be straightforward.

6.4 Further comments on Quantum Relationalism

Barbour-relationalism's indirect implementation at the classical level gives Quad and Lin . The r-approach gives Quad^r . Both these schemes are relational, thus Dir and Red schemes are both relational. Some comments are necessary.

1) QM is entrenched upon the Hamiltonian formalism and thus can be taken to be free of temporal relationalism issues at the outset (though it clearly has a Problem of Time, see Part III). This is for RPM's or GR, for which we have a frozen equation. In ordinary QM the wave equation has a right-hand-side featuring $i\hbar\partial\Psi/\partial(\text{Newtonian time})$ in the nonrelativistic case, and some other time derivative combinations in SR cases such as the Klein–Gordon equation and the Dirac equation. We cannot have such a side a priori in a relational theory, but may come to an equation of that form via emergence of an acceptable notion of time.

2) Group averaging/refined algebraic quantization looks to be a good *quantum* implementation of configurational relationalism. Given a non-configurationally-relational operator \hat{O} ,

$$\int_{\mathfrak{g}} \exp\left(i \sum_z \mathfrak{g}_{\mathbf{g}z}^{\rightarrow}\right) \hat{O} \exp\left(-i \sum_z \mathfrak{g}_{\mathbf{g}z}^{\rightarrow}\right) \mathbb{D}\mathfrak{g} \quad (508)$$

(for $\mathfrak{g}_{\mathbf{g}z}$ the action of infinitesimal generators) is a relational counterpart. One can do similarly to pass from non-configurationally relational states to configurationally-relational ones. Admittedly, there are well-definedness issues in general as regards the measure $\mathbb{D}\mathfrak{g}$ over the group \mathfrak{g} . In particular, it is not clear how the diffeomorphisms could be treated explicitly in this way. See Sec 14 for further discussion.

Note 1) Sec 1's tangibility, temporal relationalism and configurational relationalism ideas still hold here, but *implementations* are different: there are no actions now (unless one goes down the path integral route, c.f. Sec 15).

Note 2) Group averaging is essentially a variant/particular implementation of the Dirac quantization approach. Methods along these lines are fairly widely used in LQG (e.g. in the Master Constraint program [606, 210]). In this sense, LQG is configurationally relational at the quantum level.

Barbour has largely not written about QM relationalism along the above lines beyond the general idea that Dirac Quantization is an indirect scheme for it (at the very least Rovelli and Smolin stated that [539, 583]). He has on the other hand considered timeless records approach which has some relational features.

6.5 Extending primality of \mathfrak{q} to QM?

Recollect that Relationalism 3) is that \mathfrak{q} is primary, and its categorized extension is that, rather, in anticipation of quantization, it would then be $(\mathfrak{q}, \text{Point})$ that is primary. Then use just $(\mathfrak{q}, \text{Point})$ or, less limitedly, $(\text{RigPhase}, \text{Can})$ with cases with nontrivial \mathfrak{g} having additional 'A' and 'DA' options. Then applying KinQuant, we get to $(\text{RigComAl}, \text{Point})$, where RigComAl likewise allots distinction to position variables, so that the only admissible morphisms are those of the position variables.

Motivation 1) On the one hand, one often hears that QM unfolds on configuration space, on the other hand most hold that it unfolds equally well on whichever polarization within phase space, of which configuration space is but one.

Motivation 2) I note that Isham's procedure for kinematic quantization itself favours \mathfrak{q} : the canonical group comes from just this and then the \mathfrak{v} is controlled by \mathfrak{q} and its associated \mathfrak{g} and $\mathfrak{g}_{\text{can}}$ so everything is controlled by configuration space too.

Motivation 3) I would expect then also for the Pre-Hilbert space to respect Point, and for the operator-orderings of the constraints likewise. This enables a connection with a conjecture of DeWitt concerning operator-ordering in Sec 6.7 .

[Though this SSsec's somewhat outlandish suggestion is not the only possible reasoning behind that, indeed if Can plays a role, Can is an enlargement of Point, and Point is then the restriction 'the Can that act on $M_{AB}(\mathbb{Q}^C)$ ', which suffices to cover this DeWitt point.]

Note 1) Using just Point does *not* sort out all aspects of unitary inequivalence either, though it may help/may be a step in the right direction (which Can's or Point's form up into classes that are each entirely unitarily equivalent to each other?)

Note 2) RigPhase, Point has the same objects and brackets as Phase and hence the same KinQuant. But in restricting Can to Point as morphisms preserving Poisson brackets and the rigging, then ComAl's morphisms would likewise be restricted. A possible problem with this view is that QM's P's and Q's may be more alike than their classical counterparts.

Note 3) The Naïve Schrödinger Interpretation can be taken to be a $(\mathfrak{q}, \text{Point})$ scheme: $\psi(\mathfrak{q})$, and probabilities built solely out of those and a characterization of regions of \mathfrak{q} . Prerecords Theory can be taken likewise (this is defined in Sec 11.8.8). Though this may not be enough. A question then is whether using $(\text{RigPhase}, \text{Point})$ can cover all physical propositions. One can try this with $(\mathbb{Q}, \text{Point})$ or $(\text{RigPhase}, \text{Point})$. My treatment of Histories Theory in Secs 11 and 15 will comment further on this philosophy. Mistrusting Can breeds mistrust in building Can on other than \mathfrak{q} ... One can view the Histories postulate as supplanting Relationalism 3), perchance going against the rest of dyn-based physics like spacetime. But the two are different, and spacetime was in many ways fruitful, so here too I will counsel live and let live for now.

6.6 The quantum wave equations

Dirac quantization involves quantizing and then constraining. The general form for the wave equations here is

$$\widehat{\mathcal{Q}}_{\text{quad}}\Psi = 0, \quad \widehat{\mathcal{L}}_{\text{in}}\Psi = 0. \quad (509)$$

6.6.1 The GR case

For GR as geometrodynamics, the quadratic constraint at the quantum level is the Wheeler–DeWitt equation,

$$\widehat{\mathcal{H}}\Psi := -\hbar^2 \left\{ \frac{1}{\sqrt{\mathcal{M}}} \frac{\delta}{\delta h^{\mu\nu}} \left\{ \sqrt{\mathcal{M}} \mathcal{N}^{\mu\nu\rho\sigma} \frac{\delta\Psi}{\delta h^{\rho\sigma}} \right\} - \xi \text{Ric}(\mathcal{M}) \right\} \Psi - \sqrt{\hbar} \text{Ric}(h)\Psi - \sqrt{\hbar} \Lambda \Psi + \widehat{\mathcal{H}}_{\text{matter}}\Psi = 0, \quad (510)$$

where Ψ is the wavefunction of the universe. The inverted commas indicate that the Wheeler–DeWitt equation has various technical problems in addition to the Problem of Time.

1) There are problems with regularization, which is not at all straightforward for an equation for a theory of an infinite number of degrees of freedom in the absence of background structure.

2) The mathematical meaningfulness of functional differential equations is open to question.

Difference 22) RPM's do not serve to model other specific infinite dimensional/field theory issues including whether the Wheeler–DeWitt equation is well-defined.

3) There is an operator-ordering ambiguity [see Sec 6.7 of this, including for an explanation of what ξ is].

The Wheeler–DeWitt equation comes alone in basic minisuperspace models, or, more generally, accompanied by the quantum linear momentum constraint

$$\widehat{\mathcal{M}}_{\mu}\Psi = -2\frac{\hbar}{i}h_{\mu\nu}D_{\rho}\frac{\delta}{\delta h_{\nu\rho}}\Psi + \mathcal{M}_{\mu}^{\text{matter}}\Psi = 0. \quad (511)$$

There is not much scope in full traditional-variables GR for reduced quantization. LQG has some scope in this regard (knot states) [246].

6.6.2 The RPM wave equations in the Dirac formulation

For scaled RPM,

$$\widehat{\mathcal{E}}\Psi = -\frac{\hbar^2}{2}\delta^{\alpha\beta}\delta^{ij}\frac{\partial}{\partial\rho^{i\alpha}}\frac{\partial}{\partial\rho^{j\beta}}\Psi + V(\rho^{k\gamma})\Psi = E\Psi \quad (512)$$

(note that this has no operator ordering ambiguity as the configuration space metric is independent of $\rho^{i\alpha}$ and the configuration space is flat) with the quantum zero total angular momentum constraint

$$\widehat{\mathcal{L}}\Psi = \frac{\hbar}{i}\sum_{i=1}^n \underline{\rho}^i \times \frac{\partial}{\partial \underline{\rho}^i}\Psi = 0, \quad (513)$$

which parallels the GR quantum momentum constraint.

For pure-shape RPM in the geometrically natural formulation,

$$\widehat{\mathcal{E}}\Psi = -\frac{\hbar^2}{2}\text{I}\delta^{\alpha\beta}\delta^{ij}\frac{\delta}{\delta\rho^{i\alpha}}\frac{\delta}{\delta\rho^{j\beta}}\Psi + V(\rho^{k\gamma})\Psi = E\Psi, \quad (514)$$

alongside the quantum zero total angular momentum constraint and the dilational momentum constraint,

$$\widehat{\mathcal{D}}\Psi = \frac{\hbar}{i}\sum_{i=1}^n \underline{\rho}^i \cdot \frac{\partial}{\partial \underline{\rho}^i}\Psi = 0. \quad (515)$$

Difference 23) The definiteness-indefiniteness Difference 16) causes RPM wave equations to be elliptic-like rather than the hyperbolic-like Wheeler–DeWitt equation of GR. Thus the skill here is to solve constrained, or curved-space, elliptic-like equations.

Note: I credit [583] for an early article on Dirac Quantization for scaled RPM. The first such for pure-shape RPM was [20].

6.6.3 The Problem of Time

N.B. $\widehat{\text{Quad}}\Psi = 0$ as exemplified by the Wheeler–DeWitt equation or the RPM quantum energy-type equation is frozen: it is like $H\Psi = E\Psi$ (for $E = 0$) rather than like $H\Psi = i\hbar\partial\Psi/\partial t$ for some notion of time t .

Analogy 56) RPM’s manifest the frozen formalism facet of the Problem of Time We can see this from the form of (512, 514).

Part III has numerous further analogies – more facets of the Problem of Time, and proposed strategies for dealing with the Problem of Time [98, 400, 79, 80, 83, 372, 20, 22, 40, 28, 29]. Further facets of the Problem of Time are discussed in Sec 11.3.3, and strategies for dealing with the Problem of Time are discussed in Secs 12–15.4. The presence of the linear constraints complicates a number of these matters.

6.6.4 Variants on the Wheeler–DeWitt equation

Note 1) The affine approach has a different Wheeler–DeWitt equation [340, 341, 381].

Note 2) Ashtekar variables/LQG/LQC approaches have different-looking Wheeler–DeWitt equations. These include discrete versions of the Hamiltonian constraint in LQC [129], and come with an additional $SU(2)$ constraint equation in the Dirac scheme for Ashtekar variables. One may view the loop representation approach [246] as providing a more reduced version of this. However, what is toy-modelled in the present article are the geometrodynamics and conformogeometrodynamics approaches. These approaches are fine as regards conceptual consideration of the Problem of Time (and are indeed the setting for which this has been the most developed).

6.6.5 The Problem of Observables

Is there a sufficiently complete set of observables for gravitational theory, and if so what is such a set? Some literature addressing this question is [55, 198, 612, 542, 544, 545, 586, 599]. See also Sec 11.11 for generalities and Sec 16.11 for RPM examples.

6.7 Operator-Ordering Problem I

For now I assume there is no nontrivial \mathfrak{g} . The quadratic constraint’s (16) classical product combination of configurations and their momenta

$$N^{AB}(Q^C)P_A P_B \quad (516)$$

gives rise to an operator ordering ambiguity upon quantization. N.B. how one operator-orders has consequences for the physical predictions of one’s theory, and there is no established way to prescribe the operator ordering in the case of (toy models of) Quantum Gravity. One line of argument which picks out certain operator orderings is as follows.

Einstein’s general covariance of spacetime is a principle for classical physics: there is to be no dependence of classical physics on how those who study it happen to coordinatize it.

It is held to be an uncontroversial extension of this principle for this principle to also apply to space.

Question: More general covariance? What about taking this principle to hold for such as configuration space or phase space? I furthermore then ask whether there a limit to how far this principle can (or is fruitful to) be taken along the hierarchy of abstract manifolds that one encounters in physical study? If Relationalism 3) (primality of \mathfrak{q}) applies, this could signify a barrier on how far along the sequence of manifolds associated with physical systems one should take general covariance.

DeWitt’s general covariance. DeWitt [199] considered elevating the coordinatization-independence of configuration space to additionally hold at the quantum level.

My relational underpinning. Insofar as Relationalism 3) is a relational postulate and its categorification 3’) is a necessary enlargement of it with quantization in mind, relationalism underpins DeWitt’s general covariance. At the very least, this is tied to a belief that specifically Point continues to play a role in the choice of operator-ordering, which is quite deep into the quantization procedure.

[Doubts cast on canonical transformations in Appendix 4.C might suggest no further general covariance beyond this point.]

Laplacian operator ordering

$$N^{AB}(Q^C)P_A P_B \longrightarrow \triangle = \frac{1}{\sqrt{M}} \frac{\nabla}{\nabla Q_A} \left\{ \sqrt{M} N^{AB}(Q^C) \frac{\nabla}{\nabla Q_B} \right\} , \quad (517)$$

is then an implementation of DeWitt's general covariance.

Moreover, it is not a unique implementation: one can include a Ricci scalar curvature term so as to have the

ξ -operator ordering

$$\triangle^\xi := \triangle - \xi \text{Ric}(\mathbf{M}) \quad (518)$$

[199, 315, 170, 291, 480, 547] for any real number ξ . N.B. this gives inequivalent physics for each value of ξ .

Barvinsky's approximate equivalence: the physics for all of these does coincide to $O(\hbar)$ [101, 102, 103], as is relevant to approximate semiclassical physics; thus the above inequivalence is only relevant if one wishes to investigate the physics to $O(\hbar^2)$ or better.

Underlying simplicity: the above is the extent of the ambiguity only if one excludes more complicated curvature scalars e.g. by stipulating no higher-order derivatives nor higher-degree polynomials in the derivatives.

Conformal operator ordering. Among the ξ -orderings, it is then well-known [622] that there is then a unique configuration space dimension $k > 1$ -dependent conformally-invariant choice:

$$\triangle^c := \triangle - \xi^c \text{Ric}(\mathbf{M}) := \triangle - \frac{k-2}{4\{k-1\}} \text{Ric}(\mathbf{M}) . \quad (519)$$

(This furthermore requires that the Ψ itself that it acts upon itself transforms in general tensorially under \mathbf{q} -conformal transformations [622]).⁵⁸ What is the underlying conformal invariance in question? [E.g. it is *not* of space itself.]

Misner's identification [464]: it is that of the Hamiltonian constraint under scaling transformations.

$$\mathcal{H} = 0 \longrightarrow \tilde{\mathcal{H}} = 0 . \quad (520)$$

This can be generalized to invariance under

$$\mathcal{Q} = 0 \longrightarrow \mathcal{Q}^r = 0 . \quad (521)$$

My identification [32] goes one level deeper to the consideration of actions. It then so happens that it is the PPSTC invariance (197) of the relational product-type parageodesic-type action that most cleanly underlies Misner's identification. This makes it clear that it is a conformal invariance of the kinetic arc element ds alongside a compensatory conformal invariance in the potential factor W , reflecting that the combination actually present in the action, $d\tilde{s}$, is not physically meaningfully factorizable, as per Appendix 2.B.⁵⁹ As I have previously argued, these are, furthermore, appropriate in the whole-universe context that is the setting for Quantum Cosmology or toy models thereof. Thus, demanding conformal operator ordering can be seen as choosing to retain this simple and natural invariance in passing to the quantum level (which one is free to do in the absence of unrelated technical caveats to the contrary).

The time-independent Schrödinger equation following from the above family of orderings is then

$$\mathcal{Q} \Psi = 0 \Rightarrow \triangle^\xi \Psi = 2\{V - E\} \Psi / \hbar^2 . \quad (522)$$

Surveying the literature, Kuchař [394] and Henneaux–Pilati–Teitelboim [319] have advocated the Laplacian ordering itself. So have Page [498], Louko [430] and Barvinsky [101, 102, 103], however their specific examples are 2-dimensional, for which the Laplacian and conformal orderings coincide. The conformal ordering, which fixes a particular value of ξ , had been previously suggested in the quantum-cosmological context by e.g. Misner [464], Halliwell [291], Moss [480], Ryan–Turbiner [547] and I [32]. Wiltshire advocates both [642]. Christodoulakis and Zanelli [170] consider the arbitrary- ξ case, as do Hawking and Page [315], albeit the latter then also pass to a $2-d$ example for which ξ drops out (see Sec 6.8.2 for why).

An additional snag with advocating conformal ordering for geometrodynamics itself is that k is infinite so the conformally-transformed wavefunction becomes ill-defined. However in the geometrodynamical quadratic wave equation, the coefficient of $\text{Ric}(M)$ merely tends to $1/4$.

On the other hand, the k 's in working (575) continue to formally cancel, and it is the outcome of this (including its operator expectation counterpart), rather than Ψ itself, that has physical meaning. This gives my **conformal-ordered Wheeler–DeWitt equation** [32],

$$\hbar^2 \left\{ \frac{1}{\sqrt{\mathcal{M}}} \frac{\delta}{\delta h_{\mu\nu}} \left\{ \sqrt{\mathcal{M}} \mathcal{N}^{\mu\nu\rho\sigma} \frac{\delta}{\delta h_{\rho\sigma}} \right\} - \frac{1}{4} \text{Ric}(\mathcal{M}) \right\}' \Psi + \sqrt{\hbar} \{ \text{Ric}(h) - 2\Lambda \} \Psi = 0 . \quad (523)$$

⁵⁸ $k \leq 1$ is a non-issue in relational thinking as per 2.1.5.

⁵⁹ One could also consider the more complicated manifestation (198) at the level of the more usual difference-type action. Misner himself, for all that he used the words “parageodesic principle” [464], was referring to an action with integrand of the general form Liouville operator – Hamiltonian. My own contribution is to relate this conformal invariance to the *geometrically obvious* parageodesic principle, which is a relational product-type action.

6.8 r-RPM wave equations

6.8.1 Laplacian operators on shape space and relational space

For pure-shape RPM's, I denote the shape space Laplacian by

$$\Delta_{\mathfrak{S}(N,d)} = \frac{1}{\sqrt{M}} \frac{\partial}{\partial S^a} \left\{ \sqrt{M} N^{ab} \frac{\partial}{\partial S^b} \right\} \quad (524)$$

(or Δ_S for a shorthand). Then specifically, for N -stop metroland,

$$\Delta_{S^{n-1}} = \prod_{j=1}^{n-2} \sin^{j-n+1} \theta_j \frac{\partial}{\partial \theta_a} \left\{ \frac{\prod_{j=1}^{n-2} \sin^{n-1-j} \theta_j}{\prod_{i=1}^{a-1} \sin^2 \theta_i} \frac{\partial}{\partial \theta_a} \right\} = \frac{1}{\sin^{nd-1-A} \theta_A \prod_{i=1}^{A-1} \sin^2 \theta_i} \frac{\partial}{\partial \theta_A} \left\{ \sin^{nd-1-A} \theta_A \frac{\partial}{\partial \theta_A} \right\}, \quad (525)$$

(which result also readily extends to preshape space under n to nd), whilst for N -a-gonland

$$\Delta_{\mathbb{C}P^{n-1}} = \frac{\{1 + \|\mathcal{R}\|^2\}^{2n-2}}{\prod_{\bar{p}=1}^{n-1} \mathcal{R}_{\bar{p}}} \left\{ \frac{\partial}{\partial \mathcal{R}_{\bar{p}}} \left\{ \frac{\prod_{\bar{p}=1}^{n-1} \mathcal{R}_{\bar{p}}}{\{1 + \|\mathcal{R}\|^2\}^{2n-3}} \{ \delta^{\bar{p}\bar{q}} + \mathcal{R}^{\bar{p}} \mathcal{R}^{\bar{q}} \} \frac{\partial}{\partial \mathcal{R}_{\bar{q}}} \right\} + \frac{\partial}{\partial \Theta_{\bar{p}}} \left\{ \frac{\prod_{\bar{p}=1}^{n-1} \mathcal{R}_{\bar{p}}}{\{1 + \|\mathcal{R}\|^2\}^{2n-3}} \left\{ \frac{\delta^{\bar{p}\bar{q}}}{\mathcal{R}_{\bar{p}}^2} + 1 \right\} \frac{\partial}{\partial \Theta_{\bar{q}}} \right\} \right\}. \quad (526)$$

The triangleland subexample of this is then, in the barred PPSCT representation in spherical coordinates, the $\alpha, \chi \longrightarrow \Theta, \Phi$ of Sec 18.4. On the other hand, in the tilded PPSCT representation, the triangleland case's Laplacian in the flat coordinates (\mathcal{R}, Φ) takes the familiar form

$$\Delta_{\mathbb{R}^2} = \frac{1}{\mathcal{R}} \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \frac{\partial}{\partial \mathcal{R}} \right\} + \frac{1}{\mathcal{R}^2} \frac{\partial^2}{\partial \Phi^2} \quad (527)$$

(to be used approximately only, so that the global discrepancy is not an issue).

For scaled RPM's in scale-shape split form,

$$\Delta_{\mathcal{R}(N,d)} = \Delta_{C(\mathfrak{S}(N,d))} = \frac{1}{\sqrt{M}} \frac{\partial}{\partial Q^A} \left\{ \sqrt{M} N^{AB} \frac{\partial}{\partial Q^B} \right\} = \rho^{nd-d\{d-1\}/2-1} \partial_\rho \{ \rho^{-nd+d\{d-1\}/2+1} \partial_\rho \} + \rho^{-2} \Delta_{\mathfrak{S}(N,d)}. \quad (528)$$

We can use this to build all the Laplacians, e.g. in 1- d scaled case

$$-\rho^{1-n} \{ \rho^{n-1} \Psi_{,\rho} \}_{,\rho} + \rho^{-2} \Delta_{S^{n-1}}, \quad (529)$$

bar the spherical presentation of triangleland one, which is

$$\check{\Delta}_{C(\mathbb{S}^2)} = \frac{1}{I^2} \left\{ \frac{\partial}{\partial I} \left\{ I^2 \frac{\partial}{\partial I} \right\} + \Delta_{\mathbb{S}^2} \right\}. \quad (530)$$

6.8.2 Some simpler cases of Δ^ξ operator-ordering

Simplification 1) For models with 2- d configuration spaces the conformal value of $\xi^c = \{k-2\}/4\{k-1\}$ collapses to zero, so this case reduces to the Laplacian ordering and conformally invariant wavefunctions. I.e. $\Delta^c = \Delta$ for pure-shape 4-stop metroland and triangleland and for scaled 3-stop metroland.

Simplification 2) For models with zero Ricci scalar, all of the ξ -orderings reduce to the Laplacian one. Thus in particular $\Delta^c = \Delta$ for all scaled N -stop metrolands.

Simplification 3) If a space has constant Ricci scalar, then the effect of a $\xi \text{Ric}(M)$ term, conformal or otherwise, is just something which can be absorbed into redefining the energy in the case of mechanics. Parallely, were the Ricci scalar constant in a GR model, it could likewise be absorbed into redefining the cosmological constant. This clearly applies to pure-shape N -stop metrolands and N -a-gonlands.

However, almost all other minisuperspace models and relational particle mechanics models (e.g. [37]) have configuration space dimension ≥ 3 for which the choice of a value of ξ is required. [E.g. $C(\mathbb{C}P^2)$ is not even conformally flat, see Sec 3.8.7.]

6.8.3 The remaining RPM cases of Δ^c

For pure-shape N -stop metroland, $\Delta_{S^{n-1}}^c = \Delta_{S^{n-1}} - \{n-1\}\{n-3\}/4$ by simplification 3, which discrepancy from the Laplacian is incorporable via the shift

$$E \longrightarrow E_{S^{n-1}}^c = E - \hbar^2 \{n-1\}\{n-3\}/8. \quad (531)$$

For pure-shape N -a-gonland, $\Delta_{\mathbb{C}P^{n-1}}^c = \Delta_{\mathbb{C}P^{n-1}} - 2n\{n-1\}\{n-2\}/\{2n-3\}$ by simplification 3, which discrepancy from the Laplacian is incorporable via the shift

$$E \longrightarrow E_{\mathbb{C}P^{n-1}}^c = E - \hbar^2 2n\{n-1\}\{n-2\}/\{2n-3\}. \quad (532)$$

For scaled N -a-gonland,

$$\Delta_{C(\mathbb{C}P^{n-1})}^c = \Delta_{C(\mathbb{C}P^{n-1})} \Psi - \frac{3n\{2n-3\}}{4\{n-1\}\rho^2} \quad (533)$$

(see the next SSsec for comments).

6.8.4 The RPM wave equations

A master schematic time-independent Schrödinger equation for pure-shape RPM is

$$\Delta_{\mathfrak{S}(N,d)}^c \Psi = 2\{V - E\}\Psi/\hbar^2 . \quad (534)$$

The N -stop metroland case of this is

$$\Delta_{\mathbb{S}^{n-1}} \Psi - \{n-1\}\{n-3\}\Psi/4 = \Delta_{\mathbb{S}^{n-1}}^c \Psi = 2\{V - E\}\Psi/\hbar^2 , \quad (535)$$

which is rewriteable as

$$\Delta_{\mathbb{S}^{n-1}} \Psi = 2\{V - E_{\mathbb{S}^{n-1}}^c\}\Psi/\hbar^2 . \quad (536)$$

These results also readily extend to the case of wavefunctions on preshape space under n to nd .⁶⁰

The N -a-gonland case of this is

$$\Delta_{\mathbb{CP}^{n-1}} \Psi - 2n\{n-1\}\{n-2\}\Psi/\{2n-3\} = \Delta_{\mathbb{CP}^{n-1}}^c \Psi = 2\{V - E\}\Psi/\hbar^2 , \quad (537)$$

which is rewriteable as

$$\Delta_{\mathbb{CP}^{n-1}} \Psi = 2\{V - E_{\mathbb{CP}^{n-1}}^c\}\Psi/\hbar^2 . \quad (538)$$

The spherical presentation of triangleland's time-independent Schrödinger equation is

$$\Delta_{\mathbb{S}^2} \Psi = 2\{\check{V} - \check{E}\}\Psi/\hbar^2 . \quad (539)$$

On the other hand, in plane polar coordinates $\{\mathcal{R}, \Phi\}$ obtained by passing to stereographic coordinates on the sphere and then passing to the 'overlined PPSCCT representation' $\bar{\mathbf{T}} = \mathbf{T}\{1 + \mathcal{R}^2\}^2$ and $\bar{\mathbf{E}} - \bar{\mathbf{V}} = \{\mathbf{E} - \mathbf{V}\}/\{1 + \mathcal{R}^2\}^2$, the time-independent Schrödinger equation is

$$\Delta_{\mathbb{R}^2}^c = \Delta_{\mathbb{R}^2} = \left\{ \frac{1}{\mathcal{R}} \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \frac{\partial \Psi}{\partial \mathcal{R}} \right\} + \frac{1}{\mathcal{R}^2} \frac{\partial^2 \Psi}{\partial \Phi^2} \right\} = 2\{\bar{\mathbf{V}}(\mathcal{R}) - \bar{\mathbf{E}}(\mathcal{R})\}\Psi/\hbar^2 . \quad (540)$$

A master schematic time-independent Schrödinger equation for scaled RPM is

$$\Delta_{\mathcal{R}(N,d)}^c \Psi = 2\{V - E\}\Psi/\hbar^2 . \quad (541)$$

In particular, for the N -stop metroland case,

$$\Delta_{\mathbb{R}^n} \Psi = \Delta_{\mathbb{R}^n}^c \Psi = 2\{V - E\}\Psi/\hbar^2 , \quad (542)$$

with the special cases (1045) in 2- d polars with $R \rightarrow \rho$, $\chi \rightarrow \varphi$ and (1043) in 3- d spherical polars with $R \rightarrow \rho$, $\alpha \rightarrow \theta$, $\chi \rightarrow \phi$. For the N -a-gonland case,

$$\Delta_{\mathbb{C}(\mathbb{CP}^{n-1})} \Psi - \frac{3n\{2n-3\}}{4\{n-1\}\rho^2} \Psi = \Delta_{\mathbb{C}(\mathbb{CP}^{n-1})}^c \Psi = 2\{V - E\}\Psi/\hbar^2 . \quad (543)$$

Note that the conformal contribution here can no longer simply be used up shifting the energy constant. Instead, it adds to the separation constant of the shape part from the scale part in the separable case. (This is similar to the difference between monopole harmonics and ordinary spherical harmonics except that it applies to the radial equation rather than to the angular equation.)

The triangleland case's scaled triangleland time-independent Schrödinger equation is

$$\check{\Delta}_{\mathbb{R}^3}^c \check{\Psi} = \check{\Delta}_{\mathbb{R}^3} \check{\Psi} = \widehat{\mathcal{T}}_{\text{ot}} \check{\Psi} = \frac{1}{\mathbf{I}^2} \left\{ \frac{\partial}{\partial \mathbf{I}} \left\{ \mathbf{I}^2 \frac{\partial \check{\Psi}}{\partial \mathbf{I}} \right\} + \frac{1}{\sin \Theta} \frac{\partial \check{\Psi}}{\partial \Theta} \left\{ \sin \Theta \frac{\partial \check{\Psi}}{\partial \Theta} \right\} + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \check{\Psi}}{\partial \Phi^2} \right\} = \frac{2\{\check{\mathbf{V}}(\Phi, \Theta) - \check{\mathbf{E}}(\Theta)\}}{\hbar^2} \check{\Psi} . \quad (544)$$

The non-conformally transformed spherical form for triangleland is, in the conformal-ordered case, using $\text{Ric}(\mathbb{R}^3 \text{ curved}) = 3/2\mathbf{I}^2$ and particular conformal case $\xi_c \text{Ric}(\mathbb{R}_{\text{curved}}^3) = 3/16\mathbf{I}^2$,

$$\Delta_{\mathbb{R}^3(\text{non-flat})}^c = 4 \left\{ \frac{1}{\mathbf{I}^{3/2}} \frac{\partial}{\partial \mathbf{I}} \left\{ \mathbf{I}^{3/2} \frac{\partial \Psi}{\partial \mathbf{I}} \right\} + \frac{1}{\mathbf{I}^2} \left\{ \frac{1}{\sin^2 \Theta} \frac{\partial \Psi}{\partial \Theta} \left\{ \sin^2 \Theta \frac{\partial \Psi}{\partial \Theta} \right\} + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Psi}{\partial \Phi^2} \right\} - \xi_c \text{Ric}(\mathbb{R}_{\text{curved}}^3) \Psi \right\} = 2 \left\{ V - \frac{E}{\mathbf{I}} \right\} \frac{\Psi}{\hbar^2} . \quad (545)$$

Also useful below, the parabolic coordinates version of the time-independent Schrödinger equation is

$$4 \left\{ \frac{\partial}{\partial \mathbf{I}_1} \left\{ \mathbf{I}_1 \frac{\partial \bar{\Psi}}{\partial \mathbf{I}_1} \right\} + \frac{\partial}{\partial \mathbf{I}_2} \left\{ \mathbf{I}_2 \frac{\partial \bar{\Psi}}{\partial \mathbf{I}_2} \right\} \right\} + \left\{ \frac{1}{\mathbf{I}_1} + \frac{1}{\mathbf{I}_2} \right\} \frac{\partial^2 \bar{\Psi}}{\partial \Phi^2} = \frac{\{V - E\}}{2\hbar^2} \bar{\Psi} . \quad (546)$$

⁶⁰The above quantum Hamiltonians do involve suitable quantum operators (see also e.g. p 160 of [524] for an account of the properties of the Laplacian operator on \mathbb{S}^{n-1} , and also [551, 382, 383]).

6.8.5 Closure of RPM quantum equations

These above schemes are, furthermore, ‘lucky’ in Dirac’s sense [209]: the full set of constraints closes at the quantum level. Moreover this is in direct parallel with the classical closure [(78) for scaled RPM and (78,115) for pure-shape RPM] under the correspondence principle $\{ , \} \longrightarrow \{i\hbar\}^{-1}[,]$, so I do not present it.

6.9 Absence of monopole problems in N -stop metroland and triangle land

Following on from Sec 4.5’s classical monopole considerations [these pertain to scheme D]’s split of Newtonian Mechanics itself] then the charged particle toy model’s position and the canonical momentum are good Hermitian operators. However, next in looking to form $SO(3)$ objects, \underline{A} ’s positional dependence complicates the commutation relations. This necessitates, beyond what is usual in usual treatments of angular momenta, the introduction of an extra term:

$$\underline{L}^{\text{extended}} = \underline{x} \times \{ \underline{p} - e\underline{A} \} - q\underline{x}/r^3 , \quad (547)$$

for

$$q = eg = n\hbar c/2 \quad (548)$$

(the last equality being the Dirac quantization condition). Next, $L_{\text{Total}}^{\text{extended}} = \sum_{\mu=1}^3 \{L_{\mu}^{\text{extended}}\}^2 = \{\underline{x} \times \{ \underline{p} - e\underline{A} \}\}^2 + q^2$, and $L_3^{\text{extended}}\Psi = \{-i\partial_{\phi_{\text{sp}}} - q\}\Psi$ in the N-chart and $L_3^{\text{extended}}\Psi = \{-i\partial_{\phi_{\text{sp}}} + q\}\Psi$ in the S-chart, so Ψ has an $\exp(i\{m \pm q\}\phi_{\text{sp}})$ factor rather than an $\exp(im\phi_{\text{sp}})$ factor, with integer $m \pm q$. [here the suffix ‘sp’ stands for ‘spatial’, so that these are the spherical angles in their usual application.] Consequently, the azimuthal part of the Schrödinger equation then picks up two extra terms as compared to (1050):

$$-\{\sin\theta_{\text{sp}}\}^{-1}\{\sin\theta_{\text{sp}}\Psi, \theta_{\text{sp}}\}_{,\theta_{\text{sp}}} + \{\sin\theta_{\text{sp}}\}^{-2}\{m + q\cos\theta_{\text{sp}}\}\Psi = \{1\{1 + 1\} - q^2\}\Psi , \quad (549)$$

and so that ‘monopole harmonics’ replace the ordinary spherical harmonics ([647]; e.g. [466] has a 2- d counterpart of closer relevance to the present article).

N.B. however how this working collapses in the case of an uncharged particle. The monopole is then not ‘felt’, so one has the mathematically-usual form of the $SO(3)$ operator and the mathematically-usual Schrödinger equation. Furthermore, for central potentials (i.e. potentials depending on the radial variable r alone), one has the mathematically-usual spherical harmonics. This difference from [31] is one reason why [37] has had to wait for the below resolution.

Reminder then that total angular momentum plays the role of charge in this analogy, so that the zero angular momentum case of relevance to the present article is not beset by such monopoles issues.

6.10 Operator ordering II: discrepancy between Quantum Cosmology and Molecular Physics

There turns out to be a discrepancy between the ‘relational portion’ of Newtonian Mechanics (scheme D) and that of the relationalspace-reduced approaches (schemes A-B-C.I). In scheme D), there is a chain of transformations as conventionally used in Molecular Physics can be understood. This is from particle positions to relative Jacobi coordinates to spherical coordinates plus absolute angles to Dragt-type coordinates plus absolute angles. [References for this are e.g. [348, 494, 453] and the much earlier but non-geometrical [662].] Consequently scheme D has an absolute block in its configuration space metric. Then, via the participation of this in the formation of the overall volume element $\sqrt{M_{\text{abs-rel}}}$, this enters the *relational* block’s part of the Laplacian, since $\sqrt{M_{\text{abs-rel}}}$ then sits inside the $\partial/\partial Q_{\text{rel}}$ derivative. Therefore, by this means, if absolute space is assumed, it leaves an imprint on the ‘relational portion’, which is absent if one considers a relationally-motivated Lagrangian (as in schemes A-B-C.I).

In any case, it is reasonable from the relational perspective for modelling a molecule in a universe to differ from modelling those self-same particles as a whole universe toy model, since the former case possesses an inertial frame concept due to the rest of the universe. Taking the RPM-geometrodynamics analogy as primary rather than trying to describe reality as a few (or even very many) non-specially-relativistic particles, gives serious reason *not* to use Molecular Physics’ quantum equations in the whole-universe model context. That mathematical analogy ends with the classical dynamics and the quantum *kinematics*.

In greater detail, firstly note that this difference is a consequence of the nontriviality of the rotations (there is no corresponding effect for the translations as the centre of mass motion block does not contribute any further functional dependences to the volume element. The reduced–relationalspace approaches’ Laplacian is $\Delta_{\mathfrak{S}(N,d)}$ in the pure-shape case and $\Delta_{\mathcal{R}(N,d)}$ in the scaled case. The Laplacian for the only-trivially-reduced convenient starting-point of relative space is $\Delta_{\mathfrak{r}(N,d)}$. The reduced–relationalspace versus restriction of relative space distinction is not relevant to scaled N -stop metrolands by $\mathfrak{R}(N,1) = \mathcal{R}(N,1)$. However,

$$\Delta_{\mathfrak{r}(N,1)}^{\text{c}}|_{\text{S-rel part}} = \Delta_{\mathfrak{r}(N,1)}|_{\text{S-rel part}} := \Delta_{\mathfrak{r}(N,1)}|_{\partial/\partial\rho=0} = \rho^2\Delta_{\mathbb{S}^{N-2}} = \rho^2\Delta_{\mathfrak{S}(N,1)} \neq \rho^2\Delta_{\mathfrak{S}(N,1)}^{\text{c}} , \quad (550)$$

though the last inequality is but by a constant which can be absorbed into a redefinition of the energy, as per (531). Also,

$$\check{\Delta}_{\mathcal{R}(3,2)} = \Delta_{\mathfrak{q}(3,2)}|_{\partial/\partial\rho=0} , \quad (551)$$

which is given by the fourth portion of eq (544), with $\Delta_{\mathcal{R}(3,2)}$ is distinct as given by the second portion of eq (545), so that these differ by $2\partial_I = \mathcal{D}/I$. Also, scheme D in the usual Molecular Physics context (spatially 3- d) gives, for the radial part of the Schrödinger equation,

$$\partial_I^2 + 5I^{-1}\partial_I . \quad (552)$$

On the other hand, purely relational considerations give the far more usual spherical-type radial part (551)

$$\partial_I^2 + 2I^{-1}\partial_I . \quad (553)$$

Note 1) This difference can be ascribed to an absolutist imprint which occurs at the quantum level and even within the drastic $\mathcal{L} = 0$ simplification within scheme D.

Note 2) Also note that the above Molecular Physics literature result is for 3-cornerland due to the molecules being studied living in 3- d . To be even more relevant to the present article's examples, one should consider the 2- d case. Then I note that the imprint of 2- d absolute space is different from that of 3- d absolute space. In that case there is a 3 and not a 5 in (552), corresponding to there being a relational 2 plus now an absolute 1 [for $SO(2)$] rather than an absolute 3 [for $SO(3)$].

Note 3) In 3- d the shape part also differs between the split of Newtonian Mechanics and the relationalspace approach. See e.g. [348, 494, 453, 662] for various coordinatizations of the former's Laplacian. This is due to this case's absolute block depending on relational angle as well as on scale. In 2- d the situation is simpler. Thus 3-cornerland's pure-shape workings, as well as workings with scale, differ between scheme A-B-C.I and scheme D.

Now also $\Delta_{\mathbf{q}(3,2)}^c = \Delta_{\mathbf{q}(3,2)}$ but $\Delta_{\mathcal{R}(3,2)}^c \neq \Delta_{\mathcal{R}(3,2)}$ and this is now not just a shift by a constant. Finally,

$$\Delta_{\mathbf{q}(3,2)}|_{S\text{-rel part}} = \Delta_{\mathbf{q}(3,2)}|_{\partial/\partial A=0, \partial/\partial \rho=0} = \Delta_{\mathbf{s}(3,2)} \quad (554)$$

(the above difference is now removed due to containing the new constraint as a factor). Though also

$$\Delta_{\mathbf{q}(3,2)}^c|_{S\text{-rel part}} = \Delta_{\mathbf{q}(3,2)}^c|_{\partial/\partial A=0, \partial/\partial \rho=0} \neq \Delta_{\mathbf{s}(3,2)}^c , \quad (555)$$

albeit the difference is again but an absorbable constant. (Here, A is an absolute angle).

Finally, I note that the purely relational operator ordering form that I adopt has the further theoretical advantage of having (well-)known solutions in a number of cases.

6.11 Operator-ordering III: spectre of Dirac–reduced inequivalence

Dirac quantization–reduced quantization inequivalence renders all of the above orderings by themselves unsatisfactory (but the Laplacian ordering less so than the conformal ordering). The suggested way out is that one can only apply such an ordering prescription to the most reduced configuration space itself (this unfortunately then leaves one stuck in the general case, as one has no explicit form for the most reduced configuration space).

I will demonstrate below that procedure C.III) (reduction at the level of the QM constraint equations) is capable of producing a different answer from procedures A-C.I-II) (relationalspace approach, indirect approach and configuration space and phase space reductions). If one takes one's relationalism very seriously, one would favour the relationalspace approach A) foremost (and, quite possibly, the conformal ordering). But if one takes relationalism less seriously, it is worth noting that C.III) can often (not always in general but always for this article's models) be made to agree with A-C.I-II) by adopting the Laplacian ordering.

Previously, Barvinsky [101, 102, 103] investigated for what ordering Dirac and reduced approaches coincide. On the other hand, e.g. Ashtekar, Horowitz, Romano and Tate [65, 532] have argued for inequivalence of these two approaches to quantization. The Laplace ordering does so to $O(\hbar)$ and to $O(\hbar)$, Laplacian ordering coincides with conformal ordering.

In the Dirac quantization approach for RPM's, the PPST rescaling $\mathcal{L}\text{in}\Psi = 0$ to $\widetilde{\mathcal{L}}\text{in}\widetilde{\Psi}$ does cause an alteration since these constraint operators are differential operators and therefore act on the conformal factor of $\widetilde{\Psi}$. The reduced quantization scheme avoids this issue.

This SSec has its own notation: \mathcal{M}_{AB} and M_{ab} are simply the less and more restricted configuration space metrics with no scale–shape connotations intended. In more detail, there is an issue of whether $\Delta_{\mathcal{M}|\mathcal{C}}\Psi$ – the less reduced Laplacian restricted to the constraint surface \mathcal{C} – matches up with $\Delta_{\mathbf{M}}\Psi$ – the more reduced Laplacian. For simplicity whilst sufficiently illustrating this argument, I consider the case in which the configuration space dimension goes down by one in the reduction. Then there is an (Euclidean-signature for mechanical configuration spaces) ADM-type split of configuration space itself,

$$\mathcal{M}_{AB} = \begin{pmatrix} \beta_c \beta^c + \alpha^2 & \beta_a \\ \beta_b & M_{\gamma\delta} \end{pmatrix} , \quad (556)$$

where M_{ab} is the kinetic metric on the more reduced configuration space. A convenient way of working out the hypersurface geometry for this is to consider Kuchař's hypersurface formulation [395], which in the present configuration space metric

context gives $\mathcal{M}_{\perp\perp} = 1$, $\mathcal{M}_{\perp a} = 0$, $\mathcal{M}_{ab} = \mathbf{M}_{ab}$, and so $\mathcal{N}^{\perp\perp} = 1$, $\mathcal{N}^{\perp a} = 0$, $\mathcal{N}^{ab} = \mathbf{N}^{ab}$ and $\mathcal{M} = \mathbf{M}$. Then use the following formulae for the hypersurface split of a covector applied to $\partial_A \Psi$,

$$\mathcal{D}_\perp \partial_\perp \Psi = \{\delta_\beta \{\partial_\perp \Psi\} + \partial^a \Psi \partial_a \alpha\} / \alpha \quad (557)$$

and

$$\mathcal{D}_a \partial_b \Psi = \mathbf{D}_a \partial_b \Psi - \partial_\perp \Psi K_{ab} . \quad (558)$$

Here, \mathbf{D}_a is the covariant derivative corresponding to \mathbf{M} , K_{ab} the extrinsic curvature

$$K_{ab} = -\frac{1}{2\alpha} \delta_\beta \mathbf{M}_{ab} , \quad (559)$$

and $\delta_\beta = ' - \mathcal{L}_\beta$ the hypersurface derivative for $'$ the derivative with respect to the perpendicular to the hypersurface). Then

$$\triangle_{\mathcal{M}} \Psi = \mathcal{N}^{AB} \mathcal{D}_A \partial_B \Psi = \triangle_{\mathbf{M}} \Psi - K \partial_\perp \Psi + \{\delta_\beta \partial_\perp \Psi + \partial^a \Psi \partial_a \alpha\} / \alpha . \quad (560)$$

Also the doubly-contracted Gauss Law is (form 1)

$$\text{Ric}(\mathcal{M}) = \text{Ric}(\mathbf{M}) + K^2 - K_{ab} K^{ab} - 2\text{Ric}(\mathcal{M})_{\perp\perp} \quad (561)$$

or (form 2)

$$\text{Ric}(\mathcal{M}) = \text{Ric}(M) + K^2 + K_{ab} K^{ab} + 2\{\delta_\beta K - \triangle_{\mathbf{M}} \alpha\} / \alpha . \quad (562)$$

Then form 2 gives that

$$\triangle_{\mathcal{M}}^c \Psi = \triangle_{\mathbf{M}}^c \Psi - \frac{\text{Ric}(\mathbf{M})}{4\{k-2\}\{k-1\}} \Psi - \frac{k-2}{4\{k-1\}} \{K^2 + K_{ab} K^{ab}\} \Psi + \frac{1}{\alpha} \left\{ \frac{k-2}{2\{k-1\}} \{ \{-\triangle_{\mathbf{M}} \alpha + \delta_\beta K\} \Psi + \delta_\beta \partial_\perp \Psi + \partial^c \Psi \partial_c \alpha \} \right\} . \quad (563)$$

Next, suppose that the reduction involves explicitly splitting the less reduced space's coordinates Q^A into more reduced space coordinates Q^a and orbit coordinates O (indexed by the single value z). These are considered to be such that the constraint is absorbed, so $\widehat{\mathcal{L}}\text{in} \Psi = 0$, becomes

$$0 = \widehat{P}_O \Psi = \frac{\hbar}{i} \frac{\partial \Psi}{\partial O} . \quad (564)$$

(In general this will not be explicitly constructible, and it may not even exist at all e.g. due to global issues or stratification issues. But it does suffice to illustrate the point in question, and does apply to a number of concrete RPM examples considered below.) Then the reduction involves $\partial/\partial O^2$ contributions to the formula for the Laplacian being struck out,

$$\triangle_{\mathcal{M}}|_{\partial\Psi/\partial O=0} \Psi = \frac{1}{\sqrt{\mathcal{M}}} \frac{\partial}{\partial Q^A} \left\{ \sqrt{\mathcal{M}} \left\{ \mathcal{N}^{Ab} \frac{\partial \Psi}{\partial Q^b} + \mathcal{N}^{Az} \frac{\partial \Psi}{\partial O^z} \right\} \right\} \Big|_{\partial\Psi/\partial O^z=0} = \frac{1}{\sqrt{\mathcal{M}}} \frac{\partial}{\partial Q^A} \left\{ \sqrt{\mathcal{M}} \mathcal{N}^{Ab} \frac{\partial \Psi}{\partial Q^b} \right\} , \quad (565)$$

which, if in hypersurface split form, gives

$$\triangle \Psi|_{\text{red}} = \triangle_{\mathcal{M}} \Psi + \frac{\partial \sqrt{\mathcal{M}} \mathcal{N}^{zb}}{\partial O^z} \frac{\partial \Psi}{\partial Q^b} \quad (566)$$

[the other term vanishing by mixed partial equality and (564) again]. Thus there is a discrepancy if $\sqrt{\mathcal{M}} \mathcal{N}^{zb}$ is a function of O^z . This is in general an obstruction to the equivalence of Laplace-ordering-then-constraining giving the same as constraining-and-then-Laplace-ordering. Note in particular that if there is 'blockwise diagonality' in the b 's and z , then this obstruction to Laplacian order preservation is absent. [If more than 1 value for z were needed, one could either work sequentially or consider a hypersurface split of higher codimension, which are certainly also present in the Mathematical Physics literature (see e.g. [251]).]

Next, note that this striking out works even less well for $\triangle_{\mathcal{M}}^{\xi \neq 0}$ (as exemplified by $\triangle_{\mathcal{M}}$, but any other $\xi \neq 0$ has a similar problem). This is due to the $\text{Ric}(\mathcal{M}) \Psi|_{\text{red}}$ term not receiving any striking out and going to plenty of additional terms as well as $\text{Ric}(\mathcal{M})$. (And, in the case of conformal trying to be matched up with conformal, even the actual coefficient of $\text{Ric}(\mathcal{M})$ comes out wrong due to the dimension change.

Showing that this example's assumptions do permit explicit RPM examples, 1) for the passage from relational space to shape space in 4-stop metroland or triangleland ($\sigma = \rho$ or \mathbf{I}),

$$2/\sigma^2 = \text{Ric}(\mathbb{S}^2) = K^2 - K_{ab} K^{ab} = K^2/2 . \quad (567)$$

This is by, respectively, evaluation, theorema egregium and isotropy of K_{ab} due to spherical symmetry alongside the taking of traces. Thus (560) reads

$$\triangle_{\mathbb{R}^3} \Psi = \sigma^{-2} \triangle_{\mathbb{S}^2} \Psi + 2\sigma^{-1} \partial_\sigma \Psi + \partial_\sigma \partial_\sigma \Psi . \quad (568)$$

Thus here

$$\Delta_{\mathbb{R}^3}\Psi|_{\text{red}} = \sigma^{-2}|_{\sigma=\text{const}}\Delta_{\mathbb{S}^2}\Psi \quad (569)$$

– a case which has good agreement (and moreover one for which conformal ordering = Laplacian ordering too).

Example 2) The passage from N -stop metroland relational space to the shape space $\{n-1\}$ -sphere, for which the above working directly generalizes to

$$\{n-1\}\{n-2\}/\rho^2 = \text{Ric}(\mathbb{S}^{n-1}) = K^2 - K_{ab}K^{ab} = K^2\{n-2\}/\{n-1\} \quad (570)$$

(using form 1 of the doubly contracted Gauss equation alongside the higher- d space being flat in place of the Theorema Egregium), so

$$\Delta_{\mathbb{R}^n}\Psi = \rho^{-2}\Delta_{\mathbb{S}^{n-1}}\Psi + p\rho^{-1}\partial_\rho\Psi + \partial_\rho\partial_\rho\Psi, \quad (571)$$

$$\text{so } \Delta_{\mathbb{R}^n}\Psi|_{\text{red}} = \rho^{-2}|_{\rho=\text{const}}\Delta_{\mathbb{S}^{n-1}}\Psi, \quad (572)$$

so there is agreement again, but now $\Delta_c\Psi|_{\text{red}}$ is out from $\mathcal{D}_c^2\Psi$ by

$$-\frac{\{n-3\}}{4\{n-2\}}\text{Ric}(\mathbb{S}^p)\Psi = \frac{1}{\rho^2}\bigg|_{\rho=\text{const}}\frac{\{n-3\}}{4\{n-2\}}\{n-1\}\{n-2\}\Psi = \frac{\{n-1\}\{n-3\}}{4}\frac{1}{\rho^2}\bigg|_{\rho=\text{const}}\Psi. \quad (573)$$

[But it is relatively benign in that one can get around it by redefining the energy by an additive constant.]

Example 3) The curved, conformally flat \mathbb{R}^3 to \mathbb{S}^2 and \mathbb{S}^3 to \mathbb{S}^2 cases both send Laplacian ordering to Laplacian ordering as their left hand side spaces are diagonal. Thus, via both the relational space route and via the preshape space route, Laplacian ordering is inherited at all levels in the pure-shape triangle land problem. This example's steps do *not* preserve conformal ordering which was the case I argued for quantum-cosmologically.

Example 4) $\mathbb{R}^{2n} = \mathbb{C}^n$ to $\text{C}(\mathbb{CP}^{n-1})$, $\text{C}(\mathbb{CP}^{n-1})$ to \mathbb{CP}^{n-1} and \mathbb{S}^{2n-1} to \mathbb{CP}^{n-1} of N -a-gonland [435] all have suitable at-least-block-diagonal form left-hand-side spaces, and therefore preserve Laplacian ordering. They do not preserve conformal ordering. Thus via both the relational space route and via the preshape space route, Laplacian ordering is inherited at all levels in the pure-shape N -a-gonland problem, and also in the scaled N -a-gonland problem.

Thus all of this article's principal concrete examples of RPM's are, nevertheless, Laplacian ordering preserving under passage between Dirac and reduced/relationspace schemes. However, it is the conformal ordering that is relationally motivated, and this is not preserved under this passage even for some of these simple examples.

Kuchař's pro-reduced argument. One would not expect that appending unphysical fields to the reduced description should change any of the physics of the of the true dynamical degrees of freedom. Thus, if they differ, one should be more inclined to believe the reduced version.

Note 1) Another perspective on this is that the reduced configuration space provides 'inside knowledge' of the geometry which underlies a correct choice of operator-ordering.

Note 2) Additionally, DeWitt's argument of coordinatization invariance makes best sense in the case which involves just true degrees of freedom rather than a mixture of these and gauge degrees of freedom.

There is however a modicum of approximate protection.

Barvinsky's second approximate equivalence is that, to 1 loop (first order in \hbar in the Semiclassical Approximation), Laplacian ordering coincides for reduced and Dirac schemes.

Note 3) This can be composed with Barvinsky's first argument to hold to that accuracy in that regime for any ξ -ordering and thus in particular to the relationally-motivated conformal operator-ordering.

Note 4) This and the preceding SSSec are both cases of how ignoring some of the degrees of freedom in a configuration space can change the QM of the remaining subconfigurations by altering the nature of the Laplacian or related differential operators that occur in the QM wave equation.

Analogies 57) and 58) that RPM's suffice to reveal the conformal ordering argument and the midisuperspace-type problem with it.

N.B. that in the presence of nontrivial \mathfrak{g} , $\text{Point}(\mathfrak{q}/\mathfrak{g})$ invariance trumps Point invariance as a meaningful concept. For, proceeding indirectly in the presence of an \mathfrak{g} really means having \mathfrak{g} -Point invariance, so arguments based naïvely on DeWitt's use of just Point in this context are indeed open to doubt. N.B. these are 'midisuperspace' issues and thus invisible in the more usual minisuperspace studies. Thus, upon encountering discrepancies I for now keep the r-version of the argument as the one I know for now to be correctly formulated and implemented.

In general, once one is dealing with linearly constrained theories, arguments for such as conformal or Laplacian operator ordering are insufficient since these differ if applied before or after dealing with the linear constraints. Moreover, immediately below is a strong argument for it being the reduced case for which such arguments make sense, and this case is less available (in particular it is not available for full geometrodynamics).

6.12 Inner product and adjointness issues

Difference 24) Following from the indefiniteness of the GR configuration space metric (Difference 16), GR's inner product is indefinite, rendering a Schrödinger interpretation inappropriate whilst a Klein–Gordon interpretation fails on further grounds (see Sec 11.6.1 for details). On the other hand, RPM's, like ordinary QM, have a positive-definite configuration space metric, giving a positive-definite inner product for which a Schrödinger interpretation is appropriate. If one sends $H\Psi = E\Psi$ to $\overline{H\Psi} = \overline{E\Psi} = \{E/\Omega^2\}\overline{\Psi}$ (a move used in the preceding text), one's eigenvalue problem has a weight function Ω^{-2} which then appears in the inner product:

$$\int_{\mathfrak{S}} \overline{\Psi_1}^* \overline{\Psi_2} \Omega^{-2} \sqrt{M} d^k x . \quad (574)$$

Here, $\langle \mathfrak{S}, \mathbf{M} \rangle$ denotes the relationalspace portmanteau at the level of Riemannian geometry. This inner product additionally succeeds in being PPSCT invariant, being equal to (c.f. [464] for the minisuperspace case)

$$\int_{\mathfrak{S}} \Psi_1^* \Omega^{\{2-k\}/2} \Psi_2 \Omega^{\{2-k\}/2} \Omega^{-2} \sqrt{M} \Omega^k d^k x = \int_{\mathfrak{S}} \Psi_1^* \Psi_2 \sqrt{M} d^k x \quad (575)$$

in the PPSCT representation that is mechanically natural in the sense that E comes with the trivial weight function, 1.

Generally, $\widehat{H} = \overline{\widehat{H}}$ is not (in the elementary sense) self-adjoint with respect to $\langle \cdot | \cdot \rangle$, while the mechanically-natural \widehat{H} is, in a simple sense, with respect to $\langle \cdot | \cdot \rangle$. More precisely, this is in the sense that

$$\int_{\mathfrak{S}} \sqrt{M} d^k x \Psi^* \Delta \Psi = \int_{\mathfrak{S}} \sqrt{M} d^k x \{ \Delta \Psi^* \} \Psi + \text{boundary terms} , \quad (576)$$

which amounts to self-adjointness if the boundary terms can be arranged to be zero, whether by the absence of boundaries in the configuration spaces for 1- and 2- d RPM's [25] or by the usual kind of suitable fall-off conditions on Ψ . This is not shared by the Ω -inner product as that has an extra factor of Ω^{-2} . This in general interferes with the corresponding move by the product rule. (\sqrt{M} does not interfere thus above. This is because the Laplacian is built out of derivatives that are covariant with respect to the metric $M_{\Gamma\Lambda}$.) However, on the premise that solving $\overline{H\Psi} = \overline{E\Psi}$ is equivalent to solving $H\Psi = E\Psi$, the PPSCT might at this level be viewed as a sometimes-useful computational aid. The answer would then be placed in the preceding paragraph's PPSCT representation for further physical interpretation.

For triangleland in the checked representation, \sqrt{M} is the usual spherical Jacobian $I^2 \sin \Theta$, so the checked inner product weight is $I \sin \Theta / 4$. Alternatively and equivalently, the plain inner product is $(1/4I)^{3/2} I^2 \sin \Theta = I^{1/2} \sin \Theta / 8$, i.e. differing from the usual spherical one by the obvious conformal factor. Also, the checked representation in parabolic coordinates, the inner product weight is just $1/8$.

Conformal transforming prior to solving parallels that which is done by e.g. Iwai, Tachibana and Uwano [351, 598, 352] for a different problem (the 4- d isotropic HO). As they explain well, solving QM involves wavefunctions *and* inner product being found, on which grounds the current article does *not* involve hydrogen. For, it has been set up to have the same wavefunctions as hydrogen but the inner products are different. (So, e.g., normalization is different, as are expectations of operators).

6.13 The Quantum Cosmological ones

6.13.1 Closed-system/whole universe QM issues

Being whole-universe models, RPM's do already exhibit some of these. For example, RPM's having a fixed 'energy of the universe' has the effect of cutting down on the number of valid eigenvalues (a type of closed-universe feature that goes back at least to DeWitt [201]); see Secs 7 and 10 for more closed-universe features.

More deeply, **the usual Copenhagen interpretation of quantum mechanics cannot apply to the whole universe.** In conventional QM, one presupposes that the quantum subsystem under study is immersed in a classical world, crucial parts of which are the observers and/or measuring apparatus. In familiar situations, Newtonian Mechanics turns out to give an excellent approximation for this classical world. However, there are notable conceptual flaws with extending this 'Copenhagen' approach to the whole universe. For, observers/measuring apparatus are themselves quantum-mechanical, and are always coupled at some level to the quantum subsystem. Treating them as such requires further observers/measuring apparatus so the situation repeats itself. But this clearly breaks down once the whole universe is included.

An alternative, albeit a somewhat slippery one [369], is the **many-worlds interpretation**. A number of further replacements are tied to various Problem of Time strategies such as the Conditional Probabilities Interpretation, Histories Theory and Records Theory, as covered in later sections of this article.

As another alternative, in the absence of external observers, one can postulate that physics is in terms not of observables but of **beables** [111] (i.e. replacing things that can be observed by things that simply *can be*).

6.13.2 Structure formation

Analogy 59) RPM's are considered to be a qualitative conceptual model of the quantum cosmological seeding of structure formation in a semiclassical regime. Particularly for scaled RPM, this parallels the Halliwell–Hawking [296] approach to GR Quantum Cosmology. This is somewhat narrower as a Problem of Time strategy (it is an emergent semiclassical time strategy) but has further conceptual and computational applications outside of the Problem of Time context too), and of Records Theory [502, 248, 80, 83, 292, 28, 29]. I consider this in [33, 35, 36, 37, 40] and Sec 10 and 13.

6.13.3 Uniform states in (Quantum) Cosmology

Analogy 60) Uniformity is of widespread interest in Cosmology. It applies to good approximation to the present distribution of galaxies and to the CMB. There is also the issue of whether there was a considerably more uniform initial state [509]. There are further related issues of uniformizing process and how the small perturbations observed today were seeded. Sec 10.7 outlines how RPM's have qualitative counterparts of these issues, and Sec 14 studies these using the Naïve Schrödinger Interpretation.

6.13.4 Robustness of quantization to ignoring some of the degrees of freedom.

Question [Analogy 61]) RPM's are a toy model for robustness issues, i.e. the consequences of ignoring some of the degrees of freedom. This is along the lines in which Kuchař and Ryan [405] question whether Taub microsuperspace sits stably inside the Mixmaster minisuperspace as regards making QM predictions (which is a toy model of whether studying minisuperspace might be fatally flawed due to omitting all of the real universe's inhomogeneous modes). This was found to be unstable. The RPM (or, for that matter, molecular) counterparts of this have the advantage of possessing a wider range of analytically tractable examples with which to carry out such an investigation. E.g. is the $N = 3$ model is stable to the inclusion of 1 further particle, in each of 1- d and 2- d ?

6.13.5 Question: Do RPM's touch even on some Arrow of Time issues?

E.g. [311, 83, 294, 545] have some suggestions about associations between this and quantum cosmological issues. While I consider this topic to be outside of the usual Problem of Time, and make no claim to resolve it or say anything new about it, I do make some mention of it in Secs 13 and 14.

7 The simplest quantum RPM: scaled 3-stop metroland

For this model, as there are no nontrivial constraints, Dirac and r-quantization coincide. (However, this also means it is trivial as regards configurational relationalism). Also, pure-shape RPM is in this case trivial rather than simpler. These features make scaled 3-stop metroland an obvious starting point for the study of quantum RPM's. This is a simple study: the mathematics arising is very familiar, that of QM in the flat plane. However, this mathematics now has an unusual significance that renders it appropriate as a whole universe model, and this model already reveals a number of features of RPM models and of closed-universe physics. I drop (a) labels. Firstly, if one thinks in terms of the 2 subsystems corresponding to the clustering in use, $\{\rho_1, \rho_2\}$, and these give 2-d Cartesian mathematics. Secondly, if one thinks in terms of the scale-shape split $\{\rho, \varphi\}$, one is working with what are mathematically plane-polar coordinates.

7.1 Scaled 3-stop metroland free quantum problem

7.1.1 Study in $\{\rho_1, \rho_2\}$ coordinates

In terms of these, the free problem straightforwardly separates into two copies $i = 1, 2$ of⁶¹

$$\hbar^2 \psi_{i, \rho_i^2} / 2 + E_i \psi_i = 0 . \quad (577)$$

Thus the wavefunctions are $\psi_i = \exp(\pm i \sqrt{2E_i} \rho_i / \hbar)$, corresponding to a positive continuous spectrum. The solution of the relational problem may then be reassembled as (up to signs in each exponent)

$$\begin{aligned} \Psi_E = \exp(\sqrt{2}i \{ \sqrt{E_1} \rho_1 + \sqrt{E_2} \rho_2 \} / \hbar) = \\ \exp \left(\frac{\sqrt{2}i}{\hbar} \left\{ \frac{1}{\sqrt{m_2 + m_3}} \left\{ \sqrt{E_1 m_2 m_3} + \sqrt{\frac{E_2 m_1}{m_1 + m_2 + m_3}} \right\} r_{23} + \sqrt{\frac{E_2 m_1 \{m_2 + m_3\}}{m_1 + m_2 + m_3}} r_{12} \right\} \right) \end{aligned} \quad (578)$$

in terms of straightforwardly relational variables (for use in later discussions). There is moreover one unusual relational/closed universe feature: E_1 and E_2 are not independent: $E_1 + E_2 = E_{\text{uni}}$. This amounts to the collapse of a larger Hilbert space to a smaller one by the condition that the energy of the whole universe takes a fixed value. This collapse is atypical as not coming with a separate nontrivial linear constraint (unlike in the collapse that occurs in the Dirac approach).

7.1.2 Study in $\{\rho, \varphi\}$ coordinates

I separate variables using the notation $\Psi = \mathcal{I}(\rho)F(\varphi)$. This SSec concerns free models; these require $E > 0$ in order to be physically realized. Then the scale part of the solution is, in terms of Bessel functions,

$$\mathcal{I}_d(\rho) \propto J_d(\sqrt{2E}\rho/\hbar) . \quad (579)$$

The probability density function for this exhibits an infinity of oscillations in the scale direction.

7.1.3 Interpretation of the shape part (also useful in further SSecs)

For 3-stop metroland in e.g. the $\{D, M\}$ basis corresponding to the (1) cluster, the solutions are

$$\mathcal{Y}_d(\varphi) \propto \exp(id\varphi) , \quad (580)$$

corresponding to energies $E = \hbar^2 d^2 / 2$. Here, $d \in \mathbb{Z}$ is a total relative dilational quantum number. One often then takes sine and cosine combinations of (580). Then in terms of shape and size quantities,

$$\mathcal{Y}_d \propto \mathcal{T}_d(\text{RelSize}(23)) , \quad (581)$$

for $\mathcal{T}_d(X) := \{T_d(X) \text{ for cosine solutions and } \sqrt{1 - T_d(X)^2} \text{ for sine solutions}\}$, and $T_d(X)$ the Tchebychev polynomial of degree d (see Overall Appendix C).

Interpretation of the first few solutions is then as follows. For $d = 0$ – the basis-independent ground state – one has the same probability for all ratios of ρ_1 and ρ_2 . For the $d = 1$ cosine solution, 23 small compared to the separation of this cluster and particle 1 is favoured (i.e. a well-separated model universe). On the other hand, for the $d = 1$ sine solution 23 large compared to the separation of this cluster's centre of mass and particle 1 is favoured (i.e. a merged model universe). For the $d = 2$ cosine solution, both of the preceding are equally favoured with gaps in between, while for the $d = 2$ sine solution both of the preceding are equally disfavoured with peaks in between. For the $d = 3$ cosine solution, near-D configurations for all six possible D's are favoured with near-M configurations for all 6 possible M's being disfavoured, and vice versa for the $d = 3$ sine solution.

⁶¹In this Sec i denotes a particular component rather than an index to be summed over.

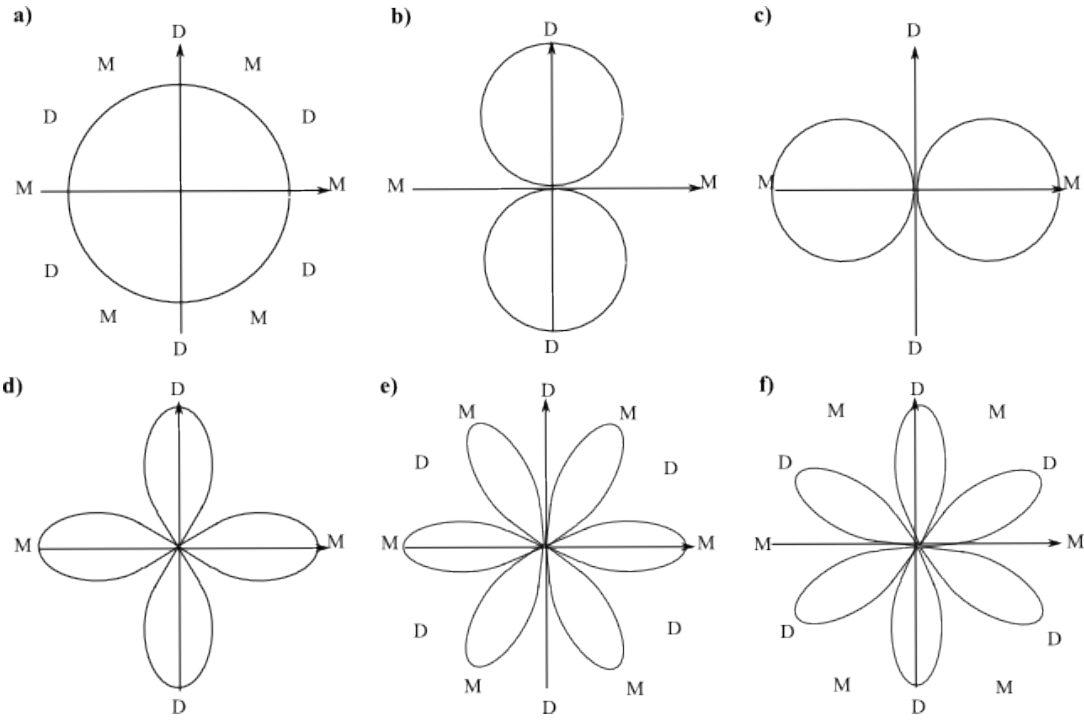


Figure 44: Separated-out shape part of the wavefunction for 3-stop metroland.

7.2 Scaled 3-stop metroland quantum multi-HO problem

In the isotropic case, I have a choice of separation in subsystem-split $\{\rho_1, \rho_2\} = \{\text{RelSize}(23), \text{RelSize}(1, 23)\}$ Cartesian coordinates and in scale-shape polars. In the non-isotropic case, only the former survives, but after solving in these one can nevertheless apply a coordinate change to pass to scale-shape split variables for the purpose of interpretation that parallels Quantum Cosmology.

7.2.1 Study of multi-HO in $\{\rho_1, \rho_2\}$ coordinates

For a single HO, without loss of generality between particles 2 and 3, one has

$$\hbar^2 \psi_{1,\rho_1\rho_1} - K_1 \rho_1^2 \psi_1 + 2E_1 \psi_1 = 0 \quad (582)$$

and the equation in ρ_2 from the preceding SSec. (582) is solved by

$$\Psi_{n_1} \propto \exp(\pm i \sqrt{2E_2} \rho_2 / \hbar) H_{n_1}(K_1^{1/4} \rho_1^2 / \sqrt{\hbar}) \exp(-\sqrt{K_1} \rho_1^2 / 2\hbar) \quad (583)$$

for H_{n_i} the n_i th Hermite polynomial and with corresponding spectrum $E_i = \sqrt{H_i/\mu_i} \hbar \{1/2 + n_i\}$, $n_i \in \mathbb{N}_0$. Finally, $n := n_1$ and E_2 are not independent, being related by $\sqrt{K_1} \hbar \{n + 1/2\} + E_2 = E_{\text{uni}}$.

For 2 or 3 HO potentials, in the special case one obtains two separated-out 1-d HO problems. The solution is then

$$\Psi_n \propto H_{n_1}(K_1^{1/4} \rho_1^2 / \sqrt{\hbar}) \exp(-\sqrt{K_1} \rho_1^2 / 2\hbar) H_{n_2}(K_2^{1/4} \rho_2^2 / \sqrt{\hbar}) \exp(-\sqrt{K_2} \rho_2^2 / 2\hbar), \quad (584)$$

with n_1 and n_2 not independent but related, rather, by $\sqrt{K_1} \hbar \{n_1 + 1/2\} + \sqrt{K_2} \hbar \{n_2 + 1/2\} = E_{\text{uni}}$. The solution here is immediately interpretable using $\rho_1 = \text{RelSize}(23)$ and $\rho_2 = \text{RelSize}(1,23)$. Early RPM work was hampered by being in terms of \underline{r}^{IJ} , which are much less conducive to having separability.

7.2.2 Isotropic multi-HO ($\Lambda < 0$, $k < 0$ vacuum or wrong-sign radiation) in scale-shape split coordinates

For 3-stop metroland with $V = A\rho^2$ $A > 0$, the time-independent Schrödinger equation gives the 2-d isotropic quantum HO problem [531, 569]. In scale-shape variables, this is *exactly* solved by [33] (645, 580) and a radial factor

$$\mathcal{I}_{N,d}(\rho) \propto \rho^{|d|} L_N^d(\omega \rho^2 / \hbar) \exp(-\omega \rho^2 / 2\hbar) \quad (585)$$

for $L_N^d(\xi)$ the generalized Laguerre polynomials. These solutions correspond to energies $E = \hbar \omega \{2N + |d| + 1\} > 0$ for $N \in \mathbb{N}_0$, $d \in \mathbb{Z}$ and $\omega = \sqrt{K_1} = \sqrt{K_2}$. These wavefunctions are finite at 0 and ∞ in scale, with N nodes in between.

7.2.3 Non-isotropic multi-HO transcribed to scale–shape split variables

These solutions are then

$$\Psi_{n_1 n_2}(\rho, \varphi) \propto H_{n_1}(\sqrt{K_1/\hbar}\rho \cos \varphi) H_{n_2}(\sqrt{K_1/\hbar}\rho \sin \varphi) \exp(-\rho^2\{A + B \cos 2\varphi\}/\hbar) . \quad (586)$$

One can think of these as rectangular/box-shaped arrays of peaks and troughs (note that peak strength is not constant). The ground state favours all D's and M's equally. Then in a basis that corresponds to a particular pair of opposite D's and a perpendicular pair of opposite M's. One of the first excited states favours the pair of D's and disfavors the pair of M's and the other does vice versa. The maximal collision O is also disfavoured in each case. Second excited states with two 0 quantum numbers and a 2 do likewise but in three lobes, so that O is also favoured while two nonzero distances have disfavoured nodes. On the other hand, the second excited state with two quantum numbers 1 disfavors all four of these D's and M's. The $B \neq 0$ case (for $|B| < A$) is then a distortion of this (stretched in one direction and squeezed in the perpendicular direction).

Note 1) There is non-agreement in general with isotropic case above: because solving in Cartesian coordinates amounts to using a different basis of solutions. However, the ground state expressions coincide. That is because the ground state is unique and thus independent of which basis one evaluates it in.

Note 2) The potential $U\{1 - \cos 2\theta\}$ for U constant occurs in modelling the rotation of a linear molecule in a crystal [505, 641, 506]. In the present 2-d-like case, this leads to [505] Mathieu's equation [1] and thus fairly standard Mathematical Physics.

7.3 Scaled 3-stop metroland quantum problems with further ‘cosmological’ potentials

A number of cases that are quantum cosmologically useful are as per the list in Sec 5.3 (which is only a Lagrangian-level analogy and so is somewhat tenuous).

7.3.1 Analogues of $\Lambda > 0$, $k < 0$ vacuum models

For $V = A\rho^2$, $A < 0$, the time-independent Schrödinger equation gives instead isotropic upside-down HO mathematics (see e.g. [531]). The solutions of these are oscillatory rather than tightly localized.

An approximate solution for e.g. the ground state of the isotropic upside-down HO in 3-stop metroland is (using separation in Cartesian coordinates [531] and then changing to scale–shape variables)

$$\Psi \propto \sin(\sqrt{-K_1/\hbar}\rho^2 \cos^2 \varphi/2) \sin(\sqrt{-K_2/\hbar}\rho^2 \sin^2 \varphi/2) / \rho \sqrt{\sin 2\varphi} . \quad (587)$$

The probability density function for this is a rectangular grid of humps that decrease in height as one moves out radially. Getting angular dependence via $B \neq 0$ continues to be straightforward in this case, readily allowing study of models for which the premises of the Semiclassical Approach apply. This is also the case for what is mechanically a mixture of HO's and upside-down HO's. For other power-law potentials, furthermore approximate or numerical work is required.

7.3.2 Analogues of $k > 0$ dust model

This (restricted to $D = 0$, i.e. no wrong-sign radiation) is solved by (580) and a radial factor

$$\mathcal{I}_N(\rho) \propto L_{N-1}(2\sqrt{-2E}\rho/\hbar) \exp(\sqrt{-2E}\rho/\hbar) \quad (588)$$

corresponding to energies $E = -k^2/2\hbar^2\{N - 1/2\}^2$.

7.3.3 Analogues of right and wrong sign radiation models

In the 2-d case, $d < \sqrt{2R}/\hbar$ is still a collapse to the maximal collision. However, $d > \sqrt{2R}/\hbar$ now has a positive root choice, which is finite as $\rho \rightarrow 0$. Finally, the critical case $d = \sqrt{2R}/\hbar$ is now also finite as $\rho \rightarrow 0$. This is somewhat different from the higher-dimensional case covered in Sec 9.

7.3.4 Summary of approximate ρ behaviours at the quantum level

Each O_1 is bound in an interval including 0 and each O_2 is bound in an interval excluding (shielded from) 0. O_3 , M_1 , M_2 are all ‘tending to free behaviour at large ρ , forbidden regions have exponential decay, and the O_2 –forbidden– O_3 and A_1 – S_1 – A_2 slices become tunnelling scenarios. This holds also for N -stop metroland and for the \mathbb{CP}^{N-2} presentation of N -a-gonland

7.4 Interpretation I: closed-universe features

1) There is **energy interlocking** between constituent subsystems. E.g. this requires the above naïvely free problem to have a segment of linear lattice rather than a quadrant's worth of lattice as its overall eigenspectrum. The single HO and free particle system to have a set of points rather than an infinity of lines as its overall eigenspectrum, and the coupled HO's give a rather smaller set of points in place of the regular lattice eigenspectrum. Moreover, unlike in the usual interpretation of few-particle QM, the energy here is the energy of the universe. This is not only fixed, but is also a separate attribute of the universe so that the fixed value it takes need bear no relation to the eigenspectra of the universe's contents. This leads to the following effects.

2) In the multiple HO example, all the energies $[E_1(j_1) \text{ and } E_2(j_2)]$ are positive. Then $E_{\text{uni}} < E_1(0) + E_2(0)$ so no wavefunctions exist. Universes failing to meet the zero point energy of their content fail to have a wavefunction.

3) Such universes could rather fail to meet the energy required for just some of the states which lie above a given energy that is greater than the zero point energy. Then one obtains a **truncation** of the conventional eigenspectrum.

4) In the 2 or 3 HO examples, one is required to solve $k_1 n_1 + k_2 n_2 = q$ for $n_i \in \mathbb{N}_0$, $k_i = \sqrt{K_i} \hbar$ and $q = E_{\text{uni}} - \hbar\{\sqrt{K_1} + \sqrt{K_2}\}/2$. Then if k_i and q mismatch through some being rational and some irrational, or even if all are integers but the highest common factor of k_1 and k_2 does not divide q , no solutions exist. This can also be set up to give **gaps** in what would otherwise look like a truncation of the conventional eigenspectrum. Similar effects can be achieved by more complicated matches and mismatches in the other examples' E_i dependences on n_i .

5) There are also then implications for what is an appropriate ensemble, which I postpone to Sec 14.9.5.

7.4.1 How large-universe 'recovery of everyday physics' is not affected by energy interlocking

6) In universes which furthermore contain free particles, because these have continuous spectra, the missing out of some conventional states by 4) cannot occur. E.g. this is so for the 1 HO example above, while 'tensoring' free particles with the 2 or 3 HO setting alleviates this 'missing state' problem in the setting of a universe with a slightly larger particle number.

7) If negative energies are possible for some subsystems (which is certainly the case for suitable power-law potentials), then subsystems can attain energies higher than E_{uni} . So if one tensors a subsystem with negative energies with an independent HO pair, one can have less truncation of that HO's spectrum. One will still have many, or all, states missing, depending on how the coefficients of the two independent subsystems' potentials are related. But if one then tensors in free particles one can have everything up to the truncation. Thus truncation can be displaced at least for some universes, by a modest increase in constituent particle number. It should be noted that there is a mismatch between e.g. the HO which has excited states unbounded from above in conventional QM and hydrogen which has negative energy states bounded from below. This apparent difficulty disappears by how very high positive energy states are unlikely to be physically meaningful. At the very least the physical validity of the model would break down due to e.g pair production and ultimately the breakdown of spacetime. To ensure one's model can attain high enough (but finite) positive energies, wells that are deep and/or numerous enough can be brought in.

8) Unlike 2)–4), energy interlocking *does not* go away with particle number increase or the accommodation of a variety of potentials within one's universe model. However, the large particle number and high quantum number aspects of semiclassicality are relevant here. Subsystems remain well-behaved and all experimental studies in practice involve subsystems. But subsystems may be taken to have conventional wavepackets insofar as these are products of their constituent separated-out problems' wavepackets.⁶² These may still be truncated rather than built out of arbitrarily many eigenfunctions along the lines of 1) to 3) above. But, this will be alleviable in practice by ensuring that sufficient additional free particles or particles whose mutual potentials are of an opposing sign to the original subsystem's. In such a framework, the correlation of a quantity within a subsystem with another in the rest of the universe is overwhelmingly likely to evade detection provided that the rest of the universe contains plenty of other particles. And of course the real universe is indeed well-populated with particles. This is not a point I involve myself much with, to me RPM's are Quantum Cosmology toys rather than likely to be approximate realizations of our universe – it is the 'relativity without relativity' form of GR that I consider to be a good candidate for a relational explanation of that.

Note: 6) to 8) run contrary to an objection of Barbour and Smolin to RPM's as QM's [98].

9) If one considers the whole universe, there is a breakdown of the cluster decomposition principle, in which Steven Weinberg places great stock (see e.g. [627, 628] in the latter of which he comments that "*I do not see how science is possible without the cluster decomposition principle*"). The absence of this principle means that highly separated experiments could in principle yield noticeably correlated results in a small closed-universe setting. The chances of this being observed in a large universe like our own, however, is very small.

Whether the WKB ansatz may additionally be applied in to whole universes is an issue relevant to the Semiclassical Approach to Quantum Cosmology in the general (rather than just RPM) context. I mostly leave this issue discussion to Sec 13 [22, 23, 40]. For the moment I show that increasing the dimension away from 1 reveals further such 'small closed universe' effects.

⁶²But why separable cases? These are, after all, not very general...

After consideration of further examples, I continue this closed-universe effects discussion in Sec 10.

7.5 Interpretation II: semiclassical issues

7.5.1 Ways of characterizing semiclassicality

While one common requirement is for the wavelength λ_Q to be smaller than some characteristic scale l_C of the problem in question, there is no universal rigorous notion of semiclassical limit for quantum theories. Various characterizations are along the following lines ([30, 31] and references therein).

A) Consider the spread (i.e width) of the wavefunctions (this is essentially what is done in [98]).

B) Furthermore investigate how localized wavepackets are as a whole (as opposed to the spread of each wavefunction in their summand/integrand).

C) Consider what happens to the system for large quantum numbers, for which the wavefunction becomes wiggly on scales much shorter than l_C .

D) Consider a WKB ansatz for the wavefunction and expand in powers of λ_Q/l_C .

7.5.2 Spread of the scaled 3-stop metroland wavefunctions

Barbour and Smolin's other objection in [98] was to the sensitivity of the spread of the wavefunctions of large-mass particles to the values of the mass of small-mass particles. This was in connection with piecewise-constant potential models with two small masses and one large one. I clarify why this is not a problem as follows [19].

Firstly, in the absolutist quantization of single particles, one thinks primarily of particle wavepackets, however in a 1- d relationalist quantization one should think of *relative distance wavepackets*. [This is arguably already how one actually thinks in Atomic and Molecular Physics. E.g. Lubkin [432] talks of the approximation by which the proton is considered fixed versus that in which it is not fixed and the spread is in proton–electron separation. Also consider where in the relatively fixed separations that one encounters in molecules it is that one considers the distribution of ‘bond electrons’ to be. [This is an example of shape/relative angle albeit also with the nuclei forming a ‘fixed background’.] In the present situation, then, intuitively, once one rephrases one's standard quantum intuition about small masses being more spread out in relational terms. Thus it is clear that the position uncertainty of the small mass dominates the relational formulation's relative separation uncertainty between that small mass and a large mass.

Barbour and Smolin's specific example has the same content as my Example 1. Then for $m_2, m_3 = m \ll M = m_1$, the wavefunction goes as

$$\Psi \sim \exp(i\{\sqrt{E_1} - \sqrt{E_2}\}\sqrt{m}r_{23}/\hbar)\exp(i2\sqrt{E_2}\sqrt{m}r_{12}/\hbar) . \quad (589)$$

Thus, indeed as Barbour and Smolin claim, the small masses dominate all the uncertainties. But by my above interpretation, reveals that these uncertainties are in fact in the separations between a big mass and a small mass, so this situation conforms to standard quantum intuitions rather than constituting some kind of impasse. The truly relevant test to establish whether there is a semiclassical limit problem is rather to check that if $m_1, m_3 = M \gg m = m_2$, so there is a big mass–big mass relative separation, that this is not influenced much by a small mass somewhere else. Now, upon performing the new approximation, and isolating the big mass–big mass separation r_{13} as the variable whose spread is of relevance, I find that

$$\Psi \sim \exp(i\sqrt{E_1}\sqrt{M}r_{23}/\hbar)\exp(-i\sqrt{E_2}\sqrt{m}r_{12}/\hbar) , \quad (590)$$

so that indeed only the big masses contribute significantly to the spread in the big mass–big mass separation. Thus basic conclusion is unaffected by having the two identical masses replaced by merely similar masses. Moreover, it holds widely throughout the models presented in this article when suitable pairs of quantities are set to be relatively large and small.⁶³ The Jacobi coordinate presentation renders all of this entirely normal.

Also, the application of polar coordinates for $d > 1$ brings out that the standard QM interpretation is close to being relational. This is most familiar in the study of the hydrogen atom. For this, a simple standard approach is to treat the proton as fixed and then consider the spread of the radial separation r between the proton (or more accurately the atom's barycentre) and the electron. All that is missing as regards obtaining a fully relational perspective is to consider the position of the barycentre not only to be uninteresting but also to be meaningless. Then one considers the spread in ρ_1 . The ready availability of this familiar picture is one reason why it is unfortunate that Barbour and Smolin restricted their study to 1- d examples.

7.5.3 Wavepackets for scaled 3-stop metroland

Piecemeal construction of wavepackets for the separated-out 1- d quantum problems does not care whether these arise from separation in relational problems or in absolutist ones (see e.g. [566, 531]). There are, however, limitations building composite wavepackets in the relational case. Unlike in the absolutist case, composition of subsystem wavepackets cannot be extended

⁶³In pure-shape RPM, the corresponding interpretation now additionally involves *spreads in relational ratios* and moreover that now the underlying formal mathematics is not standard (at least as far as I know and in the context of the Molecular Physics literature).

to include the whole system. This is due to $\sum_i E_i$ taking a fixed value, E . By energy interlocking the small universe models built up from individual problems' wavepackets additionally contain a delta function

$$\delta \left(\sum_{\Gamma \in \text{subsystems}} E_{\Gamma} - E_{\text{uni}} \right) \quad (591)$$

acting inside the sums and integrals required to build it up, which causes it to differ mathematically from e.g the direct product of subsystem wavepackets in a fully separable universe. This is analogy with conservation of energy–momentum at each vertex in path integral formulation of QFT.

7.6 Interpretation III: characteristic scales

K/ρ RPM models have a ‘Bohr moment of inertia’ (square of ‘Bohr configuration space radius’) for the model universe analogue to (atomic Bohr radius)², $I_0 = \rho_0^2$, which, in the gravitational case, goes like $\hbar^4/G^2 m^5$. Then $E = -\hbar^2/2I_0\{N-1/2\}^2$ in the 3-stop case. HO RPM models have characteristic $I_{\text{HO}} = \hbar/\omega$. Note that the first of these is limited by the breakdown of the approximations used in its derivation, while the second of these has no such problems.

7.7 Interpretation IV: expectations and spreads of the wavefunctions

7.7.1 Expectations and spreads of shape operators

As well as characterization by ‘modes and nodes’ as are evident from figures such as Fig 46, expectations and spreads of powers of r are used in the study of atoms. (See e.g. [458] for elementary use in the study of hydrogen, or [241] for use in approximate studies of larger atoms). These provide further information about the probability distribution function from that in the also-studied ‘modal’ quantities (peaks and valleys) that are read off from plots or by the calculus. They also represent a wider range of bona fide relational outputs of the quantization procedure (like the probability densities but unlike the wavefunctions themselves: only when inner products are used are the outputs physical in this sense). E.g. for hydrogen, one obtains from the angular factors of the integrals trivially cancelling and orthogonality and recurrence relation properties of Laguerre polynomials in [1] for the radial factors that (e.g. in the $l = 0$ case)

$$\langle nlm | r | nlm \rangle = \{3n^2 - l(l+1)\}a_0/2 \quad \text{and} \quad \Delta_{nlm}r = \sqrt{\{n^2\{n^2+2\} - \{l(l+1)\}^2\}}a_0/2, \quad (592)$$

where a_0 is the Bohr radius of the atom. One can then infer from this that a minimal typical size is $3a_0/2$ and that the radius and its spread both become large for large quantum numbers. C.f. how the modal estimate of minimal typical size is a_0 itself; the slight disagreement between these is some indication of the limited accuracy to which either estimate should be trusted. Also, the above can be identified as expectations of scale operators, and thus one can next ask whether they have pure-shape counterparts in the standard atomic context.

Up to normalization, they are the $3\mathcal{Y}$ integrals [416] (for \mathcal{Y} spherical harmonics, the radial parts of the integration now trivially cancelling), the general case of which has been evaluated in terms of Wigner 3j symbols [416] (here more properly termed 3d symbols). Furthermore, many of the integrals for the present article’s specific cases of interest are provided case-by-case in [468]. Shape operators for hydrogen are also considered in [67] (briefly) and [149]. Also see [50] for comments on shape operators elsewhere in Molecular Physics and a start on the corresponding question of shape operators in mini and midisuperspace.

Moreover, the context in which shape operators occur in Molecular Physics is wider than just the above.

Example 1) expectations of $\cos \beta$ for β a relative angle from inner products between physically meaningful vectors. Examples of such are i) between the 2 electron–nucleus relative position vectors in Helium. ii) In the characterization of molecules’ bonds or in nuclear spin-spin coupling (p 443 of [50]).

Example 2) one also gets expectations of $\mathcal{Y}_{20}(\theta)$ [c.f. form 4 of (404)] in spin-spin and hyperfine interactions (p 437-441) of [50] (as a shape factor occurring alongside a $1/r^3$ scale factor).

Example 3) In the study of the H_2^+ molecular ion, one uses fixed nuclear separation as a scale setter. Then one has not only 1 relative angle but also 2 ratios forming spheroidal coordinates with respect to which this problem separates, and expectations of all these things then make good sense.

We contemplate ‘mini- and midi’^{superspace} counterparts of such shape operators in the Sec 10.9.

Overlap integrals $\langle D_1 d_1 | \widehat{\text{Operator}} | D_2 d_2 \rangle$ are relevant for three applications 1) expectation and spread of shape operators (below). 2) Time-independent perturbation theory about very special multi-HO solution. 3) Time-dependent perturbation theory on space of shapes with respect to a time provided by the scale in the scale–shape split scaled RPM models in semiclassical formulation also makes use of these. This parallels Halliwell–Hawking’s work [296] and embodies one of the RPM program’s eventual goals, so I prefer to give details of computing the overlaps to giving details of application 2). 2) and 3) have the merit of extending to far more general potential terms than the harmonic oscillator-like terms discussed in the present working. Also, 2) in the scaled case, 2) survives as a subproblem in the corresponding time-independent non-semiclassically approximated scale–shape split scaled RPM.

7.7.2 Multi-HO potential example

As regards the expectation of the size operator for 3-stop metroland isotropic case in scale–shape coordinates, using the normalization result for associated Laguerre polynomials and Gaussian integral results, $\langle 0 \, d \, | \widehat{\text{Size}} | 0 \, d \rangle = \binom{2|d|+1}{|d|} \frac{\sqrt{\pi}\{|d|+1\}}{2^{2|d|+1}} \rho_{\text{HO}}$.

Thus e.g. $\langle 00 | \widehat{\text{Size}} | 00 \rangle = \frac{\sqrt{\pi}}{2} \rho_{\text{HO}}$ and the large- $|d|$ limit for $N = 0$ is $\sqrt{\hbar|d|/\omega}$ which slowly rises to become arbitrarily large for configurations possessing more and more relative distance momentum. Also, $\langle 10 | \widehat{\text{Size}} | 10 \rangle = \frac{7\sqrt{\pi}}{8} \rho_{\text{HO}}$. In comparison, the mode value for the ground state is $\rho_{\text{HO}}/\sqrt{2}$.

The spreads are an integral made easy by a recurrence relation on the generalized Laguerre polynomials, $\langle N \, d \, | \widehat{\text{Size}}^2 | N \, d \rangle = \{2N + |d| + 1\} \rho_{\text{HO}}^2$, minus the square of the expectation. Thus $\Delta_{00}(\widehat{\text{Size}}) = \frac{4-\pi}{4} \rho_{\text{HO}}^2 \approx 0.215 \rho_{\text{HO}}^2$, while the large- $|d|$ limit with $N = 0$ gives $\Delta_{0|d|}(\widehat{\text{Size}}) \rightarrow \rho_{\text{HO}}^2$. Also, $\Delta_{10}(\widehat{\text{Size}}) = \frac{192-49\pi}{64} \rho_{\text{HO}}^2 \approx 0.595 \rho_{\text{HO}}^2$.

The expectations of the 3-stop metroland shape operators are all zero by two-angle formulae and the orthogonality of Fourier modes. Likewise, the spreads of the shape operators are 1/2 in all states. All of the above calculations benefit from factorization into shape and scale parts, with insertion of a pure-shape operator rendering the scale factor trivial and vice versa.

8 Pure-shape quantum RPM

8.1 Quantum pure-shape 4-stop metroland

In the reduced scheme, the pure-shape case's configuration space geometry is easier and so this case is treated first. For 4-stop metroland, the Laplace ordering and the conformal ordering coincide, both in the Dirac and in the reduced setting. Thus there is no trouble in this case with matching up reduced calculations with Dirac ones.

8.1.1 QM solutions for pure-shape 4-stop metroland

The multi-HO time-dependent Schrödinger equation is

$$\{\sin \theta\}^{-1} \{\sin \theta \Psi_{,\theta}\}_{,\theta} + \{\sin \theta\}^{-2} \Psi_{,\phi\phi} = \{\mathcal{A} + \mathcal{B} \cos 2\theta + \mathcal{C} \sin^2 \theta \cos 2\phi\} \Psi, \quad (593)$$

where $\mathcal{A} = 2\{A - E\}/\hbar^2$, $\mathcal{B} = 2B/\hbar^2$, $\mathcal{C} = 2C/\hbar^2$ are dimensionless constants.

8.1.2 Solution in very special case

The $\mathcal{C} = 0$ case of Eq (593) separates to simple harmonic motion and the θ equation

$$\{\sin \theta\}^{-1} \{\sin \theta \Psi_{,\theta}\}_{,\theta} - \{\sin \theta\}^{-2} D\{D+1\} \Psi = \mathcal{A} \Psi + \mathcal{B} \cos 2\theta \Psi. \quad (594)$$

If $\mathcal{B} = 0$ as well (our very special problem), then from Sec 4, this has similar mathematics to ordinary QM's central potential problem. For this, the quantum Hamiltonian \hat{H} , total angular momentum $\hat{\mathcal{L}}_{\text{Total}} = \sum_{\mu=1}^3 \hat{\mathcal{L}}_{\mu}^2$ and magnetic/axial/projected angular momentum $\hat{\mathcal{L}}_3$ form a complete set of commuting operators. Thus, they share eigenvalues and eigenfunctions. Recollect that the RPM very special problem is mathematically the same as the rigid rotor, for which \hat{H} is $\mathcal{L}_{\text{Total}}$ up to multiplicative and additive constants. Thus, effectively one has a complete set of two commuting operators, whose eigenvalues and eigenfunctions are the well-known spherical harmonics and, moreover also occur as a separated-out part of the corresponding scaled relational particle model problem. However, the RPM 'rigid rotor' is in configuration space rather than in space and with total relative dilational momentum $\widehat{\mathcal{T}}_{\text{Tot}} = \sum_{i=1}^3 \widehat{\mathcal{D}}i_i^2$. These are in place of total angular momentum and projected relative distance momentum $\widehat{\mathcal{D}}i_3$ in place of axial angular momentum. These then have eigenvalues $\hbar^2 D\{D+1\}$ and $\hbar d$ respectively. Thus I term D and d respectively the *total* and *projected relative dilational quantum numbers*. (These are analogous to the ordinary central force problem/rigid rotor's total and axial/magnetic angular momentum quantum numbers).

As well as the separation involving spherical coordinates, another approach is to

- 1) use that H and $\rho \mathcal{D}$ form a complete set of commuting operators.
- 2) Then solve the linear constraint first to get that the wavefunction is independent of ρ and then knock out ρ derivatives in the H to get a pure shape problem. This approach works for all N , showing that N -stop metroland indeed has scale-shape split at the quantum level. Note however that this will not always map to conformal problem, so 5-stop metroland and so on will have a constant difference in the energy.

Our very special problem's time-independent Schrödinger equation separates into simple harmonic motion and the associated Legendre equation (in $X = \cos \theta$) i.e. the spherical harmonics equations, Thus its solutions are

$$\Psi_{Dd}(\theta, \phi) \propto \mathcal{Y}_{Dd}(\theta, \phi) \propto P_D^d(\cos \theta) \exp(\pm i d \phi) \quad (595)$$

for $P_D^d(X)$ the associated Legendre functions of X , $D \in \mathbb{N}_0$ and d such that $|d| \leq D$. Also, $D\{D+1\} = -\mathcal{A}$, which, interpreted in terms of the original quantities of the problem, is the condition $E' = E - K_3/2 = \hbar^2 D\{D+1\}/2$ on the model universe's 'energy' and inter-cluster effective spring in order to have any quantum solutions. (E is *fixed* as this is a whole-universe model so there is nothing external from which it could gain or lose energy). If this is the case, there are then $2D+1$ solutions labelled by d (the preceding sentence cuts down on a given system's solution space, other than in the multiverse sense).

For further interpretation, using a basis with sines and cosines instead of positive and negative exponentials,

$$\Psi_{D\mathcal{N}}(n^i) \propto \mathcal{N}(n^i). \quad (596)$$

Here, the D -label runs over the orbital types (s for $D=0$, p for $D=1$, d for $D=2$...) and \mathcal{N} is the 'naming polynomial' i.e. 1 for s , n_x for p_{n_x} , $n_x n_y$ for $d_{n_x n_y}$ etc. (Note that the name ' z^2 ' in $d_{n_z^2}$ is indeed *shorthand* for $z^2 - 1/3$; shorthand begins to proliferate if one goes beyond the d -orbitals. The polynomials arising in this working are also subject to being 'nonunique' under $\sum_{i=1}^3 n^i{}^2 = 1$.) That the wavefunctions are their own naming polynomials is the configuration space analogy of how the orbitals in space historically got their Cartesian names. It is also akin to representations [185] of the spherical harmonics in terms of homogeneous polynomials. Another form for the solution is

$$\Psi_{D|d|}(n^i) \propto P_D^d(n_z) \mathcal{T}_d \left(n_x / \sqrt{1 - n_z^2} \right). \quad (597)$$

The first factor is purely in $\text{RelSize}(12,34)$ whilst the second is a mixed function of this and $\text{RelSize}(12)$. Also, using the tessellation method, I can interpret the wavefunctions in terms of the metroland mechanics on the sphere itself, on which they take the particularly familiar ‘orbital’ form.

For $D, d = 0, 0$ (s -orbital), note that the axis is arbitrary so it is evident from using 2 different principal axes that the probability distribution function on-axis is not to be trusted in spherical coordinates about that axis. We conclude that the ground state does not have bias toward any particular configurations. For $D, d = 1, 0$ (p_{n_z} orbital), equatorial configurations are improbable, meaning that mergers of the $\{12\}$ and $\{34\}$ clusters [including the non $\{12,34\}$ DD’s] are disfavoured. On the other hand, near-polar configurations are probable, meaning that the $\{12\}$ and $\{34\}$ clusters being small and well-apart is favoured. For $D, d = 1, \pm 1$, in the p_{n_y} orbital case, the $n_y = \text{RelSize}(3,4)$ axis part of the equator is probable, so mergers of a small $\{12\}$ and a large $\{34\}$ are favoured. p_{n_x} is the $\{12\} \longleftrightarrow \{34\}$ of this. For $D, d = 2, 0$ ($d_{n_z^2}$ orbital), both equatorial and polar configurations are probable, so that the $\{12\}$ and $\{34\}$ clusters are either merged or small and well-apart. For $D, d = 2, \pm 1$, in the $p_{n_y n_z}$ case, equatorial configurations are improbable, so mergers of $\{12\}$ and $\{34\}$ are improbable, and also the $\{12\}$ cluster is small; DD’s are disfavoured. $d_{n_x n_z}$ is the $\{12\} \longleftrightarrow \{34\}$ of this. For $D, d = 2, \pm 2$, in the $d_{n_x^2 - n_y^2}$ case, equatorial configurations, i.e. mergers of $\{12\}$ and $\{34\}$, are probable, especially those with one but not both of the clusters are large. (I.e. configurations along one of the $\text{RelSize}(12)$ or $\text{RelSize}(34)$ axes: contents inhomogeneity), including the $\{13,24\}$, $\{14,23\}$, $\{23,14\}$ and $\{24,13\}$ DD’s. In the $d_{n_x n_y}$ case, again equatorial configurations are probable. However, now these have $|\text{RelSize}(12)| \approx |\text{RelSize}(3,4)|$ i.e. contents homogeneity, including the DD’s where the two clusters are on top of each other $\{13,24\}$ and $\{14,23\}$.

While this case is simple and not general, further motivation for the very special problem is that its exact solution serves as something about which one can conduct a more general perturbative treatment (Sec 8.1.5).

8.1.3 4-stop metroland examples

As regards good shape operators for 4-stop metroland, the kinematical quantization carries guarantees that the three n^i are promoted to good quantum operators. These can be interpreted as $\text{RelSize}(1,2)$, $\text{RelSize}(3,4)$ and $\text{RelSize}(12,34)$. It is also useful to note at this stage that n_z is not only physically $\text{RelSize}(12,34)$ but also mathematically the Legendre variable.

Then $\langle D d | \widehat{\text{RelSize}}(1,2) | D d \rangle = \langle D d | \widehat{\text{RelSize}}(3,4) | D d \rangle = 0$ and $\langle D d | \hat{\theta} | D d \rangle \approx \langle D d | \widehat{\text{RelSize}}(12,34) | D d \rangle = 0$ as an obvious result of orientational symmetry. The useful information starts with the spreads,

$$\Delta_{D d}(\widehat{\text{RelSize}}(1,2)) = \sqrt{\frac{D\{D+1\} + d^2 - 1}{\{2D-1\}\{2D+3\}}} Q_1(d) \quad , \quad \Delta_{D d}(\widehat{\text{RelSize}}(3,4)) = \sqrt{\frac{D\{D+1\} + d^2 - 1}{\{2D-1\}\{2D+3\}}} Q_2(d) \quad , \quad (598)$$

$$\Delta_{D d}(\hat{\theta}) \approx \Delta_{D d}(\widehat{\text{RelSize}}(12,34)) = \sqrt{\frac{2\{D\{D+1\} - d^2\} - 1}{\{2D-1\}\{2D+3\}}} \quad . \quad (599)$$

Here, $Q_2(d) = 1/2$ for the d cosine solution, $3/2$ for the d sine solution, and 1 otherwise, and $Q_1(d)$ the $\sin \longleftrightarrow \cos$ of this. One can then readily check that $\langle \hat{n}_x^2 + \hat{n}_y^2 + \hat{n}_z^2 \rangle = 1$, as it should be.

One case of interest is the ground state. Therein, the spreads in each are $1/\sqrt{3}$. Another case of interest is the large quantum number limit. $\Delta_{D d}(\hat{\theta}) \approx \Delta_{D d}(\widehat{\text{RelSize}}(12,34))$ which, for the maximal d ($|d| = D$), is equal to $1/\sqrt{2D+3}$ which goes as $1/\sqrt{2D} \rightarrow 0$ for D large. The hydrogen counterpart of this result is $\Delta_{11}\hat{\theta}_{\text{sp}} \approx 1/\sqrt{21} \rightarrow 0$, i.e. restriction to the Kepler–Coulomb plane (e.g. [50] outlines this, while [149] considers it in more detail). Back to the RPM problem, this result therefore signifies recovery of the equatorial classical geodesic as the limit of an ever-thinner belt in the limit of large maximal projectional relative dilational quantum number $|d| = D$. (This is ‘the rim of the disc’ in Sec 5’s classical representation of this problem, traversed in either direction according to the sign of d). In fact, as for the constant potential one can consider the axis to be wherever one pleases, this leads to recovery of *any* of the classical geodesics. Also, for $d = 0$, $\Delta_{D 0}\hat{\theta} \approx \Delta_{D 0}(\widehat{\text{RelSize}}(12,34)) \rightarrow 1/\sqrt{2}$ for D large. This means that the $s, p_{n_z}, d_{n_z^2} \dots$ sequence of orbitals does not get much narrower as D increases. Thus, for these states there is only limited peaking about clusters $\{12\}$ and $\{34\}$ both being small and well apart. This is a situation which shall be revisited in the next subsection due to its centrality to the assumptions made in, and applications of, this article. The $\text{RelSize}(1,2)$ and $\text{RelSize}(3,4)$ operators’ spreads tend to finite constant values for large D no matter what value d takes.

What of $\hat{\phi}$? Now, clearly, by factorization and cancellation of the θ -integrals, the $d = 0$ states obey the uniform distribution over 0 to 2π , with mean π and variance $\pi^2/3$ (corresponding to axisymmetry). Furthermore, $\langle D d | \hat{\phi} | D d \rangle$ is also π and cosine and sine states have

$$\Delta_{D d}(\hat{\phi}) = \sqrt{\pi^2/3 + 1/2d^2} \quad \text{and} \quad \Delta_{D d}(\hat{\phi}) = \sqrt{\pi^2/3 - 1/2d^2} \quad , \quad (600)$$

which indicate some resemblance to the uniform distribution arising for large d . This is a flower of $2D$ petals that gets more and more equatorially flat as D gets larger, thus tending to the equatorial great circle classical path (in parallel to [149] for hydrogen). (Mean and variance do not see the multimodality. But, at least, by inspection along the lines of the preceding subsection, it is *regular* multimodality for d maximal. I.e., one has, by inspection of the shapes of the standard maximal $s, p, d, f, g \dots$ orbitals, equatorial flowers of $2-d$ petals.)

8.1.4 Solution in special case – large and small regimes

Passing to stereographic coordinates, PPSC'T'ing to the flat representation and applying the small approximation, the Schrödinger equation becomes

$$-\{\hbar^2/2\}\{\mathcal{R}^{-1}\{\mathcal{R}\Psi_{,\mathcal{R}}\}_{,\mathcal{R}} + \mathcal{R}^{-2}\Psi_{,\phi\phi}\} = \mathcal{E} - \Omega^2\mathcal{R}^2/2, \quad (601)$$

which is in direct correspondence with the 2- d quantum isotropic harmonic oscillator (see e.g. [458, 569, 531] under $r \rightarrow \mathcal{R}$ (radial coordinate), $1 \longleftrightarrow$ particle mass, and with the above ω as classical frequency ($\times I$). Thus,

$$\mathcal{E} = n \hbar \omega \quad \text{for } n := 1 + 2N + |d| \quad (602)$$

node-counting quantum number running over \mathbb{N}_0 and d a ‘projected’ dilational quantum number as in the preceding subsection but now running over \mathbb{Z} . [The ‘shifted energy’ in its usual units, $\mathcal{E}' = \mathcal{E} - \mathcal{A} - \mathcal{B}$, itself goes as $\mathcal{E}' = \{n^2\hbar^2/2\}\{1 + \sqrt{1 - \mathcal{B}\{4/n\hbar\}^2}\}$, so for $n\hbar/\omega \ll 1$ (small quantum numbers as used below), $\mathcal{E}' \approx n\hbar\Omega$ for $\Omega = 2\sqrt{-\mathcal{B}}$.] The solutions are then (to suitable approximation)

$$\Psi_{Nd}(\theta, \phi) \propto \theta^{|d|} \{1 + |d|\theta^2/12\} \exp(-\omega\theta^2/8\hbar) L_N^{|d|}(\omega\theta^2/4\hbar) \exp(\pm id\phi) \quad (603)$$

for $L_a^b(\xi)$ the associated Laguerre polynomials in ξ (see Overall Appendix C). [As regards physical interpretation, the ϕ -factor of this is rewriteable as before in terms of the n^i or RelSize(12,34) and RelSize(1,2), while the θ -factor is now a somewhat more complicated function of RelSize(12,34)].

The large regime gives the same eigenvalue condition (602), and (603) again for wavefunctions except that one now uses the supplementary angle $\zeta = \pi - \theta$ in place of θ . Next, see Fig 45 for the form and interpretation of the wavefunctions.

In the small regime, the RelSize(1,2) and RelSize(3,4) operators still have zero expectation as each sign for these remains equally probable. For D , d substantially smaller than ω/\hbar so powers of the latter dominate powers of the former. [And ω/\hbar was considered to be large, so this works for the kind of quantum numbers in this subsection’s specific calculations]. Then the following mean and spread results for shape operators are derived using orthogonality of, and a recurrence relation for, Laguerre polynomials, as provided in Overall Appendix C.

$$\langle Nd | \widehat{\text{RelSize}(12,34)} | Nd \rangle = 1 - 2n\hbar/\omega. \quad (604)$$

$\Delta_{Nd} \widehat{\text{RelSize}(12,34)}$ is zero to first two orders, beyond which the approximations used begin to break down, but it would appear to have leading term proportional to \hbar/ω . These results signify that the potential has trapped what was uniform in the preceding example into a narrow area around the {12,34} DD collision. Furthermore,

$$\Delta_{Nd} \widehat{n^{\bar{a}}} \approx \sqrt{2n\hbar Q_{\bar{a}}(d)/\omega} \quad (605)$$

for $\bar{a} = 1, 2$ gives $\Delta_{Nd}(\widehat{\text{RelSize}(1,2)})$ for $\bar{a} = 1$ and $\Delta_{Nd}(\widehat{\text{RelSize}(3,4)})$ for $\bar{a} = 2$. So one can obtain strong concentration around the poles by suitable choice of springs, amounting to a tall thick equatorial barrier and polar wells. The ground state has the tightest spread in RelSize(1,2) and RelSize(3,4): $\sqrt{2\hbar/\omega}$. This has some parallels with how the Bohr radius is an indicator of atomic size, including the hydrogen–isotropic harmonic oscillator correspondence [569].

8.1.5 Perturbations about the very special solution

Begin by recasting the RPM Schrödinger equation in Legendre variables $n_z = \cos \theta$,

$$\{\{1 - n_z^2\}\Psi_{,n_z}\}_{,n_z} + \{1 - n_z^2\}^{-1}\Psi_{,\phi\phi} = \{\mathcal{A} - \mathcal{B} + n_z^2\{2\mathcal{B} - C\cos 2\phi\} + C\cos 2\phi\}\Psi, \quad (606)$$

and then study this using time-independent perturbation theory (see e.g. [416] for derivation of the formulae for this up to second order). Applying perturbation theory here means considering 1) \mathcal{C} small, which is high contents homogeneity at the level of each cluster’s (Hooke coefficient)/(reduced mass) in the sense that $K_1 - K_2$ is small compared to \hbar^2 . 2) \mathcal{B} small, in the sense that \hbar^2 is large compared to $\{K_1 + K_2\}/2 - K_3$, which collapses to $K_1 - K_3$ small in the case of $\mathcal{C} = 0$. This means that there is little difference between the inter-cluster spring and the intra-cluster springs.

E.g. perturbative study of (606) is amenable to exact calculations though involving various of trigonometric and standard/tabulated associated Legendre function integrals, or, alternatively, the aforementioned 3- \mathcal{Y} integrals. Furthermore, this continues to be the case if one includes a non-diagonal/non-normal basis’ D , E and F terms.

For the \mathcal{B} -perturbation, as both it and the unperturbed Hamiltonian commute with \mathcal{D} , the eigenvalue problem can be solved separately in each subspace \mathbf{V}_d of a given eigenvalue d of \mathcal{D} . As then in each such subspace the spectrum of the unperturbed Hamiltonian is nondegenerate, nondegenerate perturbation theory is applicable (this argument parallels e.g. p 697 of [458]). This gives (with the unperturbed problem’s \mathcal{A} playing the role usually ascribed to the energy and H' the perturbative term)

$$\mathcal{A}_{Dd}^{\{1\}} = \langle Dd | H' | Dd \rangle \quad (607)$$

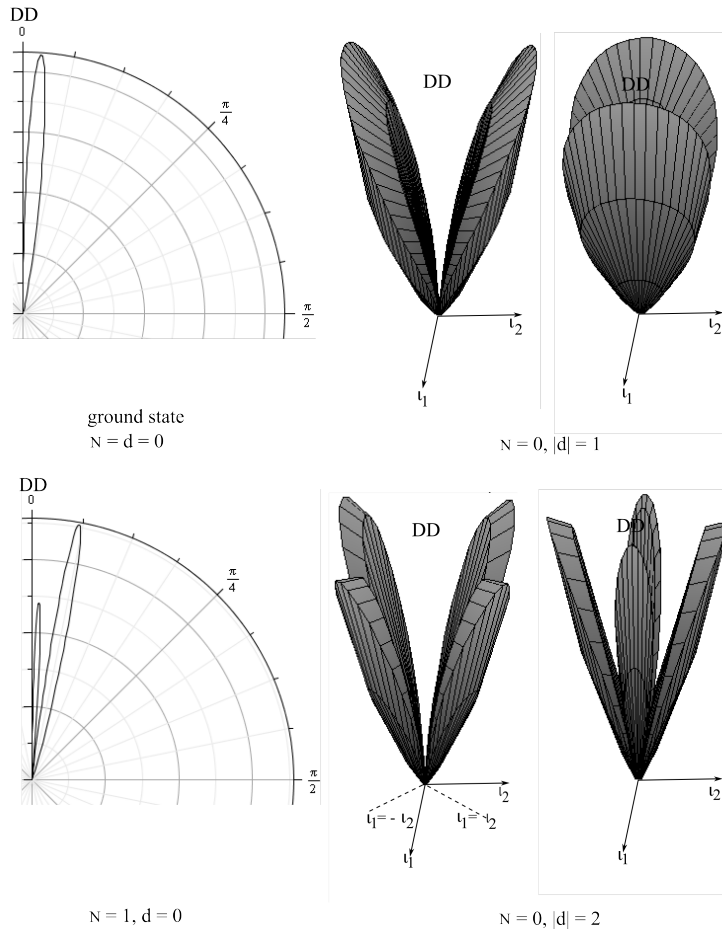


Figure 45: Probability density functions for this subsection's problem for ω/\hbar large, 400, say, plotted using Maple [440]. All $d = 0$ states are axisymmetric about the $\{12,34\}$ clustering's DD axis, i.e. all relative sizes for cluster $\{12\}$ and for cluster $\{34\}$ are equally favoured. ι^i is my former notation to ρ^i . The ground state is peaked around the $\{12,34\}$ DD collision. It is the surface of revolution of the given curve. The $N = 0$, $|d| = 1$ solutions are a degenerate pair. Each takes the form of a pair of inclined lobes – the cosine solution's is oriented about the $n_x = \text{RelSize}(1,2) = 0$ D collision and the sine solution's about the $n_y = \text{RelSize}(3,4) = 0$ D collision. These next three solutions form a degenerate triplet. The $N = 1$, $d = 0$ solution is a slender bulge around the $\{12,34\}$ DD collision, then a gap and then a second bulge in the form of a cone. This corresponds to 'a band very close to this DD collision and a band somewhat close to it being probable, while all other configurations are improbable. The $N = 0$, $|d| = 2$ solutions are tulips of four petals, the cosine one separately favouring the lunes of near $\{12\}$ D collisions and near $\{34\}$ D collisions (i.e. contents inhomogeneity). On the other hand, the sine one disfavors these and favours instead the lunes at $\pi/4$ to the preceding, which correspond to contents homogeneity of the $\{12\}$ and $\{34\}$ clusters.

The large case's approximate solution is just the reflection of the preceding about the equatorial plane with the same interpretation except that $\{34\}$ is now to the left of $\{12\}$.

at first order and

$$\mathcal{A}_{Dd}^{\{2\}} = - \sum_{D', d' \neq D, d} |\langle D' d' | H' | D d \rangle|^2 / \{ \mathcal{A}_{D'} - \mathcal{A}_D \} \quad (608)$$

at second order [416]. Then e.g. [31] double use of a standard recurrence relation [1] gives a $\Delta d = 0$, $\Delta D = 0, \pm 2$ 'selection rule' paralleling that for the Raman spectrum of a polarized linear rotor. Moreover, the terms that survive this take the following forms.

$$\langle D d | \mathcal{B}\{2n_z^2 - 1\} | D d \rangle = \mathcal{B}\{1 - 4d^2\} / \{2D - 1\}\{2D + 3\}, \quad (609)$$

which is closely related to the expectation of $n_z = \text{RelSize}(12,34)$ already computed in Ref . Two new overlaps that are more general than expectations are

$$\langle D + 2 d | \mathcal{B}\{2n_z^2 - 1\} | D d \rangle = \frac{2\mathcal{B}}{2D + 3} \sqrt{\frac{\{D + 2\}^2 - d^2}{\{2D + 5\}\{2D + 1\}}}, \quad (610)$$

and then, swapping D for $D - 2$, also,

$$\langle D - 2 d | \mathcal{B}\{2n_z^2 - 1\} | D d \rangle = \frac{2\mathcal{B}}{2D - 1} \sqrt{\frac{\{D - 2\}^2 - d^2}{\{2D + 1\}\{2D - 3\}}}. \quad (611)$$

Using these then gives the perturbed 'pseudoenergies':

$$E_{Dd}^{\{2\}} = A + \hbar^2 D \{D + 1\} / 2 + B \{1 - 4d^2\} / \{2D - 1\}\{2D + 3\} +$$

$$\frac{4B^2\{\{2D+5\}\{2D+3\}^3\{D^2-d^2\}\{\{D-1\}^2-d^2\}-\{2D-1\}^3\{2D-3\}\{\{D+2\}^2-d^2\}\{\{D+1\}^2-d^2\}\}}{\hbar^2\{2D+5\}\{2D+3\}^3\{2D+1\}\{2D-1\}^3\{2D-3\}}+O(B^3). \quad (612)$$

Note that d positive and negative are treated the same, so there is only a partial uplifting of degeneracy. Changes to the wavefunction due to the perturbations for the sign of B corresponding to 8.1.4 and to second order in B are slight bulges at the poles for the ground state (a bit of $d_{n_z^2}$ mixed in).

Unlike for triangle land, the C -perturbation cannot be turned into a B -perturbation with respect to a rotated basis. But the C perturbation can likewise be studied based on half-way stage overlaps that can be directly transcribed by the angular momentum to relative distance momentum analogy from those computed in e.g. [468]. E.g. the surviving terms are found [31] by a second standard recurrence relation [1] to obey the selection rule $\Delta d = \pm 2$, $\Delta D = 0$. Some noteworthy features of the study of the C term are that degenerate perturbation theory is now required, there is no first order contribution as $\Delta d = \pm 2$ only. Now d and $-d$ do get shifted differently corresponding to this perturbation not preserving the axis of symmetry. In nondiagonal/nonnormal form, the further D term has the same selection rule to the C term's while the E and F terms share the selection rule $\Delta d = \pm 1$, $\Delta D = 0, \pm 2$. The above 'noteworthy features' apply to these also.

8.1.6 Molecular Physics analogies for 4-stop metroland

Analogy A) (594) occurs in Mathematical Physics, e.g. from the separation of the wave equation in prolate spherical coordinates [609, 594, 479, 1]). It also has multiple applications in Molecular Physics studies of which parallel some of the studies in the present article. Examples of this in Molecular Physics are as follows.

Analogy A.1) (594) recast in terms of the Legendre variable is

$$\{\{1-n_z^2\}\Psi_{,n_z}\}_{,n_z}-\{1-n_z^2\}^{-1}D\{D+1\}\Psi=\{\mathcal{A}-\mathcal{B}+2\mathcal{B}n_z^2\}\Psi, \quad (613)$$

which is the easier of the two spheroidal equations that arise in the study of the H_2^+ molecular ion [73, 578, 564]. This and the next two analogies are for $B < 0$, although the aforementioned Mathematical Physics literature covers $B > 0$.

Analogies A.2-3) are Sec 5.2's rotation in a crystal (Pauling) and molecular polarizability (Raman).

Analogy A.4) Examples 2) of Sec 8.1.5 is another substantially developed area in the Molecular Physics literature.

Analogy B) is with the ammonia molecule NH_3 , in the following rougher but qualitatively valuable sense. NH_3 has two potential wells separated by a barrier and then is capable of tunnelling between the two at the quantum level (like an umbrella inverting in the wind). Our model for $B < 0$ is similar to this, albeit in spherical polar coordinates: there are two polar wells with an equatorial barrier in between.

This analogy then gives us some idea about how the separate solutions for the two wells compose. For NH_3 , one can start with separate solutions for each well and additional degeneracies ensue (due to the wells being identical and being able to distribute some fixed energies between these in diverse ways). However the wavefunctions tend to perturb each other toward breaking these degeneracies, forming symmetric and antisymmetric wavefunctions over the two wells [50, 578].

8.1.7 Applications of the analogies and developing an overall picture

- 1) Rotor regime: the $B < 0$ locally stable small or large regimes are the kind of regimes that are termed 'rotor-like' in analogy A.2)'s literature; both of the $SO(3)$ quantum numbers (for us, dilational quantum numbers) hold good in this regime.
- 2) Putting together the small and large θ approximations: for $B < 0$ one can use Analogy B) to form a simple picture of this. d remains a good quantum number for the unapproximated problem. Thus, one expects to need the North Pole approximation's d and the South Pole approximation's d to match and the subsequent perturbations exacted by these two approximations upon each other not to affect d . Also, prior to any recombination, one has degeneracies as follows (call the near-North Pole's node-counting quantum number N and the near-South Pole's N'). There is the one ground state $N = N' = d = 0$, then the degenerate pair $N = N' = 0, d = \pm 1$, and then the degenerate quadruplet $N = 1, N' = 0$ or $N = 0, N' = 1$ for each of $d = \pm 1$. Now if N and N' match, $\langle \text{RelSize}(12, 34) \rangle$ goes to 0 again though the wavefunction's distribution is bimodal about both poles. If they do not match, $\langle \text{RelSize}(12, 34) \rangle \neq 0$ due to the peaking near the two poles being different in detail. The flip here, as in NH_3 , is an inversion, i.e. it reverses the orientation, sending 1,2,3,4 to 4,3,2,1.
- 3) Analogy A.3) is well-known for its Raman-type ± 2 and not ± 1 selection rule, which parallels the results of Sec 8.1.5. Analogy A.3) has furthermore been studied perturbatively for what is for us the small $B < 0$ regime. This allows us to e.g. check the half-way house results (609, 610, 611) against p 271-273 of [614].
- 4) Analogy A.1)'s references [594, 479] include i) analysis of this equation's poles in the complex plane. ii) How it admits a solution in the form of an infinite series in associated Legendre functions in the vicinity of ± 1 and in Bessel functions in the vicinity of ∞ . iii) How to piece together these different representations. It is then appropriate to compare results from the expansion in associated Legendre functions against our perturbative regime (this particular working holds regardless of the sign of B). Thus the lowest four cases of (612) agree with p 1502-4 of [479]. This additionally provides the corresponding wavefunctions which are used to first order in B in Sec 14.2 to evaluate the naïve Schrödinger interpretation probabilities for these states' model universes being large.
- 5) Large regime, one cannot really put together the near-polar calculations and the perturbative calculations, because the "B small perturbative condition" goes a long way toward ω being small and then only a small amount of the wavefunction

is near the pole. Our near-polar calculations should be compared, rather, with the asymptotics for B large. Analogy A.2)'s literature covers this for what for us is the B large negative ($>> E - A - B = E'$) regime, giving, via the analogy,

$$E' \sim n\hbar\omega_{\text{large}} + O(1/\omega_{\text{large}}) \quad \text{and} \quad (614)$$

$$\Psi \propto \exp(\omega_{\text{large}} \cos\theta/\hbar) \{ \{ \tan \frac{\theta}{2} \}^{2N} \{ \sec \frac{\theta}{2} \}^{2\{d\}+1} + O(1/\omega_{\text{large}}) \} \exp(\pm i d \phi) \quad (615)$$

as the relevant asymptotic solutions, for $\omega_{\text{large}} = 2\sqrt{-B}$. Now, from (411,412) the small- θ approximate solution (603)'s $\omega = 8\sqrt{-B} = 4\omega_{\text{large}}$. Thus, $\exp(\omega_{\text{large}} \cos\theta/\hbar)$ in (614) $\approx \text{const} \times \exp(\{\omega/4\hbar\}\{-\theta^2/2\})$, which is indeed in agreement with the leading and dominant factor of (603). For the RPM model, this regime signifies that $K_3 \gg K_1, K_2$ i.e. that the inter-cluster spring is much stronger than each of the intra-cluster springs. This is a 'harmonic oscillator-like regime' – comparing (614) and the standard result for the $2-d$ isotropic harmonic oscillator makes it clear why. d alone is a good quantum number in this regime.

6) the spheroidal equation has led to many hundreds of pages of tabulations [594] and further numerical work e.g. in [1, 234, 455, 227, 228], though the most recent of this states that this study is still open in some aspects.

7) One can furthermore envisage analogy A.2) being extended to have the further parallel with this model via a rotationally-dislocated molecule in a cubic crystal having preferred directions in space of approximately the same form as this model's are in configuration space. We do not know if such a study has been done.

8) The polarization analogy A.3) has been extended [623, 13] to include the counterparts of the C, D, E and F terms. For, what one has more generally is a symmetric polarization tensor α such that $\mu_\rho = \alpha_{\rho\sigma} E_\sigma$. Then for the CO_2 model in a diagonal basis $\alpha_z = \alpha_{||}$ giving the combination $-\alpha_\perp \sin^2\theta - \alpha_{||} \cos^2\theta$ [This is a slight improvement of analogy A.3) by inclusion of the smaller $\alpha_\perp = \alpha_x = \alpha_y$]. Furthermore, this readily rearranges to the special case of the third form of (404). But for more general groups than just oxygen atoms at each end of the axis (while still remaining in a diagonal basis) $\alpha_x \neq \alpha_y$, giving $-\alpha_x \sin^2\theta \cos^2\phi - \alpha_y \sin^2\theta \sin^2\phi - \alpha_z \cos^2\theta$ which is the general case of the second form of (404). Moreover, in non-diagonal bases, the off-diagonal elements form extra terms directly analogous to those in (404). Thus there is an extended analogy between this problem and the study of polarization, with 4-stop metroland's Jacobi-Hooke coefficients forming a configuration space-indexed analogue of the spatial-indexed polarizability tensor.

8.2 Extension to QM solutions for pure-shape N -stop metroland

Here the Laplace/conformal ordered time-independent Schrödinger equation is, with HO-like potentials

$$\frac{1}{\prod_{j=1}^{A-1} \sin^2\theta_j \sin^{n-1-A}\theta_A} \frac{\partial}{\partial\theta_A} \left\{ \sin^{n-1-A}\theta_A \frac{\partial\Psi}{\partial\theta_A} \right\} = -\mathcal{E}\Psi + \sum_{\bar{p}=1}^n \mathcal{K}_{\bar{p}} n_{\bar{p}}^2 \Psi \quad (616)$$

for $\mathcal{E} = 2E/\hbar^2$, $\mathcal{K}_{\bar{p}} = 2K_{\bar{p}}/\hbar^2$ and $n_{\bar{p}}$ the unit vector (written in terms of ultraspherical angles). There is also a highly special constant potential case within the multiple harmonic oscillator like potentials. This is now a more complicated sequence of associated Gegenbauer equations as explained in Overall Appendix C; these are nevertheless also fairly standard and well-documented [1, 270]) and then study perturbations about this. Then, if the associated quantum number is not zero, one does not get the Gegenbauer pairings or the right weights straight away. Computation of first order perturbations in this case requires the Gegenbauer parameter converting recurrence relation (1058) as well as the polynomial order reducing recurrence relation (1057). Thus, the calculation is somewhat more complicated in this case.

For constant potential, the time-independent Schrödinger equation is an equation of form $\Delta\Psi = \Lambda\Psi$. Then the separation ansatz $\Psi = \prod_{\bar{p}=1}^{n-1} \psi_{\bar{p}}(\theta_{\bar{p}})$ yields the simple harmonic motion equation for θ_{n-1} and $n-2$ equations of form

$$\{1 - X_{n-p}^2\} \frac{d^2\psi_{n-p}}{dX_{n-p}^2} - \{p-1\} X_{n-p} \frac{d\psi_{n-p}}{dX_{n-p}} + j_{p-1} \{j_{p-1} + p-2\} \psi_{n-p} - \frac{j_{p-2} \{j_{p-2} + p-3\}}{1 - X_{n-p}^2} \psi_{n-p} = 0 \quad (617)$$

under the transformations $X_{\hat{p}} = \cos\theta_{\hat{p}}$, $\hat{p} = 1$ to $n-2$. These are associated Gegenbauer equations (1059) with parameter $\lambda_p = \{p-2\}/2$, where $j_{p-1}, j_{p-2} \in \mathbb{N}_0$ are picked out as eigenvalues, so that the $\{n-p\}$ th equation is solved by $C_{j_{p-1}}^{j_{p-2}}(\cos\theta_{n-p}; \{p-2\}/2)$.

Then one gets a sequence of quantum numbers taking the ranges $j_p \in \mathbb{Z}$ such that $|m_p| \leq |m_{p-1}|$ for $\bar{p} = 1$ to $n-2$ and $j_{n-1} \in \mathbb{N}_0$ (a straightforward generalization of the restriction on the two angular momentum quantum numbers in 3- d).

Notation: $\tau\dots d$ are a series of $N-2$ relative distance momentum quantum numbers subject to the usual kind of restrictions (and which are very well-known for this article's $N=3$ and $N=4$ examples). I use τ for the top relative distance momentum quantum number (which is d itself for $N=3$ and D for $N=4$).

The most special of these in each case has a constant potential and thus gives ultraspherical geodesics classically and the ultraspherical rigid rotor quantum-mechanically (solved by ultraspherical harmonics [31]). On the other hand, the next most special of these in each case has $\{N-1\}$ - d isotropic harmonic oscillator mathematics in its near-polar regime (solved by a power times a Gaussian times an associated Laguerre polynomial). Establishing a perturbative regime about each most special problem would then appear to be possible e.g. [31] by recurrence relations of the Gegenbauer polynomials [1, 270]. One technical difference is that, if one does use conformal operator ordering, then one can no longer use the configuration space

being 2- d to evoke collapse to Laplacian ordering. However, hyperspheres are of constant curvature and so of constant Ricci scalar curvature. Thus, $\xi \text{Ric}(\mathbb{S}^k)$ is just a constant, $\xi k \{k-1\}$ (our ‘E has no nonconstant prefactors PPSCT representation having the unit sphere as its configuration space). So, even in this case, the sole difference between Laplace and conformal ordering (or any other member of the Δ^ξ) family of operators) is in what is to be interpreted to be the zero of the energy. We also note that for $N = 5$ the analogy with the Halliwell–Hawking scheme is somewhat tighter, as both involve perturbative expansions in \mathbb{S}^3 ultraspherical harmonics. Finally, the next most special equation unapproximated can also be mapped to the spheroidal equation, so that the fairly standard Mathematical Physics of that equation continues to be of aid in N -stop metroland.

8.3 Quantum pure-shape triangleland

This amounts to a different reinterpretation of the mathematics of the sphere. The triangleland time-independent Schrödinger equation (539) is separable under the separation ansätze [for (a) or [a] labels, dropped]

$$\Psi(\mathcal{R}, \Phi) = \zeta(\mathcal{R})\eta(\Phi) \quad \text{or} \quad \Psi(\Theta, \Phi) = \xi(\Theta)\eta(\Phi) , \quad (618)$$

and in each case one obtains simple harmonic motion solved by

$$\eta = \exp(\pm ij\Phi) \quad (619)$$

for j an integer. This is a relative angular momentum quantum number corresponding to \mathcal{J} being classically conserved. This is in analogy with how there is an angular momentum quantum number m in the central-potential case of ordinary QM corresponding to the angular momentum \mathcal{L}_z being classically conserved. The accompanying separated-out equation is, in the tilded PPSCT representation in $\{\mathcal{R}, \Phi\}$ coordinates, the radial equation

$$\mathcal{R}^2 \zeta_{,\mathcal{R}\mathcal{R}} + \mathcal{R} \zeta_{,\mathcal{R}} - \{2\mathcal{R}^2 \{\tilde{V}(\mathcal{R}) - \tilde{E}(\mathcal{R})\}/\hbar^2 + j^2\} \zeta = 0 , \quad (620)$$

or, in the barred conformal representation in (Θ, Φ) coordinates, the azimuthal equation

$$\{\sin\Theta\}^{-1} \{\sin\Theta \xi_{,\Theta}\}_{,\Theta} - \{2\tilde{V}(\Theta) - E\}/\hbar^2 + j^2 \{\sin\Theta\}^{-2} \xi = 0 . \quad (621)$$

The conformal ordering coincides with the Laplacian one here again because 2- d .

The time-independent Schrödinger equation for the harmonic oscillator like potential problem on triangleland is then

$$\widehat{\text{Tot}}\Psi = \{\mathcal{A} + \mathcal{B}\widehat{\text{dra}}_z + \mathcal{C}\widehat{\text{dra}}_x\}\Psi , \quad (622)$$

i.e.

$$\{\sin\Theta\}^{-1} \{\sin\Theta \Psi_{,\Theta}\}_{,\Theta} + \{\sin\Theta\}^{-2} \Psi_{,\Phi\Phi} = \{\mathcal{A} + \mathcal{B} \cos\Theta + \mathcal{C} \sin\Theta \cos\Phi\} \Psi \quad (623)$$

for $\mathcal{A} = 2\{A - E/4\}/\hbar^2$, $\mathcal{B} = 2B/\hbar^2$ and $\mathcal{C} = 2C/\hbar^2$.

8.3.1 Solution in the very special case

The very special case $\mathcal{B} = \mathcal{C} = 0$ then has a potential that balances out to be constant. Recollect from Sec 5 that this is mathematically the same as the linear rigid rotor, for which the Hamiltonian is $\widehat{\mathcal{L}}_{\text{Total}}$ up to multiplicative and additive constants. Thus, effectively, this and $\widehat{\mathcal{L}}_3$ form a complete set of commuting operators whose eigenvalues and eigenfunctions are the well-known spherical harmonics. Moreover, these also occur as a separated-out part of the corresponding scaled relational particle model problem.

Note that this theoretical possibility for simple mathematics has been known for some time [56]. However the idea of using it for toy modelling GR Quantum Cosmology is new (and the simplicity does not match the distinct situation in Molecular Physics itself). However, the RPM ‘rigid rotor’ is in configuration space rather than in space. Furthermore, it has total relative rational momentum $\widehat{\mathcal{T}}_{\text{ot}} = \sum_{\Gamma=1}^3 \widehat{\mathcal{S}}_{\Gamma}^2$ in place of total angular momentum and projected shape momentum $\widehat{\mathcal{S}}_3$ in place of magnetic angular momentum. These then have eigenvalues $\hbar^2 S\{S+1\}$ and $\hbar s$ respectively; Serna and I therefore term S and s *total* and *projected ‘shape momentum quantum numbers’*. [These are analogous to the ordinary central force problem/rigid rotor total and magnetic angular momentum quantum numbers, l and m respectively, and to the total and projected ‘relative distance momentum quantum numbers’ of 4-stop metroland (Sec 8)]. Our very special problem’s time-independent Schrödinger equation thus separates into simple harmonic motion and the associated Legendre equation (in $Z = \cos\Theta$ for any axis system A) that constitute the spherical harmonics equations. Thus its solutions are

$$\Psi_{Ss}(\Theta, \Phi) \propto Y_{Ss}(\Theta, \Phi) \propto P_S^s(\cos\Theta) \exp(\pm is\Phi) \quad (624)$$

for $P_S^s(Z)$ the associated Legendre functions in Z , while $S \in \mathbb{N}_0$ and s is such that $|s| \leq S$. Also, $S\{S+1\} = -\mathcal{A}$, which here signifies that

$$E = 2\hbar^2 S\{S+1\} + K_2 \quad (625)$$

is required of the model universe's energy and inter-cluster effective spring in order for there to be any quantum solutions. If this is the case, there are then $2S + 1$ solutions labelled by s (the preceding sentence cuts down on a given system's solution space, though the more usual larger solution space still exists in the 'multiverse' sense). From (625) then, for there to be any chance of solutions one needs $E \geq \{K_1 + K_2\}/4$, so for HO-type models, $E > 0$ is indispensable. If there is a S and it is not zero, there are various degenerate solutions corresponding to different values of s . These correspond to states of different relative angular momentum between the particle 2, 3 subsystem and the particle 1 subsystem.

I consider various significant bases of orbitals corresponding to the different choices of axis systems in Sec 3.11.3. In particular, the $[a]$ bases with their equilateral principal axis (in the $4 \times$ area direction) are natural for the very special case. E.g. these are well-adapted to questions about equilateral and collinear configurations. On the other hand, in the special case with potential $B \cos \Theta^{(1)}$, the (1)-clustering's D-M axis [corresponding to $\text{ellip}(1)$] is picked out as the sole remaining axis of symmetry for the problem. In this case, one would usually prefer to be the principal axis, corresponding to using the (1)-basis (as in [31]). Each (a) basis is associated with the pure relative angular momentum quantity $\mathcal{J}_{(a)} = -i\hbar \partial_{\Phi_{(a)}}$ for $\Phi_{(a)}$ a rightness operator, and as such each has a projected pure relative angular momentum quantum number $j_{(a)}$. In contrast, the $[a]$ bases have 'mixed relative distance and relative angular momentum' rational momentum quantity $\hat{\mathcal{S}}_2 = -i\hbar \partial_{\Phi_{[a]}}$ for $\Phi_{[a]}$ an isoscelesness/regularity quantity about the $E\bar{E}$ axis. Moreover, this is clustering-independent, and as such I denote the projected shape quantum number in each case by an unadorned s . In this sense these readily afford an extension to $[\gamma]$ bases, for which the quantum number is *still* the same s . [This permits alignment with the axis picked out by the general case's $B, C \neq 0$ potential via setting $\gamma = \arctan(C/B)$ as per Sec 5.2.8.] Finally, one can also define a (γ) basis by exchanging 'y' and 'z' axis designations around; in this case one is as best adapted to the potential as is possible, but one has a different $s_{(\gamma)}$ per γ as per Sec 5.

In polar coordinate charts, one should not trust the immediate vicinity of the polar axes as the charts go bad there (the angle around the pole becomes undefined). Use 'cartesianization' near there, e.g. the corresponding stereographic chart 'cartesianized'. E.g. the ground-state probability density function $\sin \Theta$ gets sent to $\mathcal{R}/\{1 + \mathcal{R}^2\}^2$ and then to $1/\{1 + \mathcal{X}^2 + \mathcal{Y}^2\}^2$.

One is then to interpret the familiar orbital basis with respect to the axis system in use as follows. I superpose the triangleland tessellation in order to interpret each orbital in terms of triangleland mechanics. I compute expectations and spreads of shape operators.

8.3.2 Wavefunctions in the natural alias equilateral bases $[a]$

The form of the solution is here

$$\Psi_{Ss}(\Theta_{[a]}, \Phi_{[a]}) \propto P_S^s(4 \times \text{area}) \mathcal{T}_r(\text{ellip}(a)/\sqrt{1 - 4 \times \text{area}^2}) . \quad (626)$$

The 'orbitals' in this basis then receive the following configurational interpretations. The $S = s = 0$ solution (s orbital) favours all of the various configurations equally. Moreover, this ground state solution occurs for any basis. The $S = 1, s = 0$ solution ($p_{\text{dra}_z}^{[a]}$ orbital) disfavors collinearities and favours equilateral configurations. It gives all clusterings equal status (which is a general feature for all $s = 0$ solutions following from their axisymmetry). The $S = 1, s = 1$ cosine solution $p_{\text{dra}_x}^{[a]}$ favours (a)-sharp and (a)-flat and disfavors (a)-regular and the near-equilateral configurations, while somewhat disfavouring (b)- or (c)-sharp or flat and somewhat favouring (b)- and (c)-regular. The $S = 1, s = 1$ sine solution $p_{\text{dra}_y}^{[a]}$ favours (a)-regular and disfavors (a)-sharp or flat and the near-equilateral configurations, while somewhat disfavouring (b)- or (c)-regular and somewhat favouring each of (b)- or (c)-sharp or flat. The $S = 2, s = 0$ solution $d_{\text{dra}_z}^{[a]}$ favours near-equilateral solutions (polar lobes) and separately favours collinear configurations (equatorial ring). The $S = 2, s = 1$ cosine solution $d_{\text{dra}_x \text{dra}_z}^{[a]}$ disfavors near-collinear and near-equilateral configurations as well as (a)-sharp or flat; it favours (a)-isosceles over (a)-regular. The $S = 2, s = 1$ sine solution $d_{\text{dra}_y \text{dra}_z}^{[a]}$ favours (a)-sharp or flat and disfavors both near-collinear and near-equilateral solutions; it favours (a)-regular over (a)-isosceles. The $S = 2, s = 2$ cosine solution $d_{\text{dra}_x^2 - \text{dra}_y^2}^{[a]}$ disfavors near-equilateral configurations and favours those collinear ones that are on the '+' consisting of (a) being one of flat, sharp or regular. The $S = 2, s = 2$ sine solution $d_{\text{dra}_x \text{dra}_y}^{[a]}$ disfavors near-equilateral configurations and favours those collinear ones that are on the 'X' consisting of (a) being none of particularly flat, sharp or regular.

8.3.3 Wavefunctions in the clustering-following alias collinear bases (a)

The form of the solution is here

$$\Psi_{Sj_{(a)}}(\Theta_{(a)}, \Phi_{(a)}) \propto P_S^{j_{(a)}}(\text{ellip}(a)) \mathcal{T}_{j_{(a)}}\left(\text{aniso}(a)/\sqrt{1 - \text{ellip}(a)^2}\right) . \quad (627)$$

What do the orbitals in this basis mean physically? One has the same s -orbital as usual, while the $p^{(a)}$ orbitals are just permutations of the $p^{[a]}$ orbitals: $p_{\text{dra}_x}^{(a)} = p_{\text{dra}_x}^{[a]}$, $p_{\text{dra}_y}^{(a)} = p_{\text{dra}_z}^{[a]}$ and $p_{\text{dra}_z}^{(a)} = p_{\text{dra}_y}^{[a]}$. The $S = 2, j_{(a)} = 1$ cosine solution ($d_{\text{dra}_x \text{dra}_y}^{(a)}$ orbital) is just $d_{\text{dra}_x \text{dra}_y}^{[a]}$ again. The $S = 2, j_{(a)} = 1$ sine solution ($d_{\text{dra}_y \text{dra}_z}^{(a)}$ orbital) is just $d_{\text{dra}_y \text{dra}_z}^{[a]}$ again, and the $S = 2, j_{(a)}$

= 2 cosine solution ($d_{\text{dra}_x \text{dra}_y}^{(a)}$ orbital) is $d_{\text{dra}_x \text{dra}_z}^{[a]}$ again. However, the (a) bases do contain some orbitals different to those in the [a] bases: the $S = 2, j_{(a)} = 0$ solution ($d_{\text{dra}_z}^{(a)}$ orbital) has lobes favouring (a)-sharp or flat and a central ring favouring all cases of (a) near-regular (including the near-equilateral configurations). Also, the $S = 2, j_{(a)} = 2$ sine solution ($d_{\text{dra}_x^2 - \text{dra}_y^2}^{(a)}$ orbital) favours the (a) near-regular plane over (a)-sharp or flat, within which it favours in particular both near-collinear and near-equilateral solutions.

8.3.4 Wavefunctions in collinear (γ) bases

Using the (1)-axis as the axis γ is defined about and spherical trigonometry,

$$\Psi_{S_{(\gamma)}} \propto P_S^{S(\gamma)}(\text{ellip}(1) \cos \gamma + \text{aniso}(1) \sin \gamma) \mathcal{T}_{S(\gamma)} \left(\frac{\text{aniso}(1) \sin \gamma - \text{ellip}(1) \cos \gamma}{\sqrt{1 - \{\text{aniso}(1) \sin \gamma + \text{ellip}(1) \cos \gamma\}^2}} \right). \quad (628)$$

Interpretation in terms of mass-weighted space quantities is quite complicated (but can be read off the tessellation). This basis remains useful for the investigation of the general problem with $B, C \neq 0$.

8.3.5 Wavefunctions in equilateral [γ] bases

By $\Phi_{[\gamma]} = \Phi - \gamma$ and 2-angle formulae, the solutions now take the form

$$\Psi_{S_S} \propto P_S^S(4 \times \text{area}) \left\{ \mathcal{T}_s \left(\text{ellip}(1)/\sqrt{1 - 4 \times \text{area}^2} \right) \cos s\gamma + \overline{\mathcal{T}}_s \left(\text{ellip}(1)/\sqrt{1 - 4 \times \text{area}^2} \right) \sin s\gamma \right\}, \quad (629)$$

where $\overline{\mathcal{T}}_s$ of the cosine solution = \mathcal{T}_s of the sine solution and $\overline{\mathcal{T}}_s$ of the sine solution = $-\mathcal{T}_s$ of the cosine solution. Again, interpretation is complicated, though this basis is again well-adapted to questions concerning near-equilaterality and near-collinearity. The projected rational quantum number here is as in Sec 8.3.1.

8.3.6 Expectations and spreads of shape operators in the very special case

Here, $\langle S j_{(a)} | \widehat{\text{dra}}_{(a)}^k | S j_{(a)} \rangle = 0$, i.e.

$$\langle S j_{(a)} | \widehat{\text{aniso}}(a) | S j_{(a)} \rangle = \langle S j_{(a)} | 4 \times \widehat{\text{area}} | S j_{(a)} \rangle = \langle S j_{(a)} | \widehat{\text{ellip}}(a) | S j_{(a)} \rangle = 0. \quad (630)$$

This just means that there is orientation symmetry so for each positive contribution there is a corresponding negative one. The useful information starts with the spreads in the relative size shapes

$$\Delta_{S j_{(a)}}(\widehat{\text{ellip}}(a)) = \sqrt{\frac{2\{S\{S+1\} - j_{(a)}^2\} - 1}{\{2S-1\}\{2S+3\}}}, \quad (631)$$

$$\Delta_{S j_{(a)}}(4 \times \widehat{\text{area}}) = \sqrt{\frac{\{S\{S+1\} + j_{(a)}^2\} - 1}{\{2S-1\}\{2S+3\}}} Q_2(j_{(a)}), \quad \Delta_{S j_{(a)}}(\widehat{\text{aniso}}(a)) = \sqrt{\frac{\{S\{S+1\} + j_{(a)}^2\} - 1}{\{2S-1\}\{2S+3\}}} Q_1(j_{(a)}) \quad (632)$$

for $Q_2(j_{(a)}) = 1/2$ for $j_{(a)}$ cosine solution, $3/2$ for $j_{(a)}$ sine solution, 1 otherwise, and $Q_1(j_{(a)})$ the $\sin \longleftrightarrow \cos$ of this. One can then check that indeed $\langle S j_{(a)} | \sum_{k=1}^3 \widehat{\text{dra}}_{(a)}^k | S j_{(a)} \rangle = 1$, and by (266) also obtain $\langle S j_{(a)} | \widehat{\text{sharp}}(a) | S j_{(a)} \rangle = 1/2 = \langle S j_{(a)} | \widehat{\text{flat}}(a) | S j_{(a)} \rangle$ and $\Delta_{S j_{(a)}}(\widehat{\text{sharp}}(a)) = \Delta_{S j_{(a)}}(\widehat{\text{flat}}(a)) = \Delta_{S j_{(a)}}(\widehat{\text{ellip}}(a))/2$.

The ground state and very low quantum number states are of interest, as I subsequently calculate the nontrivial potential counterparts of the results for these. The ground state spreads in $\widehat{\text{ellip}}(a)$ and $\widehat{\text{aniso}}(a)$ are $1/\sqrt{3}$ each.

Large quantum number limits are also interesting. The spread in $\widehat{\text{ellip}}(a)$ goes as $1/\sqrt{2S}$ for $j_{(a)}$ maximal and S large, and as $1/\sqrt{2}$ for $j_{(a)} = 0$ and S large. The former amounts to recovery of the equatorial classical geodesic as the limit of an ever-thinner belt in the limit of large maximal relative angular momentum quantum number $|j_{(a)}| = S$ (This is traversed in either direction according to the sign of $j_{(a)}$). The latter amounts to the $s, p_{\text{dra}_z}^{(a)}, d_{\text{dra}_z^2}^{(a)} \dots$ sequence of orbitals not getting much narrower as S increases, so that there is always limited concentration on sharp or flat triangles.

For the isoscelesness/regularness relative angle shape operator $\widehat{\Phi}_{(a)}$ gives, that, by factorization and cancellation of the Θ -integrals, the R0 states obey the uniform distribution over 0 to 2π i.e. with mean π and variance $\pi^2/3$. Next, $\langle S j_{(a)} | \widehat{\Phi}_{(a)} | S j_{(a)} \rangle$ is also π and cosine and sine states have, respectively

$$\Delta_{S j_{(a)}}(\widehat{\Phi}_{(a)}) = \sqrt{\pi^2/3 + 1/2 j_{(a)}^2}, \quad \Delta_{S j_{(a)}}(\widehat{\Phi}_{(a)}) = \sqrt{\pi^2/3 - 1/2 j_{(a)}^2}, \quad (633)$$

which indicate some resemblance to the uniform distribution coming about for large $j_{(a)}$ (mean and variance do not see the multimodality. But at least, by inspection along the lines of the preceding subsection, it is *regular* multimodality for $j_{(a)}$)

maximal – equatorial flowers of 2S petals that gets more and more equatorially flat as D gets larger, thus tending to the equatorial great circle classical path).

For $4 \times \text{area}$ (which is mixed ratio/relative angle information, but nevertheless interesting in its own right). $\Delta(\widehat{4 \times \text{area}})$ goes as $1/\sqrt{2}$ for large quantum number with $j_{(a)}$ maximal and as $1/2$ for large quantum number with $j_{(a)} = 0$. That the wavefunctions have increasingly many peaks and valleys does not register unto this overall spread quantifier. This also means that there are no limiting collinear states in this basis. This $1/\sqrt{2}$ is the largest value it takes, while the smallest, $1/\sqrt{5}$, occurs for the $S = 1, j = 0$. For the ground state, $\Delta_{00}(\widehat{4 \times \text{area}}) = 1/\sqrt{3}$.

8.3.7 Shape operators acting on [a] bases' wavefunctions

In the natural or equilateral basis, one has the $4 \times \text{area} \longleftrightarrow \text{ellip}(a)$ and $j_{(a)} \rightarrow s$ counterpart of eq's (630, 631, 632). Thus in this case one does have a collinear limit for large quantum numbers, e.g. in the case of r maximal, $\Delta_{Ss}(\widehat{4 \times \text{area}}) \approx 1/\sqrt{2S} \rightarrow 0$ for s maximal and S large. On the other hand, it tends to $1/\sqrt{2}$ for $s = 0$ and S large.

8.3.8 Small asymptotics solutions for the 'special multiple HO'

I work in the tilded PPST representation. In the special case, note that these equations are self-dual in the sense of [30], so that, again, direct study of only one of the two asymptotic regimes is necessary and then the other can be read off by mere substitutions. This and the next 2 SSecs are interpreted in terms of the (1)-basis.

8.3.9 Second order asymptotics

The special case has the same mathematical form as the fairly well-known problem of the linear rigid rotor in a background homogeneous electric field in the symmetry-adapted basis for that problem. This then at the quantum level amounts to the present mathematics being analogous to that of the Stark effect for the linear rigid rotor, which is well-documented (see e.g. [614, 318, 458]. Near the {23} double collision, to second approximation, this problem gives [31] 2- d isotropic harmonic oscillator wavefunctions [458, 569, 531]. (However, it is not the same inner product in detail, due to the curved geometry). I work in the (1)-basis that is adapted to this potential, but suppress the (1) labels. I require a quantum-mechanically sizeable classical 'frequency', ω . If the bulk of the wavefunction is to lie where the small-angle approximation holds reasonably. (This is, dimensionally frequency $\times I$ in this article's formulation. $I\omega/\hbar$ of the order of 10^3 begins to work well.) Then the 'energies' are $E - K_2/2 = n\hbar\omega$ for $n := 1 + 2N + |j|$. However, ω itself depends on the shifted energy, $E' := E - A - B$, so

$$E' = 2\{\hbar^2 n^2 + n\hbar\sqrt{n^2\hbar^2 - B}\}, \quad (634)$$

which, for $n\hbar/\omega$ small i.e. $n \ll 10^3$. (This is certainly OK for the solutions below), goes as $E' \approx n\hbar\Omega$ for $\Omega = 2\sqrt{-B}$. The solutions go like

$$\Psi_{Nj}(\Theta, \Phi) \propto \Theta^{|j|} \{1 + |j|\Theta^2/12\} \exp(-\omega\Theta^2/8\hbar) L_N^{|j|}(\omega\Theta^2/4\hbar) \exp(\pm ij\Phi) \quad (635)$$

where $L_a^b(\xi)$ is the associated Laguerre polynomials in ξ . [As regards interpretation of the Φ factor, this is rewriteable as before as per Sec 8.3.3 in terms of aniso and ellip, while the Θ factor is now a somewhat more complicated function of ellip].

Again, $j = 0$ are axisymmetric and as such are totally undiscerning of isoscelesness/collinearity. $N = 0, j = 0$ favours sharp triangles. $N = 0, j = \pm 1$ are a degenerate pair, wherein the cosine solution favours sharp triangles that are approximately collinear and the sine solution favours sharp triangles that are approximately isosceles. The next 3 solutions are also degenerate. $N = 1, j = 0$ favours two separate bands: the even sharper triangles and the fairly sharp triangles. The $N = 0, |j| = 2$ cosine solution favours sharp triangles. These are either approximately collinear or approximately isosceles, while the $N = 0, |j| = 2$ sine solution favours sharp triangles that are neither approximately collinear nor approximately isosceles. The $N = 0$ states have peaks at $\Theta \approx 2\sqrt{\mathcal{J}/\omega}$; c.f. the means in the next SSSec. For the opposite sign of B , the other pole's [(1)-merger] approximate wavefunctions have the same meaning except that one uses 'flat' in place of 'sharp'.

Comparing first and second order results, how can few-peak functions be second approximations when infinite-peak functions are first approximations? This is accounted for by the few-peak functions have an extra node-counting quantum number N . Thus the first approximation can be seen as a superposition of many different values of N each with its own Q_2 . Thus a multiplicity of peaks builds up. For the values considered, there is better than 1% accuracy between the 2 approximations up to some value of \mathcal{R} of order of magnitude 0.01 to 0.1.

8.3.10 Shape operators acting on Θ near-polar wavefunctions

In the small regime, $4 \times \text{area}$ and aniso still have zero expectation, as either sign of these are equally probable. For N, j substantially smaller than ω/\hbar so that powers of the latter dominate powers of the former. (As ω/\hbar is considered to be large, this does cover the previous subsection's states.) Then the following results hold. (This subsection's results come from the orthogonality of, and a recurrence relation for, the Laguerre polynomials provided in an Appendix of [50].)

$$\langle Nj | \widehat{\text{ellip}} | Nj \rangle \approx 1 - n\hbar/2\omega \quad (636)$$

$$\Delta_{Nj}(4 \times \widehat{\text{area}}) \approx \sqrt{n \hbar Q_2(j)/2\omega} \quad , \quad \Delta_{Nj}(\widehat{\text{aniso}}) \approx \sqrt{n \hbar Q_1(j)/2\omega} \quad , \quad (637)$$

while the spread in ellip goes as $\hbar/I\omega$ though the scheme cannot really evaluate its coefficient. (The first two orders cancel, and the differential equation that was solved in the first place was only accurate to the first two orders).

As some idea of what unsigned area is typical, for the equal-masses ground state, this is

$$\Delta_{Nj}(\text{physical area}) = \frac{3I^2}{4m^2} \frac{\hbar}{2\omega} \quad . \quad (638)$$

which has an interpretation somewhat akin to the Bohr radius. (This includes the map between the isotropic harmonic oscillator and hydrogen in [569].) It is a ‘minimal quantifier’ of area, albeit of spread of area and in a basis centred on a relative angular momentum interpretable axis. Note the change in spread of $4 \times \text{area}$ due to the confining effect of the potential. The wide range of areas correspond to a spread of $1/\sqrt{3}$ (where the range is -2 to 2) to $\sqrt{\hbar/2\omega}$ for ω/\hbar large, which is smaller by a factor of $\sqrt{3\hbar/2\omega}$.

Comparison of mean angle (e.g. roughly from the expectation of $\cos \Theta$) and mode angle (from graphs along the lines of those in [31]) reveals the mean to be larger than the mode, but by not quite as much as occurs radially in hydrogen. This reflects that this case’s Gaussianity suppresses the mean-shifting tail more than radial hydrogen’s mere exponential does. Expectations and spreads of $\widehat{\Phi}$ are just like for previous Sec as the Θ -integrals trivially cancel in each case.

8.3.11 The special triple HO problem treated perturbatively for small B

Let us treat the special triple HO problem as a perturbation about the simple ‘very special triple HO’. For a perturbation \mathcal{H}' to one’s rescaled Hamiltonian [rescaled since the other calligraphic quantities in (607, 608) are] the first few objects of perturbation theory as used in this article are as follows [416]. For $\overline{\mathcal{E}}_J = \overline{\mathcal{E}}_J^{(0)}$ nondegenerate, one has counterparts of (607) and (608). The key integral underlying time-independent perturbation theory is now $\langle \Psi_{S'j'} | V' | \Psi_{Sj} \rangle$, which has as its nontrivial factor $\int_{-1}^1 P_{S'}^{j'}(X) X P_S^j(X) dX$, but there is a recurrence relation (1055) enabling $X P_S^j(X)$ to be turned into a linear combination of $P_{S''}^{j''}(X)$, whereupon orthonormality of the associated Legendre functions (755) can be applied to evaluate it. This calculation parallels that in the derivation of selection rules for electric dipole transitions [67, 629]. Then $\langle S, j' | B X | S, j \rangle = 0$ since the recurrence relation sends $X P_S^{j'}$ to a sum of $P_{S'}^{j'}$ for $j' \neq j$, so each contribution to the integral vanishes by orthogonality. Next, for $\langle S', j' | B X | S, j \rangle$, one similarly needs $S' = S \pm 1$ and $j' = j$ to avoid it vanishing by orthogonality. So two cases survive this ‘selection rule’. The first is, by direct computation,

$$\langle S+1, j | B X | S, j \rangle = B \sqrt{\{S+1\}^2 - j^2} / \{2S+1\} \{2S+3\} \quad , \quad (639)$$

while the second is

$$\langle S-1, j | B X | S, j \rangle = \langle S, j | B X | S-1, j \rangle = B \sqrt{\{S\}^2 - j^2} / \{2S-1\} \{2S+1\} \quad (640)$$

by using (639) with $S-1$ in place of S (parallelling e.g. [458]).

The eigenspectrum is thus

$$\overline{\mathcal{E}}_{S,j} = S\{S+1\} + A + \{B^2\{S\{S+1\} - 3j^2\}\} / 2S\{S+1\}\{2S-1\}\{2S+3\} + O(B^4) \quad (641)$$

so

$$E_{S,j}^{\{2\}} = 2\hbar^2 S\{S+1\} + 4A + 4B^2\{S\{S+1\} - 3j^2\} / \hbar^2 S\{S+1\}\{2S-1\}\{2S+3\} + O(B^4) \quad . \quad (642)$$

For $S=0$, one needs a separate calculation, which gives

$$E_{0,0}^{\{2\}} = 4A + 4B^2/3\hbar^2 + O(B^4) \quad . \quad (643)$$

This calculation can then be checked against its rotor counterpart (originally in [391] and which can also found in e.g. [614, 318, 458]). The corresponding eigenfunctions can be looked up (e.g. [520]) and reinterpreted in terms of the original problem’s mechanical variables.

Bearing this long-term application in mind, note that the following observation in [50] for 4-stop metroland readily extends to triangleland. That ‘halfway-house’ overlap integral computation is *common* to building up both the immediate time-independent perturbation theory for pure-shape models and to the longer-term time-dependent perturbation theory for scaled models. Namely, 1) these overlaps are relevant halfway-house computations for perturbation theory. 2) They are of 3- \mathcal{V} integral form, for which the most general case is known explicitly [416] in terms of what are, mathematically, Wigner 3j symbols’ (though for us, physically, they are ‘3s symbols’). 3) That the specific cases relevant to triangleland with harmonic oscillator-like potentials (where the sandwiched \mathcal{V} is of degree 1) are explicitly written out in e.g. [468].

8.3.12 General triple HO pure-shape RPM problem treated perturbatively for small B, C .

A further result (specific to the immediate time-independent perturbation theory of pure-shape triangle-land) is the extension of [30]’s special case second order result to the general case by use of rotated bases:

$$E_{S_{s(\gamma)}} = 2\hbar^2 S\{S+1\} + 4A + 4\{B^2 + C^2\}\{S\{S+1\} - 3s_{(\gamma)}^2\}/\hbar^2 S\{S+1\}\{2S-1\}\{2S+3\} + O((B^2 + C^2)^2). \quad (644)$$

The time-independent Schrödinger equation with general triple HO is harder because the angle-dependence of the potential results in nonseparability in these natural coordinates. This makes for a useful model of the Semiclassical Approach to the Problem of Time [40].

As in [30], start again using rotated/normal coordinates at the classical level, or switch to such coordinates at the differential equation solving stage, amounting to a choice of basis in which the perturbation is in the (new) axial ‘ z_N ’ direction. While it can be viewed as before in the new rotated/normal coordinates, nevertheless studying the original coordinates’ Φ -dependent V term remains of interest. For, it may well be appropriate for the *original* coordinates to have mechanical attributes or the heavy–light subsystem distinction underlying the Semiclassical Approach.⁶⁴ Moreover in that case, being able to proceed further in the rotated/normal coordinates can serve as a check on “standard” procedures in the original coordinates. (This is along the lines suggested in [22]).

There is now an issue in the projection that there is an additional factor due to the change in area in moving between each patch of sphere and each corresponding patch of plane. Namely, the probability density function on the sphere is $\sin\Theta|\Psi(\Theta, \Phi)|^2$. Its $\sin\Theta$ part pertains to the sphere itself. The probability density function on the stereographic plane is $|\Psi(\mathcal{R}, \Phi)|^2 \mathcal{R}/\{1+\mathcal{R}^2\}$ of which the $\mathcal{R}/\{1+\mathcal{R}^2\}$ pertains to the stereographic plane itself. Unlike doing the very special case in [31], I have already looked at that in multiple bases, and so go for the more interesting general multiple HO case.

8.3.13 Molecular analogies for triangle-land

The following are available in the rotor literature (in the spherical presentation) and could thus be straightforwardly transcribed to the various presentations for this article’s pure-shape RPM models.

- 1) For triangle-land, one of the main results of [31] is that the harmonic-oscillator-like potential problem on triangle-land has the same mathematics as the Stark effect for the linear rigid rotor (see e.g. [614, 318, 458]). In particular, the special case corresponds to the homogeneous electric field pointing in the z -direction. The general case to it pointing in the general direction in the zx -plane. The very special case is, by this analogy, the undisturbed linear rigid rotor itself. This allows for checks of the various regimes (B small, B large, $\Theta_{(1)}$ near-polar alongside further knowledge of how to patch together such regimes (see [31] for a detailed reference list). That a 2- d isotropic HO resides within the triangle-land RPM multiple HO like problem as a limit problem has counterpart in the literature on the linear rotor ([513], see also the figure in [459]).
- 2) Higher order corrections in B are in the literature for the rotor [327, 527] and so can be transcribed into the relational context [e.g. I was able to write $O(B^4)$ and not $O(B^3)$ in (642) due to this]. Alternative variational methods (using the Hellmann–Feynman and hypervirial theorems) appear in [527]. These cover higher order terms too, and can be used to show generally that only even powers of B occur. The calculation for the large B regime has both been done [520] and matched to small B regime calculations.
- 3) Numerical evaluation of eigenvalues was been done by Lamb’s [414] continued fraction method [327, 411, 574, 513] (and otherwise, e.g. [459, 527]).
- 4) The rotor literature also indicates how the C perturbation can be transformed away with a new rotated choice of coordinates in which the mathematics is again that of a B type perturbation. (This is the quantum application of the normal coordinates trick). While in the laboratory with a rotor one could choose one’s axial ‘ z ’ direction to be in the most convenient direction, there can in the new context be various Problem of Time strategy modelling reasons not just to stay in these coordinates in the SRPM case (e.g. allotting h and l statuses to nonseparable coordinates being required in the Semiclassical Approach).

⁶⁴In the laboratory, one might likewise not pick the normal coordinates of the rotor–electric field if there is e.g. also a magnetic field that picks out a *different* direction.

9 Scaled quantum RPM's

This amounts to a re-interpretation of standard coordinate systems on \mathbb{R}^3 and \mathbb{R}^n . This gives a shape part as in the preceding Section and an extra scale equation. The separation ansatz

$$\Psi = \mathcal{I}(\rho)\mathcal{Y}(\text{Shape}) \quad (645)$$

is useful in the below workings.

9.1 Quantum scaled 4-stop metroland

I consider an (H2) basis, dropping the labels.

9.1.1 Simple quantum solutions in the small-shape approximation: common separated-out shape part

This gives in general the spherical harmonics equation

9.1.2 Analogues of $k < 0$ vacuum or wrong-sign radiation

As mechanics, these are $E > 0$ and may have an approximation to the conformal potential present (to cancel off \mathcal{T}_{ot}). Overall, they are free models with zero and nonzero top relative distance quantum number respectively. They are solved by

$$\mathcal{I}_D(\rho) \propto \rho^{-1/2} J_{D+1/2}(\sqrt{2E}\rho/\hbar) , \quad (646)$$

which are spherical Bessel functions (see Overall Appendix C). The probability density function for this exhibits an infinity of oscillations in the scale direction.

9.1.3 Analogues of very special HO ($= \Lambda < 0$, $k < 0$ vacuum or wrong-sign radiation)

For 4-stop metroland with $V = A\rho^2$, gives the 3- d isotropic quantum HO problem, which, in scale-shape variables, is solved by [using (645) and the spherical harmonics]

$$\mathcal{I}_{ND}(\rho) \propto \rho^D L_N^{D+1/2}(\omega\rho^2/\hbar) \exp(-\omega\rho^2/2\hbar) , \quad (647)$$

corresponding to energies $E = \hbar\omega\{2N + D + 3/2\} > 0$ for $N \in \mathbb{N}_0$. Again, these wavefunctions are finite at 0 and ∞ in scale, with N nodes in between.

9.1.4 Analogues of $k > 0$ dust model

As mechanics, these are $E < 0$ Newton–Coulomb potential models. The 4-stop and so 3- d case of this is well-known (e.g. [416]): it is the analogue of the $l = 0$ hydrogen model,

$$\mathcal{I}_N(\rho) \propto L_{N-1}^1(2\sqrt{-2E}\rho/\hbar) \exp(\sqrt{-2E}\rho/\hbar) . \quad (648)$$

Moreover, the 3-stop (2- d) and N -stop (general- d) cases follow by straightforward generalization. (Nor is it hard to extend the above to include $\mathcal{T}_{\text{ot}} > 0$ too, though that corresponds to wrong-sign radiation, so I just present the $\mathcal{T}_{\text{ot}} = 0$ case in this article.)

9.1.5 Analogues of $k \leq 0$ dust models

As mechanics, these are $E \geq 0$ Newton–Coulomb potential models. These are ‘ionized states’ or ‘scattering problems’ and correspond to open cosmologies. The mathematics of the ionized atom (continuous spectrum part) can be found in e.g. [416, 185].

9.1.6 Analogues of right and wrong sign radiation models

Again, for this the 4-stop metroland’s 3- d mathematics is well-known [416]. It has three cases,

- i) $D + 1/2 < \sqrt{2R}/\hbar$ for which there is collapse to the maximal collision (ground state with $E = -\infty$), and this is relevant since the approximate problem’s scale part can exchange energy with the shape problem and thus use this energy exchange to run down to the maximal collision.
 - ii) $D + 1/2 > \sqrt{2R}/\hbar$ (for which it is not clear that the cutoff in [416] is meaningful in the case of RPM’s).
 - iii) The critical case $D + 1/2 = \sqrt{2R}/\hbar$ has \mathcal{I} diverge no worse than $1/\sqrt{\rho}$ as $\rho \rightarrow 0$.
- $d > 3$ i.e. $N > 4$ is an immediate generalization of this (App A.8).

9.1.7 The special quadratic potential in 4-stop metroland

In ρ_1, ρ_2, ρ_3 variables, it is solved in terms of Hermite polynomials and Gaussians, and this extends to the $B \neq 0$ case as well, which is, likewise, a distortion. For energies $E = \sum_{i=1}^3 \hbar \sqrt{K_i} \{n_i + 1/2\} + \hbar \sqrt{K_4} \{n_4 + 1/2\}$ ($k_i = \omega_i/\hbar$ and $n_i \in \mathbb{N}_0$), is solved standardly in the associated Cartesian coordinates ρ_i , $i = 1$ to 3 in Hermite polynomials. I then express the arguments of this in terms of the scale variable ρ and the shape variable θ, ϕ to obtain

$$\Psi_{n_1 n_2}(\rho, \theta, \pi) \propto H_{n_1}(\sqrt{K_1/\hbar} \rho \sin \theta \cos \phi) H_{n_2}(\sqrt{K_1/\hbar} \rho \sin \theta \sin \phi) H_{n_3}(\sqrt{K_1/\hbar} \rho \cos \theta) \\ \times \exp(-\rho^2 \{A + B \cos 2\theta + C \sin^2 \theta \cos 2\phi\}/\hbar). \quad (649)$$

One can again think of these as box-shaped arrays of peaks and troughs. The ground state favours all DD's and T's equally. Then in the DD basis centred about the 12, 34 DD collision, one first excited state favours this DD pair while also disfavouring O, while the other two favour each of the other two DD pairs. The second excited states with two quantum numbers zero and the remaining one a 2 do likewise but in 3 lobes, so that O is also favoured while two nonzero distances have disfavoured nodes. Those with two quantum numbers 1 and the remaining one zero disfavour all DD's while somewhat favouring the T's. The third excited state includes the state with all quantum numbers taking the value 1. This picks out all 8 T's while all the DD's are nodal. The $B \neq 0$ case (for $|B| < A$) is again a distortion of this (stretched in one direction and squeezed in a perpendicular direction).

9.1.8 Interpretation I: characteristic scales

K/ρ RPM models have a 'Bohr moment of inertia' for the model universe analogue to (atomic Bohr radius)², $I_0 = \rho_0^2$, which, in the gravitational case, goes like $\hbar^4/G^2 m^5$. Then $E = -\hbar^2/2 I_0 N^2$ in the 4-stop case. Paralleling what was said in Sec 7.6, HO RPM models have characteristic $I_{HO} = \hbar/\omega$. Then note that the first of these is limited by the breakdown of the approximations used in its derivation, while the second of these has no such problems also follows through.

9.1.9 Interpretation II: expectations and spreads in analogy with Molecular Physics

Example 1) the most direct counterpart of the atomic working described in Sec 7.7.1 is the $D = 0$ approximate Newtonian gravity or attractive Coulomb problem, which is the $a_0 \rightarrow \rho_0$ of it.

Example 2) as regards the expectation of the size operator for 4-stop metroland, isotropic case in scale-shape coordinates, again using the normalization result for associated Laguerre polynomials and Gaussian integral results, $\langle 0 D d | \widehat{\text{Size}} | 0 D d \rangle = \frac{2^{2D+2}}{\sqrt{\pi}} \left(\frac{2D+2}{D+1} \right)^{-1} \rho_{HO}$. Thus e.g. $\langle 0 0 0 | \widehat{\text{Size}} | 0 0 0 \rangle = \frac{1}{2\sqrt{\pi}} \rho_{HO}$, while the large-D limit (for $N = 0$) is $\langle 0 D d | \widehat{\text{Size}} | 0 D d \rangle \rightarrow \sqrt{D+1} \rho_{HO}$. Again, the latter gradually grows to infinity along a sequence of configurations with ever-increasing relative distance momentum. Also, $\langle 1 0 0 | \widehat{\text{Size}} | 1 0 0 \rangle = \frac{3}{\sqrt{\pi}} \rho_{HO} \approx 1.69 \rho_{HO}$. In comparison, the mode value for the ground state is ρ_{HO} .

The spreads are again an integral made easy by a recurrence relation on the associated Laguerre polynomials, $\langle N D d | \widehat{\text{Size}}^2 | N D d \rangle = \{2N + D + 3/2\} \rho_{HO}^2$, minus the square of the expectation. Thus $\Delta_{0 0 0}(\widehat{\text{Size}}) = \frac{6\pi-1}{4\pi} \rho_{HO}^2 \approx 1.42 \rho_{HO}^2$. On the other hand, the large-D limit with $N = 0$ gives $\Delta_{0 D d}(\widehat{\text{Size}}) \rightarrow \rho_{HO}^2/2$, and $\Delta_{1 0 0}(\widehat{\text{Size}}) = \frac{7\pi-18}{2\pi} \rho_{HO}^2 \approx 0.635 \rho_{HO}^2$.

For expectations and spreads of the 4-stop metroland shape operators, [50] gives these immediately by the scale-shape split. Furthermore these overlap integrals are of 3- \mathcal{Y} integral form, for which the most general case is known explicitly [416] in terms of what are, mathematically, Wigner 3j symbols'. (Though for us, physically, they are '3d symbols'.) The specific cases relevant to 4-stop metroland with harmonic oscillator-like potentials (where the sandwiched \mathcal{Y} is of degree 2) are explicitly written out in e.g. [468].

9.1.10 Further perturbative treatment

The Cosmology-RPM analogy in this article gives (Sec 5) multiple power-law potential terms with some of them furthermore admitting sensible small-shape expansions, which are of interest through being partly tied to the admission of a long-term-stable semiclassical regime. The negative power cases are not expected to work well already from the classical analysis: as further discussed in the Conclusion, shape just is not secondary in some parts of configuration space in such cases... [Also, there are extra layers of control for HO's: one can set up to possess semiclassicality apply everywhere, and one can solve these models exactly even in cases for which there is not separability in the h-l variables.]

Next, I note that the above exact solution work can be perturbed about, with a number of cases of this producing standard mathematics as follows. The perturbation integrals split into scale parts and shape parts. The scale integrals give $\langle \rho^\alpha \rangle$ for α an integer (taken to be positive if one is to avoid classical instability). Such integrals are of the same type as those used in the evaluation of expectations and spreads in Sec 7.2 (and so are adaptable from results in e.g. [458]). The shape integrals for 4-stop metroland involve $\theta \mathcal{Y}_{Dd}(\theta, \phi) \mathcal{Y}_{Dd}(\theta, \phi)$ and $\phi \mathcal{Y}_{Dd}(\theta, \phi) \mathcal{Y}_{Dd}(\theta, \phi)$. The latter is trivial and the former is straightforward, at least state-by-state. For 3-stop these are of the form $\varphi \exp(i d \varphi)$, and thus trivial again. Instead, in [50] Franzen and I treated the further HO terms not as a small-shape approximation expansion but rather as an exact perturbation.

9.2 Quantum scaled N -stop metroland

The general quantum equation for this is (529). Then for V of the form $V(\rho)$ (or suitably approximated thus), the equation is separable into scale and shape parts via $\Psi(\rho, \theta_{\bar{\tau}}) = \mathcal{I}(\rho)\mathcal{Y}(\theta_{\bar{\tau}})$.

9.2.1 The common angular part

This is common to all the $V = V(\rho)$ problems (as arise e.g. approximately within the scale dominates shape approximation) and is as per Sec 8.2. It is now the hyperspherical harmonics equation.

9.2.2 The general scale equation

The above working fixes the scale equation's separation constant to be $-\tau\{\tau + N - 3\}$, so that the scale equation is

$$-\hbar^2\{\rho^2\mathcal{I}_{,\rho\rho} + \{N - 2\}\rho\mathcal{I}_{,\rho} - \tau\{\tau + N - 3\}\mathcal{I}\} + 2\rho^2V(\rho)\mathcal{I} = 2E\rho^2\mathcal{I} . \quad (650)$$

Moreover, this article considers (subcases of) $V(\rho) = A\rho^2 - K/\rho - R/\rho^2$ arising from the RPM-cosmology analogy.

9.2.3 Small approximation to the general scale equation

In this regime, (650) reduces to

$$\rho^2\mathcal{I}_{,\rho\rho} + \{N - 2\}\rho\mathcal{I}_{,\rho} + \{2R/\hbar^2 - \tau\{\tau + N - 3\}\}\mathcal{I} = 0 , \quad (651)$$

which is a Cauchy-Euler equation. In absence of the R -term, it is solved by

$$\mathcal{I} = \rho^\tau , \quad \mathcal{I} = \rho^{-\tau-N+3} , \quad (652)$$

of which the latter is discarded due to being divergent at the origin (maximal collision). [The above excludes $\tau = 0$ for $N = 3$ (repeated roots) for which the second solution is $\ln \rho$ which is likewise discarded for being divergent at the origin.] With the R term present, there are 3 possible behaviours (paralleling [416]): i) $D + \{N - 3\}/2 < \sqrt{2R}/\hbar$ for which there is collapse to the maximal collision (ground state with $E = -\infty$). This is relevant since the approximate problem's scale part can exchange energy with the shape problem and thus use this energy exchange to run down to the maximal collision. ii) $D + \{N - 3\}/2 > \sqrt{2R}/\hbar$ (for which it is not clear that Landau's cutoff [416] is meaningful in RPM's for $N > 3$). iii) The critical case $D + \{N - 3\}/2 = \sqrt{2R}/\hbar$ has \mathcal{I} diverge no worse than $1/\rho^{\{3-N\}/2}$ as $\rho \rightarrow 0$.

9.2.4 Large approximation to the general scale equation

Let α be the highest power in the potential. For $\alpha > 0$, one gets an approximate large solution of the form

$$\mathcal{I}(\rho) \approx \exp(-2\sqrt{2k}\rho^{1+\alpha/2}/\hbar\{\alpha + 2\}) . \quad (653)$$

For $\alpha \leq 0$, one gets instead an approximate large solution of the form

$$\mathcal{I}(\rho) \approx \exp(-\sqrt{-2E}\rho/\hbar) . \quad (654)$$

9.2.5 Solution in the free case

The scale part's equation is now

$$\rho^2\mathcal{I}_{,\rho\rho} + \{N - 2\}\rho\mathcal{I}_{,\rho} + \{2E\rho^2/\hbar^2 - \tau\{\tau + N - 3\}\}\mathcal{I} = 0 . \quad (655)$$

For $E > 0$, this is solvable by mapping to the Bessel equation (see Overall Appendix C.2) , giving

$$\mathcal{I} \propto \rho^{\{3-N\}/2} J_{\tau+\{N-3\}/2}(\sqrt{2E}\rho/\hbar) \quad (656)$$

('ultraspherical' generalization of well-known spherical Bessel functions). [For $E < 0$, it is instead a modified Bessel function, which has an unphysical blow-up, while for $E = 0$, one gets (652) again instead].

9.2.6 Solution for the $E = 0$ single arbitrary power law case

Denote this potential by $k\rho^n$. The scale part's equation is now

$$\rho^2\mathcal{I}_{,\rho\rho} + \{N - 2\}\rho\mathcal{I}_{,\rho} + \{2k\rho^{2+n}/\hbar^2 - \tau\{\tau + N - 3\}\}\mathcal{I} = 0 , \quad (657)$$

which for k negative is also solvable by mapping to the Bessel equation (see Overall Appendix C.2), giving

$$\mathcal{I} \propto \rho^{\{1-N\}/2} J_{\{2\tau+N-3\}/\{\alpha+1\}}\left(2\sqrt{-2k}\rho^{\{n+2\}/2}/\{\tau+2\}\hbar\right) . \quad (658)$$

[for k positive this is instead a modified Bessel function (with an unphysical blow-up), while for $k = 0$, one has (652) again].

9.2.7 Solution for the Milne in AdS analogues

As mechanics, these are HO's. The scale equation here has the form of a unit-mass $\{N - 1\}$ - d isotropic HO (with ρ as radial variable),

$$-\hbar^2 \{\rho^2 \mathcal{I}_{,\rho\rho} + \{N - 2\} \rho \mathcal{I}_{,\rho} - \tau \{\tau + N - 3\} \mathcal{I}\} + \omega^2 \rho^4 \mathcal{I} = E \rho^2 \mathcal{I} . \quad (659)$$

Then using the large and small approximands as factors, $\mathcal{I}(\rho) = \rho^\tau \exp(-\omega \rho^2 / 2\hbar) f(\rho)$, and a rescaling, this maps to the generalized Laguerre equation, allowing one to read off the solution

$$\mathcal{I}_{N\tau}(\rho) \propto \rho^{|\tau|} \exp(-\omega \rho^2 / 2\hbar) L_N^{\tau+N/2-3/2}(\omega \rho^2 / \hbar) \quad (660)$$

for the eigenvalues

$$E = \hbar \omega \{2N + \tau + \{N - 1\}/2\} , \quad (661)$$

where $N \in \mathbb{N}_0$ is a node-counting quantum number.

9.2.8 Approximate solution for the dust universe analogues

As mechanics, these are approximations to the Newton–Coulomb problem. The scale equation here has the form of a unit-mass $\{N - 1\}$ -dimensional Newtonian Gravity or attractive Coulomb problem with ρ in place of r ,

$$-\hbar^2 \{\rho^2 \mathcal{I}_{,\rho\rho} + \{N - 2\} \rho \mathcal{I}_{,\rho} - \tau \{\tau + N - 3\} \mathcal{I}\} - 2K \rho \mathcal{I} = E \rho^2 \mathcal{I} . \quad (662)$$

For $E > 0$, using the large approximand as a factor (the small one is trivial), $\mathcal{I} = \exp(-\sqrt{-2E}\rho/\hbar) f(\rho)$, and a rescaling of the dependent variable, this maps to the generalized Laguerre equation, so this article's $\tau = 0$ solution is

$$\mathcal{I}_N(\rho) \propto L_{N-1}^{N-3}(2\sqrt{-2E}\rho/\hbar) \exp(-\sqrt{-2E}\rho/\hbar) \quad (663)$$

for the eigenvalues

$$E = -k^2/2\hbar^2 \{N - \{N - 4\}/2\}^2 = -\hbar^2/2\rho_0^2 \{N - \{N - 4\}/2\}^2 , \quad (664)$$

where $N \in \mathbb{N}_0$ is a node-counting quantum number and ρ_0 is the ‘Bohr configuration space radius’.

9.3 Quantum scaled triangleland

While this case is on \mathbb{R}^3 again (at least some times requiring excision of the origin), the configuration space geometry is curved (but conformally flat). In particular, as per Sec 3, the Ricci scalar is $6/I$, and so the conformal and Laplacian (and other ξ) orderings now differ, and by more than just a constant energy shift. In the conformal-ordered case, one is entitled to change PPST representation since the QM remains PPST-invariant. Thus, due to conformal flatness, one can pass to flat geometry, in which the kinetic part of the time-independent Schrödinger equation takes a very well-known form. For “central” problems/approximations to more general problems, there is a scale–shape split, with the preceding Sec’s working now occurring as the shape part. (This is shared by all ξ -orderings, since the Ricci scalar is pure scale). This now arises alongside a new equation for the mass-weighted scale variable I (which is different for each ξ , though they can be solved together with ξ as a parameter). I drop (a) labels.

9.3.1 Very special HO case $B = C = 0$ in spherical coordinates

I approach the QM via the correspondence with the mathematics for the Kepler–Coulomb problem of Sec 5. It corresponds to positive spatial curvature dust cosmology. This gives us the same wave equations as for the atomic problem and thus the usual separation and solvability in spherical and parabolic coordinates.

Note 1) Contrast with [36]: there the Coulomb problem is approximate and subject to small angle approximation breakdown, while in the present article it is an exact solution work.

Note 2) On the other hand the present article’s analogy breaks down at the level of having a different inner product from the hydrogen problem.

Hydrogen has principal, angular momentum and magnetic quantum numbers, while the current system has analogues of these, with relative rational momentum quantum numbers s and S playing the roles of magnetic and total angular momentum quantum numbers m and l respectively. I use N for the new principal quantum number that is associated with total moment of inertia of the system, which takes values such that (using $K_1 = K_2 = K$ for the very special case)

$$K = E^2/\hbar^2 N^2 = 4\hbar^2/I_0^2 N^2 \text{ for } N \in \mathbb{N} , \quad (665)$$

Here, $4\hbar^2/E = I_0 = \text{SIZE}_0$ (the meaning of which is explained in Sec 9.3.7). Thus one requires

$$E/\sqrt{K} = N\hbar , \quad N \in \mathbb{N} \quad (666)$$

as a consistency condition on the universe-model's energy and contents.

The corresponding wavefunctions are of the form

$$\check{\Psi}_{\text{NSj}}(\mathbf{I}, \Theta, \Phi) \propto L_{N-S-1}^{2S+1}(2\mathbf{I}/\mathbf{I}_0) \exp(-\mathbf{I}/\mathbf{I}_0) \{ \mathbf{I}/\mathbf{I}_0 \}^S P_S^j(\cos \Theta) \exp(ij\Phi) , \quad (667)$$

for L_α^β the associated Laguerre polynomials. In terms of scale-shape quantities,

$$\check{\Psi}_{\text{NSj}}(\text{SIZE}, \text{aniso}, \text{ellip}) \propto L_{N-S-1}^{2S+1} \left(\frac{2 \text{SIZE}}{N \text{SIZE}_0} \right) \exp \left(-\frac{\text{SIZE}}{N \text{SIZE}_0} \right) \left\{ \frac{\text{SIZE}}{\text{SIZE}_0} \right\}^S P_S^j(\text{ellip}) \mathcal{T}_j \left(\text{aniso} / \sqrt{1 - \text{ellip}^2} \right) . \quad (668)$$

I comment on the differences that the unusual inner product used here makes to the probability densities of various wavefunctions in Fig 46. Thus this article's analogy is less extensive than [30, 31]'s linear rigid rotor analogy. In this way the inclusion of scale complicates matters. Also, nontrivial PPSCT's are needed. These have more implications for a 3- d configuration space like that of the present Sec rather than for the 2- d configuration spaces like in [31, 34], for which a number of cancellations occur.

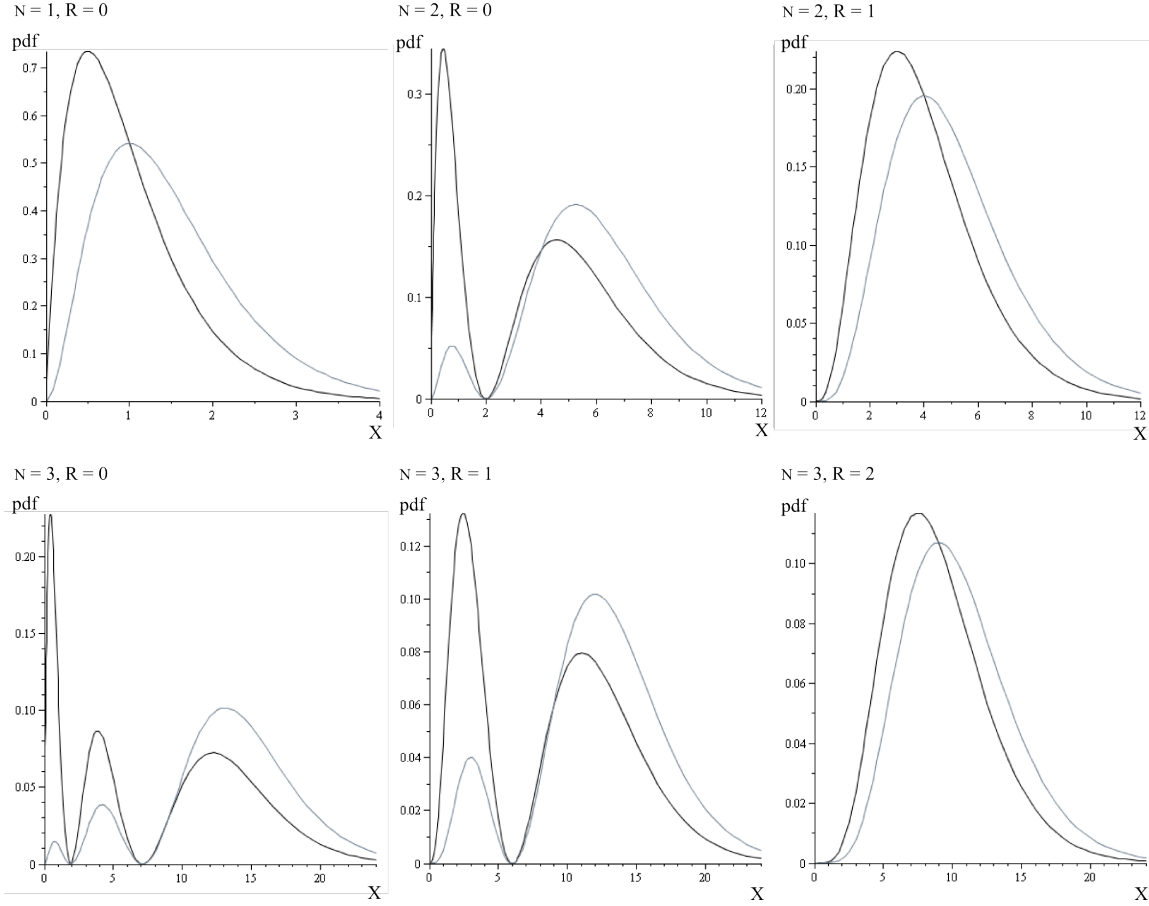


Figure 46: How the probability density functions (pdf's) of this article (black) compare to the hydrogenic ones (grey). R is my former notation for S. These are plotted using Maple [440] in terms of a dimensionless variable X , which is \mathbf{I}/\mathbf{I}_0 for the present article's case and r/a_0 for the hydrogenic case. Note that hydrogen's inner product is more suppressive closer to the origin than the present article's. This furthermore means that a rather more sizeable amount of the present article's ' s^2 ' probability density function is inside the ' s^1 ' one than in hydrogen, and likewise for the inner lobes of the ' s^3 ' and ' p^3 ' orbitals.

While one could proceed to investigate (as in [31]) the special case in spherical coordinates asymptotically and as a perturbation around the very special case, I have however found an alternative exact method, which I build up to over the next two SSecs.

9.3.2 Free problem in parabolics

In this case one obtains SHM and two split-out equations in I_1 and I_2 that map to the Bessel equation, giving an overall solution

$$\bar{\Psi}(\rho_1, \rho_2, \Phi) \propto J_j(\rho_1/\rho_1^c) J_j(\rho_2/\rho_2^c) \exp(ij\Phi) \quad (669)$$

for constants $\rho_c^{\bar{r}} = \sqrt{2\hbar/\sqrt{-E_{\bar{r}}}}$ and $E_1 + E_2 = E$. The interpretation of the probability density functions has the usual multiplicity of peaks of a free problem. These now correspond to a sequence of (cluster) separations that are probable alternating with a sequence that are improbable. As is usual, have non-normalizability; in any case prefer models with potential terms (the latter is probably more primary, and non-normalizability is then a consequence of not doing so).

9.3.3 Very special and special HO problems in parabolic-type coordinates

The very special case matches the atomic problem in e.g. [416, 457, 318, 531], under the correspondence \mathcal{E} to $-K/4$ (which is the right sign to get the bound states) and $e^2/\pi\epsilon_0$ to E .

Now, the quantum numbers are given by

$$N_{\bar{p}} = -\{|j| + 1\}/2 + N\beta_{\bar{p}} \quad (670)$$

for N as before, $N_{\bar{p}}$ parabolic quantum numbers and $\beta_{\bar{p}}$ constants such that $\beta_1 + \beta_2 = 1$. Here

$$N_1 + N_2 + |j| + 1 = N. \quad (671)$$

The time-independent Schrödinger equation separates into simple harmonic motion in Φ , while, for the other variables, one gets the same mathematics as for the standard atomic separated-out 1- d parabolic coordinate problem,

$$\frac{d}{dI_{\bar{p}}} \left\{ I_{\bar{p}} \frac{d}{dI_{\bar{p}}} \Xi_{\bar{p}} \right\} - \frac{m^2}{4I_{\bar{p}}} \Xi_{\bar{p}} - \frac{KI_{\bar{p}}}{8\hbar^2} \Xi_{\bar{p}} = -\beta_{\bar{p}} \frac{E}{8\hbar^2} \Xi_{\bar{p}}. \quad (672)$$

Thus the wavefunctions for the present problem up to normalization are (see e.g. [416, 318])

$$\bar{\Psi}_{N_1 N_2 j}(I_1, I_2, \Phi) \propto L_{N_1}^{|j|}(I_1/NI_0) L_{N_2}^{|j|}(I_2/NI_0) \exp(-\{I_1 + I_2\}/2NI_0) \{I_1 I_2\}^{|j|/2} \exp(ij\Phi), \quad (673)$$

for $I_0 = 8\hbar^2/E$.

Or, in terms of shape and size variables,

$$\bar{\Psi}_{N_1 N_2 j}(\text{SIZE}, \text{aniso}, \text{ellip}) = L_{N_1}^{|j|}(\omega_1 \text{SIZE}\{1 - \text{ellip}\}/4\hbar) L_{N_2}^{|j|}(\omega_2 \text{SIZE}\{1 + \text{ellip}\}/4\hbar) \exp(-\text{SIZE}\{\{\omega_1 + \omega_2\} + \{\omega_2 - \omega_1\}\text{ellip}\}/8\hbar) \\ \text{SIZE}^{|j|}\{1 - \text{ellip}^2\}^{|j|/2} \mathcal{T}_j \left(\text{aniso}/\sqrt{1 - \text{ellip}^2} \right), \omega_{\bar{r}} = \sqrt{K_{\bar{r}}}. \quad (674)$$

Next, the special case can be done essentially like the very special one. The separation continues to work out as before except that what is the same energy constant in each separated out parabolic problem now takes a different value for each, $-K'_{\bar{p}}/4$.⁶⁵ Now the quantum numbers come out to be as before, though each $N_{\bar{r}}$ now has a distinct form of N :

$$N_{(\bar{r})} = E/\sqrt{2K_{\bar{r}}2\hbar}, \quad (675)$$

so that there is not a simple relation like (671), but, rather,

$$\frac{N_1 + \{|j| + 1\}/2}{N_{(1)}} + \frac{N_2 + \{|j| + 1\}/2}{N_{(2)}} = 1. \quad (676)$$

This corresponds to this less symmetric case not having a principal quantum number analogue. The wavefunctions are then

$$\bar{\Psi}_{N_1 N_2 j}(I_1, I_2, \Phi) \propto L_{N_1}^{|j|}(I_1/N_{(1)}I_0) L_{N_2}^{|j|}(I_2/N_{(2)}I_0) \exp(-\{I_1/N_{(1)} + I_2/N_{(2)}\}/2I_0) \{I_1 I_2\}^{|j|/2} \exp(ij\Phi). \quad (677)$$

Note 1) compared to the hydrogen ones, the difference in inner product causes the probability density functions for these to behave differently for $m = 0/j = 0$ (out problem's then peak at the origin, while hydrogen's go to zero there).

Note 2) the significance of j for parabolic orbitals is as follows. $j = 0$ is the same probability for all relative angles.

$j = 1$ has one state favouring the near-collinear configurations and another state favouring the near-right configurations.

$j = 2$ has one state favouring both of these and one state favouring neither of them.

9.3.4 General HO problem in parabolic-type coordinates

Take (677) with normal coordinate N -labels on it and then apply the rotation in Sec 5.2.8. Then, in a basis with first axis aligned with the potential and E as second axis [with (γ) labels suppressed],

$$\bar{\Psi}_{N_1 N_2 s}(\text{SIZE}, \text{aniso}, \text{ellip}) = L_{N_1}^{|s|}(\omega_1^N \text{SIZE}\{f - B \text{ellip} - C \text{aniso}\}/2g) L_{N_2}^{|s|}(\omega_2^N \text{SIZE}\{f + B \text{ellip} + C \text{aniso}\}/2g) \text{SIZE}^{|s|} \\ \{f^2 - (B \text{ellip} + C \text{aniso})^2\}^{|s|/2} \exp(-\text{SIZE}\{f\{\omega_1^N + \omega_2^N\} + \{\omega_2^N - \omega_1^N\}\{B \text{ellip} + C \text{aniso}\}\}/2g) \mathcal{T}_r \left(\frac{C \text{ellip} - B \text{aniso}}{\sqrt{f^2 - \{B \text{ellip} + C \text{aniso}\}^2}} \right) \quad (678)$$

for $\omega_i^N = \sqrt{K_i^N}$ (normal mode frequencies), $f = \sqrt{B^2 + C^2}$ and $g = 2\hbar\sqrt{B^2 + C^2}$.

⁶⁵The counterpart of this unusual extension was already remarked upon at the classical level in Sec 5.

9.3.5 $A < 0$, $E = 0$ example later used in Semiclassical Approach

I.e. the negative-curvature vacuum cosmology analogue. The exact solution here is

$$\begin{aligned} \Psi_{Sj} &\propto I^{-1/2} J_{2S+1} \left(\frac{\sqrt{-2A}}{\hbar} I^{1/2} \right) P_S^j(\cos \Theta) \exp(ij\Phi) \\ &\propto \sqrt{\frac{\text{SIZE}_0}{\text{SIZE}}} J_{2S+1} \left(\sqrt{\frac{\text{SIZE}_0}{\text{SIZE}}} \right) P_S^j(\text{ellip}) \mathcal{T}_j(\text{aniso}/\sqrt{1 - \text{ellip}^2}) , \end{aligned} \quad (679)$$

for $\text{SIZE}_0 = \hbar^2 / -2A$.

9.3.6 Cases with further cosmology-inspired potentials

A list of potential contributions that directly parallel other work in ScaleQM are 1) the upside-down HO's that map to ionized atoms, and correspond to negative spatial curvature in cosmology. 2) Extra $1/I^2$ -type potentials, which allow for wrong-sign radiation to approximately become right-sign radiation. 3) I^2 -type potentials, which correspond to cosmological-constant terms of whichever sign. 3) is covered in Sec 9.3.9 at the level of perturbations about this SSec's main hydrogen-analogue model.

9.3.7 Interpretation: Bohr moment of inertia

The mathematical analogy implies the following.

$$(\text{Bohr radius}) a_0 = 4\pi\epsilon_0 \hbar^2 / m_e e^2 \longleftrightarrow I_0 = 4\hbar^2 / E \quad (\text{a new 'Bohr total moment of inertia' scale}) \quad (680)$$

which has the same kind of interpretation as the typical minimum quantity or effective size – the overall ground state has a moment of inertia distribution with 'characteristic width' I_0 . N.B. The analogy with hydrogen at the quantum level is part-false, since the inner products do not match up.

9.3.8 Interpretation: Expectations and spreads

Now, in spherical coordinates for the very special case, by the scale-shape split, and the orthogonality relation and a recurrence relation for the generalized Laguerre polynomials,

$$\langle N S j | I | N S j \rangle = \text{SIZE}_0 N^2, \Delta_{N S j} = \sqrt{\langle N S j | \text{SIZE}^2 | N S j \rangle - \langle N S j | \text{SIZE} | N S j \rangle^2} = \text{SIZE}_0 N \sqrt{N^2 + S\{S-1\}} / \sqrt{2}. \quad (681)$$

Thus, in particular for the ground state, the expectation is the characteristic Bohr moment of inertia $I_0 = \text{SIZE}_0$ and the spread is $\text{SIZE}_0 / \sqrt{2}$. Differences from hydrogen's expectation and spread in r are due to the difference in the inner product in each case, noting that the present article's case is easier to compute. (It requires merely one use of the aforementioned recurrence relation. Also, the expectation turns out not to depend on the total rational quantum number in this case, unlike for the atomic one.)

On the other hand, in parabolic-type coordinates, for the special case,

$$\langle N_1 N_2 j | \text{SIZE} | N_1 N_2 j \rangle = \text{SIZE}_0 \{N_{(1)} N_1 + N_{(2)} N_2 + \{N_{(1)} + N_{(2)}\} \{|j| + 1\} / 2\}, \quad (682)$$

$$\Delta_{N_1 N_2 j} = \text{SIZE}_0 N \sqrt{N_1^2 + N_2^2 + \{N - |j|\} \{|j| + 1\} / \sqrt{2}}. \quad (683)$$

Note that for very special case ground state, these reduce to the above results, as they should.

9.3.9 Interpretation: perturbative treatment of additional cosmologically-inspired potential terms

E.g. $|r^{IJ}|^6$ terms are cosmology-motivated, by corresponding to cosmological constant terms. These can be treated as small perturbations about the preceding Sec's problem. The perturbation theory expands in a series of powers of S , so then there are also $O(S^2)$ terms (and the previously mentioned corrections become $O(S^a, S)$). They have a lead scale part $S I^2$ and then $O(S^a)$ terms. Then, using the same $\langle N S j | I^2 | N S j \rangle$ integral as in the preceding SSec one obtains the first-order perturbation correction to the analogue of \mathcal{E} to be

$$K = 4\hbar^2 / N^2 I_0^2 + S I_0^2 N^2 \{3N^2 + S\{S-1\}\} / 8 + O(S^a, S) + O(S^2), \quad (684)$$

so an approximate inversion to look at the effect on E gives that

$$E^{\{1\}} = N \hbar \omega - S \{ \hbar \omega \}^3 N \{3N^2 + S\{S-1\}\} + O(S^a, S) + O(S^2). \quad (685)$$

Thus $S > 0$ lowers E , while $S \leq 0$ raises E .

10 QM Conclusion

I solved concrete N -stop metroland and triangleland models. My study involved ‘modes and nodes’ accounts of the probability density functions against the back-cloth of the tessellation of the relationalspace by the physical interpretation. I also considered expectations and spreads for these quantum solutions and the tractability of further cosmologically-inspired perturbations about them which are a toy model for quantum-cosmological structure formation. The scaled models I investigated possess characteristic scales: a ‘Bohr moment of inertia of the model universe’ analogue of the Bohr radius of the atom and an oscillator scale.

10.1 More on closed-universe effects

Closed-universe and finite-universe effects are manifest in RPM’s. Some of these involve collapse in the number of admissible combinations of quantum numbers.

Example 1) The subsystems’ energies are interlocked by each model having an overall energy of the universe, which leads to gaps, truncations, sometimes even total nonexistence in what would otherwise be the eigenspectrum (Sec 7.4).

Example 2) The subsystems’ angular momenta are counter-balanced so as to comply with the zero angular momentum constraint. This is clearly in line with the solution via second reduction, or third reduction in the Laplace-ordered case. For scaled triangleland, Dirac-approach solutions based on base–apex subsystem splits depend on one less quantum number than one might naïvely expect [19]. One can get to this via $L_i\psi_i = m_i\psi_i$ $i = 1, 2$ arising in each separated problem, but then by $0 = \hat{\mathcal{L}}\Psi = \hat{L}_1 \otimes \hat{1}_2\Psi + \hat{1}_1 \otimes \hat{L}_2\Psi = m_1\Psi + m_2\Psi$, so $m_1 = -m_2 := j$, so that

$$\Psi_{jn}(\rho_1, \rho_2, \theta_1, \theta_2) = \exp(ij\{\theta_1 - \theta_2\})J_j(\rho_1/\rho_1^0)J_{-j}(\rho_2/\rho_2^0) . \quad (686)$$

One can alternatively get to this via p_Φ (for $\Phi = \phi_2 - \phi_1$) being cyclic and carrying a relative angular momentum quantum number.

Example 3) Is there a dilational counterpart of this in pure-shape RPM’s? The preceding counterbalance is tied to the very simple form of the phase-angle wavefunctions (exponentials as opposed to exponentials times polynomials times powers, which produce complicated expressions on being acted upon by quantum constraint operators that are linear in the momenta). On the other hand, rewriting in scale–shape split coordinates, the dilational constraint candidly becomes $\rho\partial\Psi/\partial\rho = 0$ i.e. that the wavefunction is independent of ρ . In this case then the only survivor in a separated-out ρ -piece is the constant solution. Thus this is indeed a collapse in number of admissible quantum numbers but not a counterbalance. On this occasion, the argument is independent of all of N , d and V .

Example 4) There is still angular momentum counterbalancing in the pure-shape triangleland case This can be demonstrated by classically eliminating the dilational constraint but retaining the zero angular momentum constraint which still leaves SHM in the two phase angles, thus continuing to exhibit the characteristics of a counterbalance. This is solvable e.g. in ρ_1 , ϕ_1 and ϕ_2 , leading to dependence solely on $\Phi = \phi_2 - \phi_1$.

Note that Barbour and Smolin suggested [98, 83] that (Barbour’s) relationalism requires radically different QM theory whilst also not possessing a good semiclassical limit, partly on the basis of the above Example 1). I argue against this, by showing how standard QM structure and standard QM solutions made out of the usual Methods of Mathematical Physics suffice for the study of the simpler quantum RPM’s. This is achieved via the various key choices of coordinates (Jacobi coordinates, spherical coordinates...) and the good fortune of being able to reduce one’s way down to the physical configuration space for these examples. There are still closed-universe effects in the models such as examples 1 to 4, but these do not affect the ‘recovery of everyday physics’ for a large universe with sufficiently varied contents. This is via the presence of free particles and of potentials of both signs serve to overcome gaps and truncation. And via the practicalities of experimentation involving only small subsystems means that interlocking and counterbalancing would be likely to go unnoticed. These things said, this article concerns, rather, RPM’s as toy models of geometrodynamics.

Example 5) In closed-universe models, the elsewhere entirely theoretically solid cluster decomposition principle is globally violated: one cannot just tensor subsystems together right up to the inclusion of the entire universe if there are closed-universe features such as RPM model universes’ energy interlocking and overall angular momentum counterbalancing.

10.2 Quantum Cosmology versus Atomic/Molecular Physics

This article analogue-models some fairly conventional Quantum Cosmology models. It is unsurprising that this is different in at least some of the details from modelling commonly encountered molecules/ions such as H_2^+ and NH_3 [67]. Some similarities and differences are as follows.

- 1) Molecules are but tiny and unisolated pieces of a much larger actual universe. (In particular, in [34], I consider Molecular Physics as studied in laboratories on Earth, where there is good control over initial conditions but certainly not isolation from the rest of the universe, in particular in the aspect of determining what the inertial frames are). Thus in Molecular Physics, some coordinates can effectively refer to the larger set-up. E.g. this is true of ‘electric field parallel to the z-axis’ in the Stark effect for the linear rigid rotor. It is also true if one uses the spatial angles $(\theta_{\text{ph}}, \phi_{\text{ph}})$ for the same mathematics as in triangleland or 4-stop metroland but now laced with absolute rather than purely relational meaning. These features are clearly less desirable for GR-like quantum cosmological modelling than RPM’s relational context.
- 2) Useful coordinate choices/kinematics are largely shared between quantum cosmological RPM’s and molecular models, e.g. the use of Jacobi coordinates and of Dragt-type coordinates in the present article.
- 3) Then, at the level of the solutions, scaled models with multi-HO potentials have the obvious standard mathematics. Pure-shape 4-stop metroland with multi-HO-type potentials has mathematical analogies with Pauling’s study of rotations in molecules, the polarization theory behind the Raman effect and the H_2^+ problem. Scalefree triangleland with multi-HO-type potentials has a mathematical analogy with the Stark effect for the linear rigid rotor. Scaled N -stop metroland models with a range of cosmologically-motivated potentials also give rise to a range of further mathematical models that are (fairly) common in QM textbooks. Examples of such are upside-down HO’s, and $\{N - 1\}$ - d isotropic HO’s, the ordinary HO, and hydrogen-type problems. Scaled triangleland with the very special multi-HO-type potential’s wavefunctions can be made to coincide mathematically with hydrogen’s. Here, the special and general multi-HO-type potentials continuing to be accessible by extension of the hydrogen problem’s well-known solvability in parabolic coordinates. Finally, interpreting many of the above solutions involves work similar to/identical with the evaluation of radial expectations and spreads for hydrogen, and of the corresponding shape integrals (usually known as 3- \mathcal{Y} integrals).
- 4) However, molecules have rather tight physics. E.g. they are invariably held together by Coulombic forces. In contrast, as regards cosmological models, different matter types contribute to different mechanical-analogue power-law potential contributions [35]. Nor by any means is there a concept of fixed bond length in analogue models of Quantum Cosmology. Furthermore, molecules also have a precise mass hierarchy following from the large ratio between the electron and nucleon masses. This is by no means appropriate for quantum cosmological modelling. (Though one would like this to possess *some sort of* mass hierarchy if one is attempting to obtain time as an emergent semiclassical concept, c.f. Sec 13). Also, molecules’ constituent elementary particles have a precise character as regards statistics and distinguishability, while such properties would appear to be more optional than requisite for (analogue) Quantum Cosmology models.
- 5) In various cases, the analogies break down if pushed far enough, due to, firstly, the 2-body approximation only sitting unstably inside the 3-body problem for N -stop metroland with negative power-law potentials. Secondly, the very special HO for triangleland has the same wavefunctions as for hydrogen but not the same inner product. Thirdly, there are differences in the ways in which the next levels of structure complicate things. E.g. various nuclear and SR effects occur in molecules [416, 627], while one may wish to build in certain analogue-GR/quantum cosmological details into RPM’s [35] (see also SSec 10.3).
- 6)–8) See also the Operator Ordering, absolutist imprints and multiverse interpretation SSecs below.

Difference 25) I do not know of any meaningful analogue of the Hartle–Hawking type condition on Ψ in RPM’s.

10.3 ‘Multiverse’ differences between RPM’s and molecular models

One difference between this article’s specific models and their Molecular Physics analogues is at the level of which ‘multiverses’ correspond to each. (For Molecular Physics models the corresponding multiverse can be thought of as a collection of laboratory setups with different parameter values. Multiverses in the RPM context are at least aesthetically more appealing in distancing themselves from Copenhagen interpretation connotations.)

Example 1) the set-up for the Stark effect for the linear rigid rotor can have an electric field in any 3- d direction, while the corresponding direction in the triangleland harmonic oscillator problem cannot be rotated out of the collinearity plane. Moreover, in triangleland there are 3 (DM) axes of particular physical significance within this plane. Thus each corresponds to a different multiverse setting.

Example 2) Both the 4-stop metroland harmonic oscillator and the crystal problem have privileged directions; these can be set up in particularly close correspondence if the crystal had cubic symmetry.

Example 3) On the other hand, the ‘Raman’ multiverse does not have such a tight similarity with the 4-stop metroland multiverse. What the theory of Raman spectroscopy does have [614, 623, 13] is further analogies which extend to the general problem [i.e. the C, D, E and F terms of Sec 5] [50].

As regards the issue [201, 83] of whether stationary quantum universes have single or multiple states, I comment that this article’s models do exhibit some degenerate states (both among simple exact solutions and perturbatively to second order). However one needs a much more extensive study of RPM model universes before one can begin to say whether these are, however, non-generic.

A related issue concerns placing a closed-universe interpretation on the perturbed problem. Sec 7, 8 and 9's solutions are universes of fixed energy E_{uni} and not modes within a particular universe. Sometimes [201], this corresponds to no allowed S, j , sometimes to one and at least sometimes to more than one. For example, $E_{0,0} = E_{1,0}$ for $B = \sqrt{15/2}\hbar^2$, which is perturbatively acceptable provided that $A \gg B$. I use this further in Sec 14 toward Statistical Mechanics (SM)/Entropy/Information construction.

10.4 Relational nontriviality at the quantum level

As regards a classical system with only one degree of freedom: in passing to the quantum level, there may be further latent quantum degrees of freedom like spin, and via 1 degree of freedom sufficing to have a nontrivial wavefunction and subsequently multiple observables such as the n th moments, which could furnish enough of a sense of correlation to have meaningful physics. Since the point particle has gained structure such as spread (and, in mixed states, the possibility of multimodality, a lesser number of quantum particle-waves can suffice to furnish nontrivially relational physics [I thank Don Page for first making me aware of this point.] This is conceptually in parallel with Appendix 5.C.2's point 6)'s looking closer at point particles revealing that they are planets alters the relational status of the model.

10.5 Operator-ordering issues and Dirac–reduced comparison

1) I presented DeWitt's argument for operator-ordering and explained how the outcome of it is the family of ξ -orderings $\Delta^\xi = \Delta - \xi \text{Ric}(M)$ for M the configuration space metric and Δ the Laplacian built out of it. I showed how Misner's subsequent argument for conformal operator ordering rests on the PPST-invariance of relational product-type actions (or on a more complicated invariance of the more usual difference-type actions); this ordering is for a particular configuration space dimension-dependent value of ξ .

2) I also showed that the presence of an absolute sector affects the form of the Laplacian Δ of the relational sector, and thus, by extension that of the Δ^ξ operators, which include the conformal operator. This accounts for differences at the quantum level between RPM whole-universe models and conventional molecular models, which bear an absolute imprint by which they are implicitly subsystems within a much larger universe.

3) Finally, I showed that the Dirac formulation with conformal ordering is in general inequivalent to the reduced formulation with conformal ordering. This is then highly unfortunate, due to Kuchař's strong argument for favouring reduced quantization whenever the two disagree, alongside this not being available in the general geometrodynamics case. Thus it only makes sense to apply the conformal-ordering argument to minisuperspace models for which there no linear constraints causing a Dirac–reduced distinction (and possibly some other specialized models for which this distinction does not affect the operator-ordering, for which this article gave a few simple RPM examples). [The argument to the same end that conformal ordering in any case becomes ill-defined in infinite theories is partly diffused in this article; this is a good example of the value of RPM's as partly midisuperspace-like models that disentangle linear constrainedness from the theory having an infinite number of degrees of freedom. *Both* linear constrainedness and infiniteness spell trouble for conformal ordering in sufficiently general theories.]

10.6 Discussion of absolutist imprints

There is no physical distinction between the relational sector of $\mathcal{L} = 0$ absolute mechanics and RPM's. Nor as regards QM monopole effects (unlike in $\mathcal{L} \neq 0$ Newtonian Mechanics, for which such show up). There is however then a quantum-level distinction between the relational sector of $\mathcal{L} = 0$ absolute mechanics (taken to be a subsystem study) and RPM's (taken to be a whole-universe setting) at the level of which operator orderings are to be used in forming the Schrödinger equation. (In being based on an overall determinant having a factor coming in from a different sector, there is a loose parallel between this and the Faddeev–Popov determinants being imprints of the ghost species that 'live' along the unphysical orbits [576].) Most of the Molecular Physics literature does not, as far as I have seen, make a connection between the ordering used in e.g. the 3-body problem and the absolute versus relative motion. Aharonov and Kaufherr [5] do however show awareness of this issue; comparing that work and the present article represents interesting work in progress.

Point: $\mathbf{q} \rightarrow \mathbf{q}$ makes various appearances at the quantum level – Isham's kinematic quantization procedure; underlying DeWitt's example.

10.7 Uniform states

RPM's are also a useful toy model for notions of uniformity that are of widespread interest in Cosmology. This applies to good approximation to the present distribution of galaxies and to the CMB. There is also the issue of whether there was a considerably more uniform quantum-cosmological initial state [509]. Finally, there are related issues of uniformizing process and how the small perturbations observed today were seeded.

There is no distinction between scaled and pure-shape theories' notions of uniformity as these are pure-shape notions and scaled theories admit a scale-shape split (if one excludes the maximal collision from the definition of uniform in the scaled case).

Study of the 3-stop metroland case makes for a good parent for bigger models' larger complex of notions of uniformity. In the equal-mass case of this, there is one notion of merger M and this coincides with equally-separated-out particles (which is the most natural-looking notion of uniform for an N -stop metroland). The corresponding notion of least uniformity are the double collisions D. 4-stop metroland then has numerous notions of merger as discussed in Sec 3, one point among which additionally involves the four equal masses being equally spaced out.

For triangleland with equal masses there are the equilateral triangle states at the poles that are the most uniform states. This is a 2-d version of equally-spaced-out particles, and it is now, more satisfactory, a cluster-independent i.e. democracy invariant notion. That is characterized by it being the maximum of $4 \times \text{area}$; note that this is a democratic invariant, $|\text{demo}(3)|$. It is a unique maximum (in plain shape case, signed area is involved, and there is a maximum and a minimum value of $\text{demo}(3)$ corresponding to the two orientations of the equilateral triangle. Then the collinear configurations are a well-defined opposite notion, corresponding to the minimum value of $|\text{demo}(3)|$. One could then use the 3-stop metroland version of least uniform to further discern the least uniform of these. Uniformity is further investigated using the naïve Schrödinger interpretation in Sec 14.2.

In this quadrilateralland case, there are 3 squares in each hemi- \mathbb{CP}^2 as opposed to the single equilateral triangle in each hemisphere of triangleland; these are the obvious particularly uniform configurations. That there are three per hemisphere reflects the presence of a further 3-fold symmetry in quadrilateralland; choosing to use indistinguishable particles then quotients this out. These are also characterized as the extremum value of the corresponding democracy invariant, $\text{demo}(4)$, however in this case they are not the only extrema: there is a whole extremal curve per hemi- \mathbb{CP}^2 . In $\{N^e\text{-aniso}(e)\}$ coordinates, this is given by $\text{aniso}(1) = 0 = \text{aniso}(2)$ and $|\text{aniso}(3)| = 1$, i.e. (1)-isosceles, (2)-isosceles and maximally (3)-right or (3)-left i.e. (3)-collinear, with $N_3 = 1/2$ and N_1, N_2 varying (but such that the on- S^5 condition $N_1 + N_2 + N_3 = 1$ holds). In Gibbons-Pope type coordinates, the uniformity condition is $\phi = 0, \psi = 4\pi, \chi = \pi/4$ and β free, i.e., in the H-coordinates case, freedom in the contents inhomogeneity i.e. size of subsystem 1 relative to the size of subsystem 2. On the other hand, in the K-coordinates case, β free signifies freedom in how tall one makes the selected $\{12, 3\}$ cluster's triangle.

10.8 Structure formation

Does structure formation in the universe have a quantum-mechanical origin? In GR, studying this requires midisuperspace or at least inhomogeneous perturbations about minisuperspace, which are of great difficulty. There are also a number of difficulties associated with closed system physics and observables, speculations on initial conditions, the meaning and form of Ψ (e.g. the discussion of uniformity below) and the origin of the arrow of time. RPM's Semiclassical Approach scheme are useful toy models of midisuperspace Quantum Cosmology models that investigate the origin of structure formation in the universe. (E.g. the Halliwell-Hawking model toward Quantum Cosmology seeding galaxy formation and CMB inhomogeneities). Thus RPM's are valuable conceptually and to test whether one should be *qualitatively* confident in the assumptions and approximations made in such schemes. This involves the shape wavefunctions provided as the l-physics, alongside t^{em} -dependent perturbations of these, a small start on which is made in Sec 14.

I can ask questions about clustering-independent questions as well as clustering-dependent properties. This could be via the good fortune of having democratic invariants, or by more generally if less neatly summing/averaging over clusterings.

Moreover, in some cases one *does* want to consider specific clusters (e.g. the model has 2 heavy particles and a light one and the observer considers the light one to be their galaxy). The advantages of 4-stop metroland are in its cleaner concept of contents inhomogeneity, which arises through being able to partition the universe into 2 clusters of 2 particles each. In triangleland, one can only make such considerations by comparing clusters of 2 particles that overlap as regards which clusters they include. This point is further developed in the naïve Schrödinger interpretation Sec 14.2.

Note 1) Upgrading to N -stop metroland and N -a-gonland models is likely to improve one's capacity to use particle clumps to model inhomogeneities.

Note 2) The spherical presentation of triangeland is of limited use due to not extending to higher N -a-gonlands. However, one use for it is in allowing the shape part to be studied in S^2 terms which more closely parallel the Halliwell-Hawking [296] analysis of GR inhomogeneities over S^3 . This application will be more fully investigated in [40].

Note 3) Moreover, Kuchař has also argued [403] that Halliwell and Hawking's perturbative treatment of inhomogeneous GR is likewise only a model of Quantum Cosmology, because quantum fluctuations are far smaller than the classical universe's inhomogeneities. In models with inflation, this criticism may at least in part be circumvented.

10.9 Parallel mini- and midi-superspace investigations

Another longer-term goal would be to export insights acquired by the RPM program to 'mini and midi'superspace; are there then useful analogues of shape operators for these (anisotropy operators, inhomogeneity operators?) Anisotropy operators are considered in e.g. [405, 9, 47]. Inhomogeneous operators I leave as an open question.

10.10 Extending the arena to quadrilateralland

For less standard solutions to arise, one would need to check further, more complicated examples, such as the below (which have various further uses as indicated).

Question: Free solution. Take MacFarlane's [434] study of QM on \mathbb{CP}^2 and reinterpret this as the plain shape space of quadrilaterals. This looks to be a straightforward task and I will perform it in 2011.

Question: HO-like potential solution. I do not as yet know whether inclusion of potentials of this kind will allow for the shape part to remain analytically tractable. This may require a 'differential equations on the complex plane' analysis to establish.

Question: Perturbations about the free solution. This will be another exercise in the methods of Mathematical Physics, this time concerning recurrence relations for, and integrals involving, the Jacobi polynomials [1, 270] and the Wigner D-functions [217] that arise from MacFarlane's work [434]. By this the structure formation counterpart in Quadrilateralland remains around as analytically tractable as it is for triangleland.

Question: Quantum scaled quadrilateralland. The scale–shape split into the cone now gives MacFarlane's mathematics for the shape part and the standard sort of radial equation already familiar from earlier chapters, so I can solve this scaled problem to the same extent that I can solve the preceding. This is work in progress.

10.11 Otriangleland and 3-cornerland

The Ashtekar Variables approach adjoins degenerate configurations. Above experience with kinematical quantization could be viewed as reason to suspect an approach that does that. On the other hand, it is a new choice of variables with its own kinematical quantization.

Analogy 62) There are parallels between the O-choice of shapes and the affine approach [381, 340, 341] to geometrodynamics, as an extension of the half-line toy model for the affine case [332].

Question Study Otriangleland as a toy model of affine geometrodynamics.

Question There is a difference between Otriangleland and 3-cornerland as regards the form and includability of the degenerate configurations (the collinear configurations). This may be of relevance to the foundations of LQG. The situation there is that one extends to include degenerate configurations and then these play an important role in the quantization, leading to e.g. the discreteness of quantum area result. Are we absolutely sure then that the nondegenerate part of the metric is being fully taken into account in such an approach? Are we absolutely sure that we have an understood and unambiguous procedure as regards quantization of a set of states consisting of degenerate and nondegenerate parts? [Chris Isham did not point me to an answer to this issue when I discussed it with him.]

Question 3 In the light of 3-cornerland being less relational than Otriangleland, interpret the differences between the two models at the quantum level as another form of absolutist imprint.

RPM's as toy models of the Problem of Time in QG

11 Introduction to the Problem of Time

11.1 What is the Problem of Time?

The Problem of Time in QG [633, 397, 398, 400, 335, 403, 372, 545, 586, 39] occurs because the ‘time’ of GR and the ‘time’ of Quantum Theory are mutually incompatible notions. This causes difficulty in trying to replace these two branches of physics with a single framework in regimes in which neither Quantum Theory nor GR can be neglected, such as is needed in parts of the study of black holes or of the very early universe. The Problem of Time is pervasive throughout sufficiently GR-like attempts at doing Quantum Gravity. It is multi-faceted [400, 335, 39]. Resolving this incompatibility is of clear importance if theoretical physics is to form a coherent whole. Study of the Problem of Time is also important toward acquiring more solid foundations for the gradually-developing discipline of Quantum Cosmology (see e.g. [298, 294, 642, 375]).

11.2 Time in philosophy and physics; desirable properties for a candidate timefunction to indeed be a time

It is unfortunately often tacitly assumed in the literature that once one allots the word ‘time’ to a variable or parameter, then this quantity will then satisfactorily serve as a such. I hold that, rather, such an allocation is of a **candidate time**, and only detailed study of it will reveal whether it is a successful candidate. Of course, philosophers have had conflicting views about what time is since the beginning of civilization; Heraclitus and Parmenides were already at odds over whether the world contains a ‘flow of time’, whilst this and the next SSec tackles Saint Augustin’s [68] question

“What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks, I do not know.”

Moreover, what time means in modern physics is theory-dependent: each grouping Newtonian Mechanics and Quantum Mechanics, SR and QFT, and GR, has a different notion of time. Consequently what time is to mean in Quantum Gravity is heavily disputed...

11.2.1 Some properties often ascribed to time

i) **Ordering** [421, 354]. Simultaneity enters here, i.e. a notion of present that separates a notion of future from a notion of past, each of these two being perceived differently (e.g. one remembers the past and not the future). The structure of that instant is the theory’s (largest) NOS.

ii) **Causation** [421, 354], i.e. that one phenomenon causes another to occur.

iii) **Temporal logic**: “and then” and “at time t ” constructs.

iv) In dynamics, one encounters the idea of **change in time** (so that time is a container): a parameter of choice with respect to which change is manifest. Newtonian absolute time is a such (and external to the system itself and continuous). One state of a system “**Becoming**” another state of a system is another phrasing of this dynamical facet of time.

Duration is defined to be magnitude of time, the container property that parallels ‘extent’ in the case of space. See e.g. [516, 118, 70] for emphasis on the importance of this property.

Note 1) The statements in iv) should furthermore be contrasted with Mach’s [Relationalism 6]) viewpoint that *time is abstracted from change*. Such a secondary time would, moreover, be expected to have many of the other properties listed in this SSec. Mach’s viewpoint is considerably developed in Sec 12.

Note 2) There is an **arrow of time** notion in ordering, causation and temporal logic. This distinction is observed in practise and yet is not manifest in fundamental dynamical equations (which are time-reversal invariant or CPT invariant [512]). In such cases, the arrow is built in by hand at the level of the solutions selected (and the correlations in direction of various aspects of the arrow remain mysterious).

v) Mathematically, time is often taken to be modelled by the real line or an interval of this or a discrete approximation of this (as opposed to e.g. an identification of the real line, as in cyclic time). Though time can easily be *position-dependent* in field theories: $t(x^\mu)$. Some notions of time are more complicated (e.g. parallel times and branching times in Fig 47 or many-fingered times in GR in the next SSec).

vi) Time is additionally habitually taken to be **monotonic** (rather than direction-reversing) This makes sense in the context of time having further ordering and causation properties. It is also a part of the arrow of time property, in that there is a direction involved (the further part being that the various directions are then correlated). Apart from this SSec, see e.g. [616, 308, 661] for emphasis on many other properties of time.

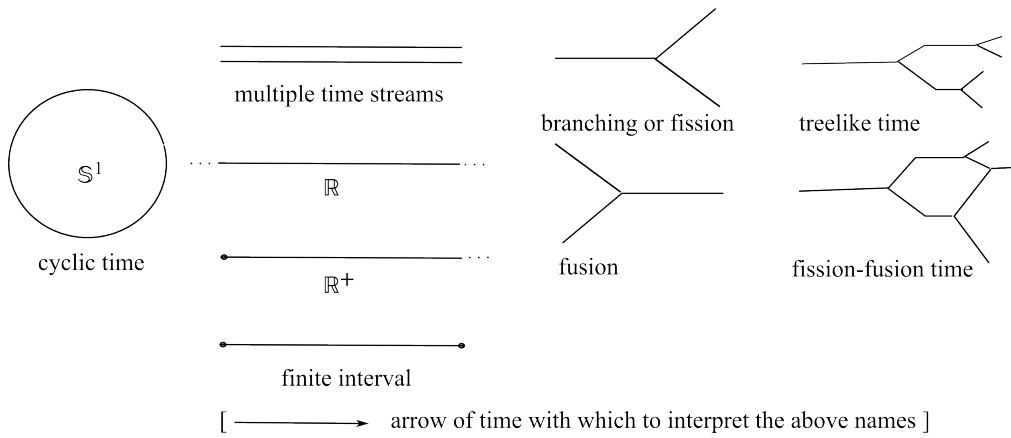


Figure 47: Different topologies proposed for single-function time. As compared to space, these are rather restricted by time’s dimension being one. The options then are the real line, the half-real line (e.g. from big bang to heat death), the finite interval (e.g. big bang to big crunch), the cycle (as in Hindu philosophy or the ekpyrotic universe), multiple time streams and then the following non-Hausdorff options. *Branching* (or *fission*) is solely to the future, whilst the time-reversal of that is termed *fusion*; if multiple time streams are possible, then one or both of these features might also be present, tying time streams together.

vii) There is to be freedom in prescribing a timefunction as to the **choice of calendar year zero** and of **tick-duration**, i.e. if t is a timefunction, so is $A + Bt$ for A, B constants.

viii) A good time function is **globally valid** [400, 567, 308] both over time (antagonist to the half-finite and finite interval times unless there is a good physical reason for this) and over space (made necessary by field theories and, to a greater extent generic curved space/GR).

ix) It also makes sense for a time function to be **operationally meaningful** (computable from observable quantities – tangible and practically accessible).

A perhaps-related aspect is that one might also expect that it only makes sense for one subsystem to furnish a time for another if the two are dynamically coupled, else how would the one subsystem know about the other. However, this is a disputed point to which I return later on in this Sec.

Note: timeless and histories conceptions also have property lists: the former has present, instant, being, being at some time and simultaneity notions, whilst the latter additionally has some form of time-ordering and causation notions as well, albeit phrased in a paradigm in which the notion of history replaces the notion of time.

11.2.2 Time in the main physical theories

That Newtonian time a *fixed* background parameter is then reflected in the role time plays in ordinary Quantum Theory. This is how time comes to enter QM as a background parameter that is

- 1) used to mark the evolution of the system.
- 2) It is represented by an *anti-hermitian* operator (unlike the representation of other quantities).
- 3) The time–energy uncertainty relation $\Delta t^{\text{Newton}} \Delta E \geq \hbar/2$ is also given an entirely different meaning to that of the other uncertainty relations.
- 4) There is unitary evolution in time, and only as one of two separate processes.
- 5) The other process is collapse of the wavefunction (this is held to occur in quantum theory despite it not being described by the evolution equation of the theory).

SR brings in the notion of

- 1) a proper time corresponding to each observer, and of
- 2) time as another coordinate on spacetime, as opposed to the external absolute time of Newtonian physics.
- 3) Though in SR, time is also external and absolute in the sense of there being (via the existence of Killing vectors) a presupposed set of privileged inertial frames (the quantum theory can be made independent of a choice of frame if it carries a unitary representation of the Poincaré group). In that sense SR’s time remains absolute, and it is only GR that frees one of this, so that the SR to GR conceptual leap is in many ways bigger than the one from Newtonian Mechanics to SR (see also Sec 2.6.4). Thus the passage to SR and QFT does not greatly affect the role of time as compared to nonrelativistic mechanics and QM.
- 4) On the other hand, the notion of simultaneity (and how to set up the simultaneity convention) changes in passing from Newtonian Mechanics to SR.

In GR,

- 1) space itself evolves which is relevant in terms of the nature of the instants/simultaneity.

2) Time is the *general* coordinate in GR clashes with QM's time the strange extraneous thing. In this respect, Tempus Ante Quantum and Tempus Post Quantum Problem of Time strategies favour QM, whereas Tempus Nihil Est is more of a harbinger of paradigm shift.

3) Time as a general choice of coordinate is embodied geometrically in how spacetime is sliced into a **sequence of spacelike hypersurfaces** of constant time. That there is one timefunction per choice of foliation is why, overall, time in GR is said to be '**many-fingered**'. Physically, these choices correspond to the perspectives of different families of observers. [A truer picture is a congruence of material worldlines, a sample of which are occupied by observers, who might possess clocks and might have set up a synchronization procedure, which those particular worldlines may or may not maintain.] Conversely, spacetime encompasses space and time, and in doing so attributes a highly nonunique character to time. At the classical level, predictions about a given constant t_1 hypersurface do not depend on how one foliates the spacetime between a given constant t_0 data hypersurface and the t_1 hypersurface. Thus the element of choice four sentences back is not as large as one might naïvely suppose. However, at the quantum level, it is a very serious problem.

4) **The generic spacetime has no timelike Killing vector**, so much of the preceding QM structure ceases to have an analogue. *The passage from Newtonian theory to SR merely swaps Newton's notion of absolute time for another absolute notion – that associated with the timelike Killing vector of Minkowski spacetime* (see also Sec 2.6.4). In this sense, it is the passage to *general* relativity that is the one which brings about a Problem of Time (see [376] for more). One has to pass from having privileged frame classes to dealing with the spacetime diffeomorphisms. It is not so clear where histories stand in this regard, being part QM-like in path integral mode, but also capable of harbouring GR-like features. This may bode well for these to be (part of) a reconciliation.

5) In GR, time has the ordering property, whilst causality reigns where simultaneity reigned in Newtonian Mechanics as extension of the situation in SR except now matter and gravitation influence the larger-scale causal properties.

6) GR additionally has a time non-orientability notion [622], and a closed timelike curve notion (both of these are usually held to be undesirable features for a physical solution to possess).

Indeed, are *spacetime* or any of its subspects meaningful in Quantum Gravity, and how do the other aspects emerge in the classical limit? Is (spatial or spacetime) classical *geometry* or any of its subspects meaningful in Quantum Gravity, and how do the other aspects emerge in the classical limit. Is there something resembling the classical notion of *causality* in Quantum Gravity, and if so, which aspects of classical causality are retained as fundamental, and how do the other aspects emerge in the classical limit?

External time is furthermore incompatible with describing truly *closed* systems, the ultimate of which is closed-universe Quantum Cosmology. Here, Page and Wootters [502] have given convincing arguments that such a system's only physical states are *eigenstates* of the Hamiltonian operator, whose time evolution is essentially trivial.

If there is more than one plausible conceptual approach providing a timefunction, one is then interested in whether the various timefunctions are aligned. In the GR setting, this is for a given sequence of slices, which correspond to a family of observers moving in a particular way.

Returning to ordinary QM, the idea of events happening at a single time plays a crucial role in the technical and conceptual foundations of Quantum Theory, as follows.

1) *Measurements* made at a particular time are a fundamental ingredient of the conventional Copenhagen interpretation (which is anchored on the existence of a privileged time). In particular, an **observable** is a quantity whose value can be measured at a 'given time'. On the other hand, a 'history' has no direct physical meaning except in so far as it refers to the outcome of a sequence of time-ordered measurements.

How should QM be *interpreted* for the universe as a whole (Sec 6.13.1 is relevant here again).

2) One of the central requirements of the *scalar product* on the Hilbert space of states is that it be conserved under evolution in time. This is closely connected to the requirement on the QM evolution of **unitarity**, i.e. that probabilities always sum to one: inner product leads to conserved probability currents.

More generally, a key ingredient in the construction of the Hilbert space for a quantum system is the selection of a complete set of observables that are required to *commute* at a fixed value of time – i.e. **equal-time commutation relations**.

The above extend to SR by again considering not one Newtonian time but the set of relativistic inertial reference frames and then one's QM can be made independent of frame choice through the carriage of a unitary representation of the Poincaré group. For a relativistic QFT, the above equal-time commutation relations point is closely related to requiring microcausality, i.e.

$$[\hat{\phi}(X), \hat{\phi}(X')] = 0 \quad (687)$$

for all spacelike-separated spacetime points X and X' . Observables and some notion of time-equal commutation relations are then big problems in GR context.

11.2.3 The extra issues in using the word 'clock'

The following property list is (part of) what is desirable for anything claimed to be a 'clock' to possess.

- 1) Clocks count occurrences that are held to be regular in time. One finds out which occurrences are regular by comparison between candidate clocks, and by extent of their predictive power in the study of other dynamical systems.
- 2) We wish to use a notion of time in terms of which the motion is simplest, and then desires clocks that read off such a time. I.e. clocks can be argued to be as convenient conventions [488]; in antiquity, uniform rotation was argued to be the best standard *because it is easiest to count* [58], and the circular motion of the heavens was an excellent such for that epoch. Clocks are usually⁶⁶ defined in terms of cyclical processes.
- 3) Clocks should actually read the purported timefunction rather than whatever they please; how this comes about (material versus spacetime property alignment) would not appear to be entirely clear. If this fails, it is a bad clock one has, or a timefunction that is at best secondary in practise (if no clock can be found that reads it).
- 4) Clocks should actually be physically constructible within the relevant regime (and this is difficult to envisage in early universe regimes, parts of black hole spacetimes and in the extremely small).
- 5) Clocks should be accurate enough to deal with the physics at hand.

Note 1) Barbour follows Leibniz (in the particular sense explained e.g. around p.41 of [635]) in suggesting that the only perfect clock is the whole universe (at the classical level). This should be contrasted with Newton's position that the universe *contains* clocks. I add that it is entirely impractical to use the whole universe as a clock, since it takes considerable effort to monitor and one only has very limited knowledge of many of its constituent parts,

Note 2) One should also bear in mind the difference between what is a convenient clock to read (e.g. a wrist-watch or an atomic clock) and what it takes for a clock to be reliably calibrated (see Sec 12 for more). It also looks paradoxical to regard any subsystem as imperfect, since in getting to the stage of including everything in the universe so as to 'perfect' one's clock, one has transcended to physics beset by the Frozen Formalism Problem. Finally, there are reasons other than non-whole-universe considerations not to believe in perfect clocks at the quantum level (see e.g. Unruh and Wald's account [616] in which it is demonstrated that all quantum clocks occasionally run backwards).

Note 3) Once one has accurate clocks,⁶⁷ one can consider nonlocal simultaneity from a practical perspective [354]. Thus clocks come before, from an operational perspective, position-dependent timefunctions. That what it reads is synchronized with other such of relevance [354]. Indeed, then, in an operational sense, clocks come before timefunctions (as opposed to mere mathematical imagining of foliations of a spacetime without thought as to how to populate them with suitable observers and clocks).

Note 4) One argument (see e.g. [244, 522]) is that a good clock should be in very little interaction with the subsystem under study. I have however already expressed the counter-point it is hard to conceive of how a clock can keep time for a subsystem if it is not coupled to it at all. Though nothing can shield gravity, so this limit of absurdity is not realized. On the other hand clock \leftrightarrow system interactions are not usually believed to be preeminantly gravitational.

A wider problem in conceptual distinction is between timefunctions versus clouds of clocks (observables of a particular kind) versus clouds of observers. All of these are different! Can the use of each, and the relationship between each, be carefully justified? See the Conclusion for more detailed treatment of this.

I next consider a number of facets of the Problem of Time (and various technical issues that become entwined with these).

11.3 Facets of the Problem of Time

11.3.1 The Frozen Formalism Facet of the Problem of Time

One notable facet of the Problem of Time shows up in attempting canonical quantization of GR (or many other gravitational theories that are likewise background-independent).

Frozen Formalism Facet The canonical approach at the classical level gives a constraint that is quadratic in the momenta whilst containing no linear dependence on the momenta. For GR, this is the Hamiltonian constraint,⁶⁸

$$\mathcal{H} := \mathcal{N}_{\mu\nu\rho\sigma} \pi^{\mu\nu} \pi^{\rho\sigma} / \sqrt{h} - \sqrt{h} \{ \text{Ric}(x; h) - 2\Lambda \} + \mathcal{H}^{\text{matter}} = 0. \quad (688)$$

Then promoting an equation with a momentum dependence of this kind to the quantum level does not give a time-dependent wave equation such as (for some notion of time t and some quantum Hamiltonian \hat{H})

$$i\hbar D\Psi/Dt = \hat{H}\Psi \quad (689)$$

⁶⁶Some (very poor) clocks can be otherwise, e.g. based on repeatable processes that need re-setting – the hourglass and the water-clock.

⁶⁷Portable mechanical clocks suitable for such purposes passed from theoretical design to practical sufficiently accurate engineering via Galileo, Huygens, Gemma and Harrison [354]. [The heavens will not serve for this purpose, since at a given moment of time they will look different from different points on the surface of the earth: the problem of keeping track of time/position at sea.] See [608, 219, 176] for further significant contributions to the theory of simultaneity.

⁶⁸ $\mathcal{H}^{\text{matter}}$ is proportional to the energy density of the universe model's matter, and $\mathcal{M}_{\mu}^{\text{matter}}$ is proportional to the momentum flux of the universe model's matter.

as one might expect, but rather a stationary, i.e. frozen, i.e. timeless equation

$$\hat{\mathcal{H}}\Psi = 0 . \quad (690)$$

In the case of GR, this is a Wheeler–DeWitt equation,

$$\hat{\mathcal{H}}\Psi := -\hbar^2 \left\{ \frac{1}{\sqrt{\mathcal{M}}} \frac{\delta}{\delta h^{\mu\nu}} \left\{ \sqrt{\mathcal{M}} \mathcal{N}^{\mu\nu\rho\sigma} \frac{\delta\Psi}{\delta h^{\rho\sigma}} \right\} - \xi \text{Ric}(h; \mathcal{M}] \right\} \Psi - \sqrt{h} \text{Ric}(x; h) \Psi + \sqrt{h} 2\Lambda \Psi + \hat{\mathcal{H}}^{\text{matter}} \Psi = 0 \quad (691)$$

(modulo the discussion in Sec 6.6). This suggests, in apparent contradiction with everyday experience, that nothing at all *happens* in the universe! Thus one is faced with having to explain the origin of the notions of time in the laws of physics that appear to apply in the universe; this paper reviews a number of strategies for such explanations. [Moreover timeless equations such as the Wheeler–DeWitt equation apply *to the universe as a whole*, whereas the more ordinary laws of physics apply to small subsystems *within* the universe, which does suggest that this is an apparent, rather than actual, paradox. See also the comments on contextualism in Secs 14 and 16 as regards the pervasive role of subsystems in Physics.]

This is at least a partial argument against the frozen formalism. Approaches requiring a radical solution outside of the standard thinking and methodology of Theoretical Physics, should not be necessary, at least as regards recovery of standard physics for subsystems within the universe.

Reminder of Analogy 56) For RPM’s, the energy constraint (42) likewise manifests the **Frozen Formalism Facet** of the Problem of Time.

One follow-on Facet from the Frozen Formalism Facet as manifested by a frozen equation like the Wheeler–DeWitt equation of GR is the **Hilbert Space/Inner Product Problem**, i.e. how one is to turn the space of solutions of that frozen equation into a Hilbert space. This is a time problem via the tie between inner products and unitary evolution.

What about alternative theories? E.g. [143, 83, 293]) that successors to the Wheeler–DeWitt equation through GR being supplanted would likely continue to have a frozen formalism. These things I largely agree with, for all that I mention a rare counterexample in Sec 11.5.5.

11.3.2 Some accompanying complications in attempting canonical quantization of gravity

Further discussion of Problem of Time Facets requires the following further details of the classical canonical approach to GR. 1) C.f. Sec 6.6 on the Wheeler–DeWitt equation’s technical problems. The RPM counterpart is nicer mathematically in these ways, albeit some ordering ambiguities remain: Secs 6.7, 6.10 and 6.11.

2) The GR momentum constraint often becomes entwined in technical problems that befall Problem of Time strategies; many such entwinings furthermore specifically concern diffeomorphisms (e.g. foliation dependence, the thin-sandwich problem, the extra problems in finding an internal time-and-spatial frame). The RPM zero total angular momentum constraint affords parallels of some (but not other) of these entwinings.

3) The canonical approach considers a sequence of spatial 3-surfaces that constitute a foliation of spacetime. These correspond to a cloud of observers distributed throughout space moving in a particular fashion. Moreover, invariance under 4- (rather than 3-)diffeomorphisms $\text{Diff}(M)$ of the spacetime M is altogether harder to find/account for in canonical approaches to GR. In particular, the Hamiltonian constraint does not directly account for the ‘missing’ diffeomorphisms. This is reflected e.g. in the Dirac algebra of the Hamiltonian and momentum constraints (168) being distinct from, and much less mathematically tractable than, the algebra of spacetime 4-diffeomorphisms. In particular, this algebra involves not structure constants but more general structure functions that depend on the 3-metric variable. At least this algebra has the useful property of closing at the classical level without further constraints becoming necessary to do so. Also, in classical GR, one can foliate spacetime in many ways, each corresponding to a different choice of timefunction. This is how time in classical GR is ‘many-fingered’, with each finger ‘pointing orthogonally’ to each possible foliation. Furthermore, classical GR has the remarkable property of being foliation independent [322], so that going between two given spatial geometries by means of different foliations in between produces the same region of spacetime and so the same answers to whatever physical questions can be posed therein. See Difference 23) in Sec 11.3.4 for the partial lack of RPM counterparts of this.

11.3.3 That the Problem of Time has further facets: the ‘Ice Dragon’

Over the past decade, it has become more common to argue or imply that the Problem of Time is the Frozen Formalism Problem. However, a more long-standing point of view [400, 335] that the Problem of Time contains a number of further facets; I argue in favour of this in this article. It should first be stated that the problems encountered in trying to quantize gravity largely interfere with each other rather than standing as independent obstacles. Kuchař talked of this in [401] as a ‘many gates’ problem, in which one attempting to enter the gates in sequence finds that they are no longer inside some of the gates they had previously entered. (The object being described is presumably some kind of enchanted castle, or, at least, a topologically nontrivial one). Additionally, the various of these problems that are deemed to be facets of the Problem of Time do bear conceptual and technical relations that makes it likely to be advantageous to treat them as parts of a coherent

package rather than disassembling them into a mere list of problems to be addressed piecemeal. For, these facets arise from a joint cause, i.e. the mismatch of the notions of time in GR and Quantum Theory.

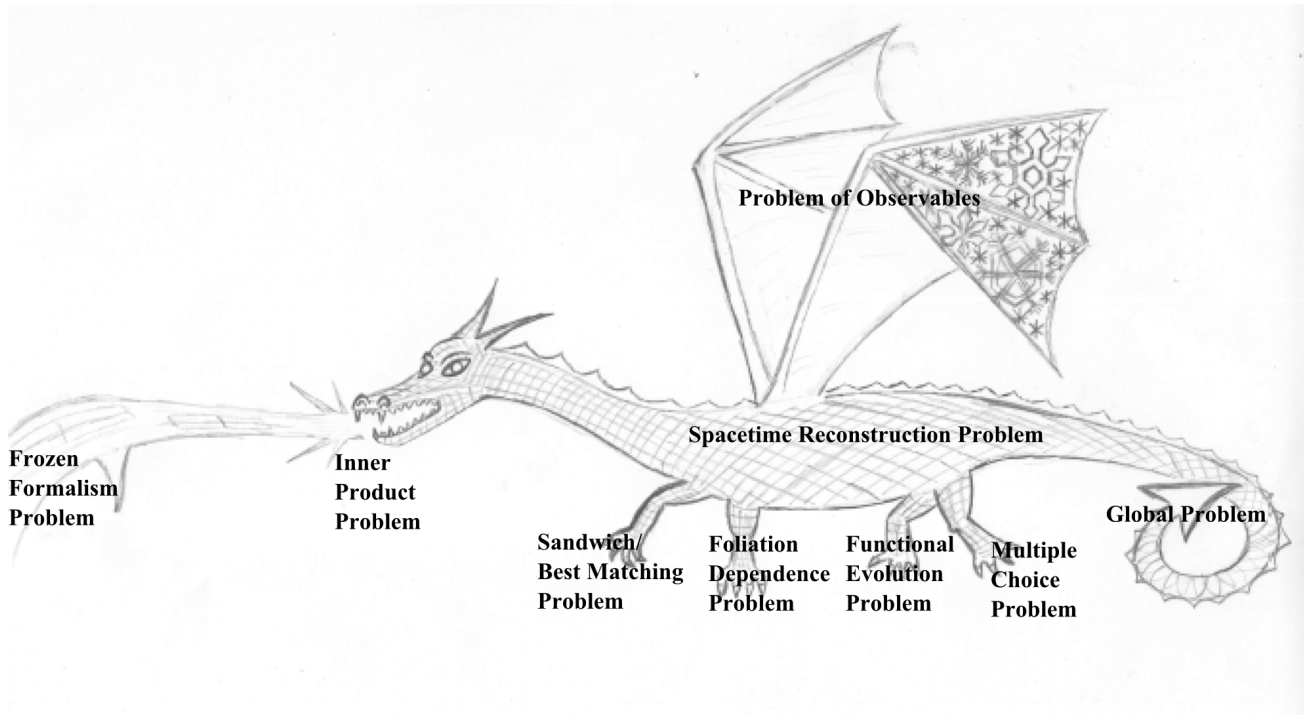


Figure 48: The Ice Dragon as a mythological mnemonic for the multi-facetedness of the Problem of Time: multiple problems underlied by one common cause, the conceptual incompatibility between what one calls ‘time’ in each of GR and QM. I hold this mnemonic to be *particularly* useful given the growing proportion of theoretical physicists who do not know even this much about the time-related foundational incompatibilities between GR and QM, despite these theories being the key cornerstones upon which the entirety of the rest of the field is founded upon.

As such, I put forward Fig 48’s mythological mnemonic for the multi-facetedness of the Problem of Time. One notices that the physical equations get frozen. One reaction is to set about trying to unfreeze them. However, another perspective is that an Ice Dragon may be on the loose, a beast [446] that not only possesses the ‘freezing breath’ of the Frozen Formalism (and ensuing ‘teeth’ of the Inner product Problem that one usually encounters next right behind the source of the frozenness) but which is the joint cause of other observed devastations, through coming simultaneously equipped with

- 1) ‘many-fingered claws’ on four ‘limbs’ that are the somewhat interlinked
 - i) Sandwich Problem, that I argue generalizes to a Best Matching Problem.
 - ii) Foliation Dependence Problem,
 - iii) Functional Evolution Problem,
 - iv) Multiple Choice Problem, and

[These are a useful grouping of four by their interconnections, and also have ties to ‘many-fingeredness’].

- 2) The ‘thunderbolt tail’ of the Global Problem of Time,
- 3) the ‘scaled armour’ of the Spacetime Reconstruction Problem, and
- 4) the ‘wings’ of the Problem of Observables.

Nor do the attributes of the Ice Dragon take an entirely fixed form. E.g. in path-integral rather than canonical approaches, one has to reckon with a Measure Problem instead of some of the difficulties of interpreting a frozen wave equation. The joint cause, or unity, of the Ice Dragon indeed stems from the common origin of these facets from the conceptual disparity between the GR and ordinary Quantum Theory notions of time. Now, setting about unfreezing the physics is likely more straightforward than defeating an Ice Dragon. Some might even claim to be defeating the Ice Dragon by just unfreezing the physics, by taking its rather feared ‘Problem of Time’ name and rebranding it to mean just its ‘Frozen Formalism Facet.’ However, this amounts to leaving oneself open to the ‘claws’, ‘tail’, ‘scales’ and ‘wings’ by choosing not to take any advantage of the presence of a joint cause – the mismatch of the notion of time in GR and Quantum Theory – as a warning that these, too, would also be expected to be present as obstacles needing overcoming before one’s Quantum Gravity program has genuine long-term viability. (Though it may take long for such deficiencies in Quantum Gravity programs to be demonstrated, through requiring detailed analysis of many difficult calculations). Finally, there are the problems of other beasts: there are other multi-faceted incompatibilities between GR and ordinary Quantum Theory each anchored on its own common cause (e.g. the differences in role played by observers, see Sec 16.17).

I next outline the nature of these other facets of the Problem of Time.

11.3.4 The further facets in detail

The **Sandwich Problem** that I argued is, more generally, the **Best Matching Problem** is already present at the classical level (see Sec 2.6.3) and becomes involved in some of the approaches that start with manipulations at the classical level prior to quantizing. It makes contact with time issues as follows. The generating function for the canonical transformation corresponding to the evolution of the gravitational field needs to be a function of the initial and final slices' metrics in the classical configuration representation. However, solving this is not a unique or known-to-be-mathematically-well-behaved procedure in the absence constraints for the auxiliary variables [110, 100].

Recollect Difference 14): the RPM Best Matching Problem is, for 1- and 2- d RPM's, not a problem but rather a resolved situation (Sec 3.14) that opens up extra paths and checks not available for geometrodynamics.

For all that one would like a quantization of GR to retain the nice classical properties of refoliation invariance and closure of the constraint algebra, at the quantum level there ceases to be an established way of guaranteeing these properties. Thus there is a **Foliation Dependence Problem**. This is obviously a time problem since such foliation is orthogonal to a GR timefunction: each slice corresponds to a moment of time for a cloud of observers distributed over the slice. Each foliation corresponds to the cloud of observers moving in a particular way.

Difference 26) RPM's do not have anything like the Foliation Dependence Problem, since the foliation of spacetime/embedding into spacetime meaning of GR's Dirac Algebra of constraints is lost through toy-modelling it with rotations (and/or dilations).

Functional Evolution Problem alias **possibility of anomalies**. This is the quantum-level problem that the commutator of constraints is capable of manifesting. [As such it is a second round of a posteriori rejection at the level of the commutator algebra.] Dirac [209] said that one had to be lucky so as to avoid this. Note that only some of the anomalies that one finds in physics are time-related. However, in the present case of the quantum counterpart of the Dirac algebra of GR constraints, this is a time issue due to what these constraints signify. In particular, non-closure here is a way in which the Foliation Dependence Problem can manifest itself, through the non-closure becoming entwined with details of the foliation. [Non-closure is also entwined with the Operator Ordering Problem, since changing the ordering gives additional right-hand-side pieces not present in the classical Poisson brackets algebra.]

Difference 27) RPM's are 'lucky in Dirac's sense', i.e. they have no Functional Evolution Problem/anomalies.

Foliation does retain some meaning, in that RPM's have a lapse/instant-like notion and a point identification map [593] corresponding to such as the auxiliary rotations in moving along the dynamical curve in the redundant setting. However, the GR lapse and shift (or instant and frame) are *more* than just naïve lapse and point identification map 'struts' of this nature in that they additionally pack together with the 3-metric configurations to form a unified spacetime 4-metric, which then turns out to have a number of additional features such as having the 3-metric embedded into it and being refoliation invariant.

The **Multiple Choice Problem** alias **Kuchař's Embarrassment of Riches** is the purely quantum-mechanical problem that different choices of time variable may give inequivalent quantum theories. [The riches are then the multiplicity of such inequivalent quantum theories.] It is a subcase of how making different choices of sets of variables to quantize may give inequivalent quantum theories, which follows from e.g. the Groenewold–van Hove theorem. Foliation Dependence is one of the ways in which the Multiple Choice Problem can manifest itself. Moreover, the Multiple Choice Problem is known to occur even in some finite toy models. It has been suggested that one way out of the Multiple Choice Problem is to specify the lapse α and shift β^μ a priori (e.g. in [271]). However, such amounts to foliation fixing, which is not what done in practise, nor desirable in principle, since one would like to be able to ascribe physical meaning to whichever foliation (and consider e.g. whether two distinct foliations between two given hypersurfaces match in their physical predictions at the quantum level). Also, foliation issues are not the only source of the Multiple Choice Problem.

Analogy 63) The Groenewold–Van Hove phenomenon indeed already occurs for finite theories and thus can be expected to indeed occur for RPM's just as it occurs in minisuperspace [335]. Both RPM's and geometrodynamics manifest the Multiple Choice Problem facet of the Problem of Time [400, 335]. [Despite the fairly widespread 'belief' otherwise this phenomenon is not QFT-specific but already occurs in finite systems.]

The **Global Problem of Time** alias **Kuchař's Embarrassment of Poverty** is already present at the classical level. The spatial part of this problem consists in the separation into true and embedding (space frame and timefunction) variables' having a capacity for being globally impossible, for reasons that closely parallel the Gribov effect [280] in Yang–Mills theory. This impossibility explains Kuchař's name for this problem. The temporal part of this problem concerns cases in which such a split can be defined but cannot be indefinitely continued as one progresses along a foliation (or simpler models' timefunctions eventually going astray). The tail is a 'topological obstruction' that does not have a fixed location but has to be 'somewhere', in the manner of a Dirac string (this renders lucid the choice of 'tail' image for the Global Problem). One can think of it as

a chart breakdown at the classical level (very natural in differential geometry); however the equivalent of ‘meshing charts’ at the quantum level in using multiple time coordinates is far from clear, see Sec 16.1.

Analogy 64) As regards globality in time, equations like GR’s CMC LFE (Sec 2.7) provides guarantees of existence and monotonicity. For scaled RPM equations along the lines of the Lagrange–Jacobi relation of Celestial Mechanics provide similar guarantees. Both cases work for sizeable classes of examples but not for all examples. Sec 12 is required prior to being able to make sharper statements as regards which equations, which GR-RPM analogies and which guaranteed cases.

Analogy 65) RPM’s have Globality in space issues do to possessing meaningful notions of localization/clumping.

The **Problem of Observables**. This involves construction of a sufficient set of observables for the physics of one’s model, which are then involved in the model’s notion of evolution. These ‘wings’ represent a particular source of danger (allowing for trouble to pop up in many other places in physical thought), though, as we shall see, whether the Ice Dragon has wings remains a debated point.

Analogy 66) RPM’s are likely to be a useful example as regards the Problem of Observables (this is tied to evolving constants of the motion/perennials/partial observables approaches [400, 335, 539] to the Problem of Time).

The **Spacetime (Reconstruction or Replacement) Problem** is as follows. Internal space or time coordinates to be used in the conventional classical spacetime context need to be scalar field functions on the spacetime 4-manifold. In particular, these do not have any foliation dependence. However, the canonical approach to GR uses functionals of the canonical variables, and which there is no a priori reason for such to be scalar fields of this type. Thus one is faced with either finding functionals with this property, or coming up with a new means of reducing to the standard spacetime meaning at the classical level. There are further issues involving properties of spacetime being problematical at the quantum level. Quantum Theory implies fluctuations are unavoidable, but now that this amounts to fluctuations of geometry, these are moreover too numerous to fit within a single spacetime (see e.g. [633]). Thus (something like) the superspace picture (considering the set of possible 3-geometries) might be expected to take over from the spacetime picture at the quantum level. It is then not clear what becomes of causality (or of locality, if one believes that the quantum replacement for spacetime is ‘foamy’ [633]). There is also an issue of recovering continuity in suitable limits in approaches that treat space or spacetime as discrete at the most fundamental level (see e.g. [640, 511, 588, 130, 289, 182, 8]). The choice of the ‘scales’ for this part of the Problem of Time ‘Ice Dragon’ is lucid insofar as 1) classical spacetime a particularly solid and intermeshed structure. 2) The last task faced is often to see the extent to which one’s approach successfully deals with this reconstruction, so it makes sense to view this as the innermost layer of defense for those who manage to dance past the frozen breath, teeth, wings, tail and claws; unfortunately none of that virtuosity counts for much if one cannot plant one’s lance through this final barrier... Issues in LQG of recovery of semiclassical limit, or of flat spacetime limit for the purpose of the recovery of standard Particle Physics results, could be viewed along such lines.

Difference 28) There is no spacetime reconstruction problem for RPM’s since they have no nontrivial spacetime notion.

One further issue that is usually considered to be not part of the Problem of Time mismatch but rather a further time problem is the **Arrow of Time**. This is a further issue since it concerns not ‘what happens to the notion of time upon jointly considering Quantum Theory and GR’ but rather ‘why is there a (theory-independent and in practise observed) consistent direction to time corresponding to a very tangible distinction between past and future?’ Quantum Gravity or Quantum Cosmology are then sometimes evoked in attempts to better explain this than one has managed to do with more everyday physical theories. As regards RPM’s being useful for investigating the Arrow of Time, Barbour conjectures about this in [83], but there is no concrete evidence for it that I have seen.

Difference 29) In Isham’s words, “*The prime source of the Problem of Time in Quantum Gravity is the invariance of classical GR under the group $\text{Diff}(\mathcal{S})$ of diffeomorphisms of the spacetime manifold \mathcal{S}* ” [335]. This, I stress, is a central and physically sensible property of GR, and it binds together a lot of the Problem of Time ‘Ice Dragon’. It would amount to ignoring most of what has been learnt from GR to simply change to a theory which does not have these complications (such as perturbative string theory on a fixed background). Both the LQG program [607] and the more recent M-theory development of string theory [122] do (aim to) take such complications into account. The extent to which RPM’s are valuable models is largely constrained by this difference. E.g. Isham and Kuchař [342, 343] and much of Isham’s review [332] involve issues not captured by RPM’s. $2 + 1$ GR and the bosonic string embody more of the character of the diffeomorphisms, with midisuperspace bearing an even closer parallel. From (42), it follows that the Frozen Formalism Facet also occurs for RPM’s. Moreover, the various RPM counterparts of the Wheeler–DeWitt equation (6) do not exhibit the well-definedness problems of the full GR case.

Thus, in summary, compared to the full Quantum GR Ice Dragon, RPM’s are a 1-legged Ice Dragon with freezing breath but no teeth, a tail but no scales and but small wings (see inside Figure 56); minisuperspace is a comparably-powered but

anatomically different creature ([46] will contain such a characterization of all toy models used to date for the Problem of Time).

11.4 Classifications and properties of Problem of Time strategies

Over the years, many conceptual strategies have been put forward to resolve the Problem of Time. However, none of those proposed to up to 1993 worked upon detailed examination ([400, 335], the present article), while not all work since has been investigated in such detail [46])... Certainly each of the 10 strategies covered by Isham and Kuchař (Fig 1) has problems if examined in detail [400, 335]. Taking these as precedent, I consider it likely that subsequent strategies will also have great difficulty in being genuinely satisfactory; this however remains undemonstrated in some cases. A resolution for the Problem of Time being acceptable would entail it to work in the general case for a fully viable theory of gravitation rather than just working for special cases or toy models. I in no way claim that this article is exhaustive as regards what problems various strategies have (e.g. [400, 335, 306, 403, 39, 46] cover plenty more such issues).

11.4.1 Classifications of strategies

Some of the differences in Problem of Time strategies result from what are (see e.g. [194]) two well-established and conflicting philosophical positions concerning time.

1) that time is fundamental, or

2) that time should be eliminated from one's conceptualization of the world.

These two perspectives can be related in various ways to schemes for the world, and the scientific enterprise therein, in which question-types involving '**becoming**' fundamentally make sense, or only those involving '**being**'.

"**Being at a time**" has an intermediate theoretical status, in that some schemes of type 2) go further than this in the elimination of time.

On the other hand, philosophers consider the following McTaggart series classification [452], consisting of the following layers of structure assumed.

A-series, for which time is dynamic - temporal passage is real: from past to present to future.

B series, which involves less: notions of earlier, simultaneous and later (i.e. ordering notions).

C-series (less well-known), which is an even more minimalistic timeless set-up.

I comment here that McTaggart's B-series is actually a slightly broader criterion than "being at a time", in that is in that it explicitly allows for constructs like "being at a later time". On the other hand, as far as I can tell, 2) coincides with McTaggart's C-series.

In Kuchař [400] and Isham [335]'s classification of Problem of Time strategies, a further element is whether the identification of what is the time is to be done before or after quantization (if time is to be identified at all). Strategies come in the following divisions.

I) **Tempus Ante Quantum** strategies (Sec 12) identifying time for their general case prior to quantization. Such strategies include hidden time in pure geometrodynamics, a matter time arising from the adjunction of matter to geometrodynamics, and unimodular gravity (in which a dynamical cosmological constant provides a time-standard). This attempts to demonstrate that the GR-QM incompatibility is removable by careful reformulation of classical GR.

II) **Tempus Post Quantum** strategies (Sec 13) follow 2) insofar as there is no time in general at the classical level, but follow A) insofar as one concedes that one is to restrict attention to cases for which a suitable notion of time does then emerge at the quantum level and hope that this covers much, or all, of the actually observable physics. Such strategies include superspace time and an associated Klein-Gordon-type inner product, Third Quantization and the Emergent Semiclassical Time Approach.

III) **Tempus Nihil Est** strategies (Sec 14) follow 2) insofar as they deny a role for time and then see how far one can do physics without. If there is no time, there is no clash between time notions in GR and QM. The onus, rather, is in reformulating these without time in a form general enough to account for the semblance of dynamics around us. I consider these to come in two distinct types.

Type 1: Naïve Schrödinger Interpretation (Sec 14.2), Conditional Probabilities Interpretation (Sec 11.8.5) and various forms of Records Theories (Sec 11.8.8 and 14).

Type 2 ‘Rovelli’: in terms of evolving constants of the motion/complete observables and partial observables (Sec 11.11).

IV) I note moreover that Histories Theory supplants the notion of time by the notion of history. This has somewhat of a different status, insofar as i) one passes from configuration space and the associated phase space being regarded as primary and being subjected to quantization to it being histories that are regarded as primary and subjected to quantization. ii) Histories Theory has spacetime primality connotations, whilst the strategies I term “timeless” have spatial primality connotations. Due to this distinction, I myself consider Histories Theory to lie outside of my own use of ‘Tempus Nihil Est’, as a fourth division, which I dub **Non Tempus Sed Historia**. The issue then is is there GR–QM incompatibility in this ‘histories’ replacement of the time notion?

Additionally, various combinations of these strategies have begun to appear in the literature (see below for examples).

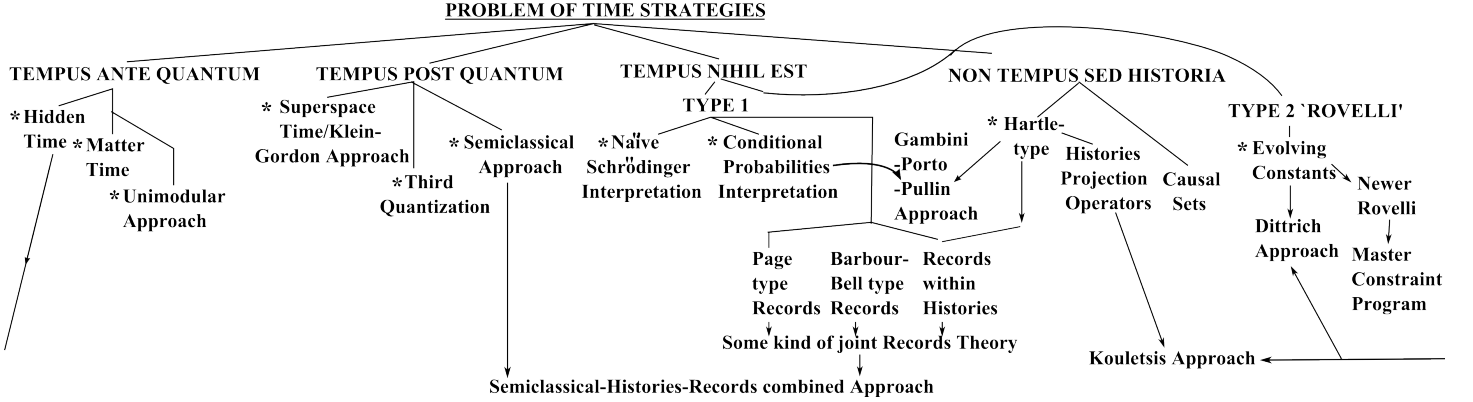


Figure 49: The various branches of strategy and their relations, in a diagram with ‘cylindrical topology’. * indicates the 10 strategies covered by Kuchař and Isham’s reviews [400, 335]. This article’s order of presentation is roughly from left to right.

11.4.2 More recent developments of the strategies

The main developments since Kuchař and Isham’s reviews from the early 90’s are as follows [39]. Firstly, Histories Theory has diversified (Sec 11.9). Secondly, each of Type 1 and Type 2 ‘Rovelli’ Tempus Nihil Est approaches has further expanded and diversified. For Type 1, various forms of timeless Records Theory have appeared (Sec 11.8), including Barbour’s [80], Page’s [499, 500] and mine ([28] and Sec 16), and also Records Theory within Histories Theory (work of Halliwell [292] building on work of Gell-Mann and Hartle [248], see Secs 11.10 and 15.4).

3) For Type 2, most of the advances (by Rovelli, Thiemann and Dittrich, see Sec 16 for a summary and 16, [545, 39] for discussion) are tied to further developments in Loop Quantum Gravity (though many of the other types of Problem of Time strategy are also considered in this arena, see [39] for a selection).

4) Additionally, one can consider further combinations of some of the types of strategy; see e.g. Secs 15.4, 16.13 and [39] for summaries of a number of others.

Thus, all in all, one could view the present situation as the 22 strategies in Figure 49. See Sec 16.13 for further discussion of ‘combined’ strategies.

11.4.3 A note on universal strategies

Some Problem of Time strategies are *universal* in the sense that they take formats that exist no matter what the underlying theory is, at least for a large number of steps. By which the RPM and geometrodynamical coverage of the present paper would not substantially change for these strategies if one were to pass e.g. to Brans-Dicke, higher curvature theory, supergravity, M-theory or the affine-geometrodynamical or Ashtekar/LQG canonical formulations of GR. By this, Part III of the present article is, in fact, of much wider interest in gravitational and quantum gravitational theory. This is true of emergent JBB and semiclassical times and of the timeless and histories approaches; another case of this is that approaches based on mastering the diffeomorphisms are widely blocked from progress.

11.4.4 The timelessness-indefiniteness cancellation hypothesis

Canonical GR has an indefinite kinetic term and time is missing. Perhaps then these two defects can cancel each other out. We know from the shape-definiteness lemma of Sec 2.7.1 that it is the overall scale which causes indefiniteness. So might overall scale, or some close relation of it, be tied to the isolation of, or emergence of, time?

11.4.5 Background-dependent/privileged slicing alternatives

These are approaches with times with sufficient significance imposed on them that they cannot be traded for other times (more like the ordinary Quantum Theory notion of time than the conventional lore of time in GR). The obvious example of a background-dependent theory is perturbative string theory. This manifests no Problem of Time. However, technical issues then drive one to seek nonperturbative background-independent strategies and then Problem of Time issues resurface. As regards privileged slicing theories, these are affected by how the classically significant notion of diffeomorphisms to have a meaningful counterpart.

Thus background structures and privileged slicings are to be viewed at least with quite some suspicion. Loop Quantum Gravity has a greater degree of background independence than perturbative string theory. Namely, it is independent of background *metric* structure. Ridding ourselves of further background structure, e.g. to the point of having no background spatial topology, is probably too ambitious for present-day physics. (See however Appendix 14.A.) However, it may be necessary in order to genuinely free oneself of unwanted background structures. (There are few reasons why the wish to rid oneself of background metric structure should not also extend to background topological structure, other than considerable mathematical inconvenience. M-theory is also expected to be background-independent.

11.5 Tempus ante quantum strategies

Ante Postulate. There is a fundamental time to be found at the classical level for the full (i.e. untruncated) classical gravitational theory (possibly coupled to suitable matter fields).

Note that, contrary to the preceding SSsec, such a time is to be derived from within a theory that a priori has no such structure, rather than imposed externally on a theory that has none or involving supplanting a theory having none by a theory that does have a such.

11.5.1 The simple but blocked-off Jacobi–Barbour–Bertotti implementation

Barbour’s relational approach to physics gives rise to an emergent Jacobi–Barbour–Bertotti (JBB) time

$$\mathfrak{t}^{\text{em(JBB)}} = \underset{\mathfrak{g} \in \mathfrak{G}}{\text{extremum}} \int ||d\mathfrak{g}||/\sqrt{W} . \quad (692)$$

This is universal (for theories with Riemannian or Randers configuration space geometries at least, covering the usual accepted theories of physics). As pointed out in Sec 2, in the mechanics context, this arises as a simplifier of the relational approach’s classical equations and amounts to a recovery of Newtonian time from relational first principles; while, in the context of GR, it amounts to recovery of proper time [and of cosmic time, in the case of (approximately) homogeneous cosmologies].

Problem JBB-1. Unfortunately while this approach yields a time function, it does not in any way unfreeze the GR Hamiltonian constraint or its energy constraint RPM analogue.

I discuss further features of this candidate timefunction in Sec 12.1. Moreover, it makes contact with another strategy (the Semiclassical Approach) in which this same timestandard (in its approximate form) emerges in an unfreezing context as the emergent semiclassical (alias WKB) time. This is also a powerhouse as regards the development of further timeless approaches: such a time that is actually made purely out of configurational information turns out to be of interest as regards which timeless questions to ask of one’s system.

11.5.2 The also-simple but likewise blocked-off superspace time implementation

Take the indefinite direction in superspace to be picking out a timefunction, in parallel to how the indefinite direction in Minkowski space does for e.g. Klein–Gordon theory. This is one of the ways of implementing the timelessness-indefiniteness cancellation hypothesis. It amounts to one of the momenta on which \mathcal{H} depends quadratically to be taken to be conjugate to a timefunction, leading to a Klein–Gordon type wave equation depending on the double derivative with respect to this timefunction. This approach is usually billed as Tempus Post Quantum, through choosing to make this identification of a time after the quantization, but the approach is essentially unchanged if this identification is made priorly at the classical level. Regardless of when this identification is made, however, this approach fails, for reasons I defer to the discussion in its more habitual Tempus Post Quantum setting (Sec 13). Schematically,

$$\Delta_{\mathcal{M}} \text{ is actually a } \square_{\mathcal{M}} . \quad (693)$$

Difference 30) There is no RPM counterpart of this as a consequence of kinetic term positive-definiteness.

11.5.3 Implementation by obtaining a part-linear form

While the above straightforward schemes fail, it may still be that a classical time exists but happens to be harder to find. We now consider starting one's scheme off by finding a way of solving Quad to obtain a part-linear form,

$$P_{\mathbf{t}^{\text{ante}}} + H^{\text{true}}[\mathbf{t}^{\text{ante}}, \mathbf{Q}_{\text{other}}^{\text{A}}, P_{\text{A}}^{\text{other}}] = 0, \quad (694)$$

where $P_{\mathbf{t}^{\text{ante}}}$ is the momentum conjugate to a candidate classical time variable, \mathbf{t}^{ante} which is to play a role parallel to that of external classical time. By passing as soon as possible to having an object that plays such a role, Tempus Ante Quantum can be viewed as the most conservative family of strategies [335, 613]. H^{true} is then the 'true Hamiltonian' for the system. Given such a parabolic form for \mathcal{H} , it becomes possible to apply a conceptually-standard quantization that yields the time-dependent Schrödinger equation

$$i\hbar D\Psi/D\mathbf{t}^{\text{ante}} = \hat{H}_{\text{true}}[\mathbf{t}^{\text{ante}}, \mathbf{Q}_{\text{other}}^{\text{A}}, P_{\text{A}}^{\text{other}}]\Psi, \quad (695)$$

which unfreezing is accompanied, at least formally, by the obvious associated Schrödinger inner product, rendering the Ice Dragon toothless (a frontal assault!) The above scheme either amounts to the accompanying momentum constraint having been reduced away or being absent through consideration of a simpler toy model, or requires enlarging the scheme to involve solving the 4-vector $\mathcal{H}_{\mu} := [\mathcal{H}, \mathcal{L}_{\mu}]$ of constraints to obtain a part-linear form

$$P_{\mathcal{X}^{\Gamma}} + H_{\Gamma}^{\text{true}}[\mathcal{X}^{\Gamma}, \mathbf{Q}_{\text{other}}^{\text{A}}, P_{\Gamma}^{\text{other}}] = 0. \quad (696)$$

Here, $P_{\mathcal{X}^{\Gamma}}$ are the momenta conjugate to 4 candidate embedding variables \mathcal{X}^{Γ} , which form the 4-vector $[\mathbf{t}^{\text{true}}, X^{\mu}]$, where X^{μ} are 3 spatial frame variables. Also, $H_{\Gamma}^{\text{true}} = [H^{\text{true}}, \Pi_{\mu}^{\text{true}}]$, where Π_{μ}^{true} is the true momentum flux. Then one passes to a 4-vector enlargement of the time-dependent Schrödinger equation,

$$i\hbar \nabla \Psi / \nabla \mathcal{X}^{\Gamma} = \hat{H}_{\Gamma}^{\text{true}}[\mathcal{X}^{\Gamma}, \mathbf{Q}_{\text{other}}^{\text{A}}, P_{\Gamma}^{\text{other}}]\Psi. \quad (697)$$

This implementation, while less straightforward than the preceding SSecs', contains some unblocked cases. Part-linear forms occur for parametrized nonrelativistic particle and parametrized field theory toy models respectively [400].

11.5.4 The hidden time Internal Schrödinger Approach

A first such suggestion is that there may be a *hidden* alias *internal* time [652, 397, 400, 335] within one's gravitational theory itself. I.e. that the apparent frozenness is a formalism-dependent statement, to be removed via applying some canonical transformation. In the first part-linear form scheme, this sends GR's spatial 3-geometry configurations to 1 hidden time, so $t^{\text{ante}} = t^{\text{hidden}}$ plus 2 'true gravitational degrees of freedom' [which are the form the 'other variables' take, and are here 'physical' alias 'non-gauge']. The general canonical transformation is here, for the single-time-variable case,

$$(\mathbf{Q}^{\text{A}}, P_{\text{A}}) \longrightarrow (\mathbf{t}^{\text{hidden}}, p_{\mathbf{t}^{\text{hidden}}}, \hat{\mathbf{Q}}_{\text{true}}^{\text{A}}, P_{\text{A}}^{\text{true}}) \quad (698)$$

and in the GR embedding variables case,

$$(\mathcal{h}_{\mu\nu}(x^{\gamma}), \pi^{\mu\nu}(x^{\gamma})) \longrightarrow (Q^{\Gamma}(x^{\gamma}), P_{\Gamma}(x^{\gamma})); (q_{\text{true}}^{\text{A}}(x^{\gamma}), P_{\text{A}}^{\text{true}}(x^{\gamma})), \quad (699)$$

for $q_{\text{true}}^{\text{A}}(x^{\gamma})$ here the true gravitational degrees of freedom. Thus one gets a hidden time-dependent Schrödinger equation of the form (695) or (697) with the above significances attached to the variables. Within geometrodynamics, some candidates that have been considered are as follows.

1) *Intrinsic time candidates*, though these have largely lacked in successful development [400]). One family of examples involve using some scale variable from Sec 3.8.1 as a time (e.g. Misner time in Quantum Cosmology). These are the most obvious attempts at implementing the timelessness-indefiniteness cancellation hypothesis. Unfortunately, such scales often make for poor time candidates due to not being monotonic in recollapsing universes.

2) *Extrinsic time candidates*. One among these is probably our best candidate for an internal time in GR: *York time* [652, 397, 400, 335], so I present this example in more detail. Perform a size-shape split of GR and then exchange the scale variable \sqrt{h} and its conjugate relative distance momentum object via a canonical transformation so that the latter is the timelike standard,

$$t^{\text{York}} := 2h_{\mu\nu}\pi^{\mu\nu}/3\sqrt{h} \quad (\text{proportional to the constant mean curvature}). \quad (700)$$

Dilational times such as York time are then also viewable as modified attempts at implementing the timelessness-indefiniteness cancellation hypothesis. Then the Hamiltonian constraint is replaced by $p_{\text{true}} = H^{\text{true}} = \sqrt{h} = \phi^6$. Here, ϕ is the solution of the conformally-transformed Hamiltonian constraint, i.e. the Lichnerowicz–York equation (161). (For brevity, I present the vacuum case of this, whilst schematically ignoring the momentum constraint; more generally, one would formally solve the whole GR initial-value problem.) Then quantization gives, formally

$$i\hbar \delta \Psi / \delta t^{\text{York}} = \widehat{H^{\text{true}}} \Psi. \quad (701)$$

The obstruction to this particular resolution is that how to solve the complicated quasilinear elliptic equation (161) is not in practice generally known. Thus the functional dependence of H^{true} on the other variables is not known (or at best known implicitly via the solution of such a partial differential equation). And thus the quantum ‘true Hamiltonian’ $\widehat{H^{\text{true}}}$ cannot be explicitly defined as an operator (or, at best, is very prone to having the horrendous operator-ordering and well-definedness issues that only knowing its form implicitly entails.) It is also the case that simpler models than full geometrodynamics give severe difficulties just beyond where geometrodynamics has this impasse.

Another extrinsic time is *Einstein–Rosen time* in the context of cylindrical wave midisuperspace models [392].

Some problems with this SSec’s hidden times are as follows.

Hidden Problem-1. The canonical transformation in question is hard to perform in practise.

Hidden Problem-2. Having a list of candidates rather than a single candidate, the Multiple Choice Problem may apply.

Hidden Problem-3. The Global Problem of Time is present, as follows. Torre showed that the right-hand-side space of (699) is a manifold while the left-hand-side space is not (due to the occasional existence of Killing vectors). This limits the sense in which such a map could hold (it certainly does not hold globally). Hájíček and Kijowski [290] have furthermore established that such a map is not unique. As regards globality in time, for GR, existence of constant mean curvature slices and propagation and monotonicity for these does hold for some (but not all) examples.

Also, two Spacetime Reconstruction issues with this approach are as follows [372].

Hidden Problem-4. The bubble time version [along the lines of (10)] describes a hypersurface in spacetime only after the classical equations have been solved. Thus its significance at the quantum level is not understood (a foliation issue).

Hidden Problem-5. t^{true} is required to be a spacetime scalar and thus obey $\{t^{\text{true}}, \mathcal{H}\} = 0$. That would look to overrule internal times based to purely geometrodynamical phase space variables (this has Problem of Observables and Spacetime Reconstruction Problem connotations).

Thus we see that a ‘frontal assault’ not only has to brave the ‘frozen breath’ but also the Ice Dragon ferociously strikes back via most of its other body parts. This is unfortunately characteristic no matter what type of strategy one assaults the Ice Dragon with.

As regards RPM toy models of the hidden time approach, I note for now that time candidate status can be ascribed to these’s scales and dilational objects conjugate to scales (such as \mathcal{D} , which in this context I term the *Euler dilational time candidate*). I leave the details of this analogy to Sec 12 since they are both quite lengthy and new to the present article

11.5.5 Implementation by unhidden time

There is a higher-derivative theory in which a natural variables set contains an already-explicit internal time [137, 323].

11.5.6 Implementation by matter time

Here, typically in minisuperspace for 1-component scalar matter, one simply isolates the corresponding momentum to play the role of the time part of the subsequent wave equation. This is ‘the alternative’ (often stated as such in the LQG and minisuperspace literatures, and also in the recent paper [419] to using the scale variable, although the momenta conjugate to each of these represent 2 further possibilities, and then, if canonical transformations are allowed, a whole further host of possibilities become apparent. On the whole, matter time has the setback of not being well-motivated choice from first principles, proneness to the Multiple Choice Problem, and from not extending well to multi-component matter in inhomogeneous GR (unless perhaps one chooses to privilege a particular matter species).

11.5.7 Implementation by reference fluids

There are also non-geometrodynamical internal time candidates: *Reference Matter Time Approaches*. Here, the part-linear form (696) is attained with $t^{\text{ante}} = t^{\text{matter}}$ and $q_{\text{other}}^{\Gamma} = h_{ij}$, or (better) the 4-component version, i.e. extending the set of variables from the geometrodynamical ones to include also matter variables coupled to these, which then serve so as to label spacetime events. Then one passes to the corresponding form (697). Examples are Gaussian reference fluid [406, 400] and the reference fluid corresponding to the harmonic gauge [408].

A further type of matter time approach involves additionally forming a quadratic combination of constraints for dust [145], more general perfect fluids [146], and massless scalar fields [404]. The point of such (see also [442]) quadratic combinations is that they result in strongly vanishing Poisson brackets. As this article’s RPM work for now makes but scant contact with matter clocks/reference matter, I refer the reader to [400, 335, 39, 46] for this approach’s problems, except for the comment that some candidates involve unphysical matter (physical energy condition violating) and/or intangible matter; clearly then relationalism furnishes an additional objection to the latter. Tangible and otherwise physical reference fluid matter candidates are not exhaustively ruled out, however.

Sec 12.4 represents the first steps in conceiving whether there is an analogue of these for RPM’s.

11.5.8 Implementation by unimodular gravity

One can also regard an undetermined cosmological constant Λ as a type of reference fluid [616]. This is the so-called *unimodular gravity* program, which Isham and Kuchař consider as a separate entry among their 10 strategies. [587] is a more recent review of this, while [286] discusses how an approach of this type arises from Barbour-type relational considerations. Here, one does not consider the lapse to be a variable that is to be varied with respect to. Instead, \mathcal{M}_μ has $\mathcal{H}_{,\mu}$ as an integrability [c.f. (168)] , which, upon integrating, gives $\mathcal{H} + 2\Lambda = 0$ with \mathcal{H} the vacuum expression and Λ now interpreted as a constant of integration.

The unimodular approach has problems of its own as a Problem of Time resolution (see [399, 400, 335] for more).

Unimodular Problem-1. The cosmological constant itself plays the role of isolated linear momentum [c.f. eq. (694)] that, in the quantum version, gets promoted to the derivative with respect to the unimodular internal time function $\mathfrak{t}^{\text{uni}}$ by the presence of which the quantum theory is unfrozen. However, there is a mismatch between this single time variable and the standard generally-relativistic concept of time, which is ‘many-fingered’ with one finger per possible foliation. [In other words, the derivative with respect to $\mathfrak{t}^{\text{uni}}$ is a partial derivative, as opposed to the functional derivative that one expects to be present in a Problem of Time resolution of the form (695).] The geometrical origin of this mismatch is that a cosmological time measures the 4-volume enclosed between two embeddings of the associated internal time functional $\mathfrak{t}^{\text{int}}$. However, given one of the embeddings the second is not uniquely determined by the value of t^{uni} (since pairs of embeddings that differ by a zero 4-volume are obviously possible due to the Lorentzian signature and cannot be distinguished in this way).

Unimodular Problem-2 The 3-metric operator does not commute with the approach’s constraints, by which this scheme’s interpretation of its wavefunction of the universe as a probability distribution for the 3-metric is not tenable. I further contribute to this in Sec 12.5 from a relational perspective.

There is a further issue of non-correspondence of the unimodular approach with observational cosmology. Bertolami resolved this issue in [125] by generalizing to a nondynamical scalar field model, and after the above critiques. However, he admits his resolution is only consistent with a rather special class of metrics (whilst the Problem of Time’s intended scope is for generic metrics). He does not attempt to further address the above-mentioned mismatch either, by which Kuchař and Isham’s Problem of Time non-resolution critiques continue to apply to Bertolami’s otherwise-improved scheme.

Some overall advantages of this SSec’s strategies are that these get round Hidden Problem-1 by no longer requiring some complex canonical transformation of the geometrodynamical variables themselves; Hidden Problem-6 and Torre’s part of the Global Hidden Problem-3 are also absent.

11.6 Tempus post quantum: time not fundamental but emergent under some circumstances

Post Postulate In strategies in which time is not always present at the fundamental level, time is nevertheless capable of emerging in the quantum regime. Because this is an emergence, it means that the Hilbert space structure of the final quantum theory is capable of being (largely) unrelated to that of the Wheeler–DeWitt equation-type quantum theory that one starts with. Such emergent strategies are of the following types.

11.6.1 Attempting a Klein-Gordon Interpretation based on superspace time

The GR supermetric on superspace is indefinite like spacetime is, so analogous time-like notions exist for it. This suggests thinking about the Wheeler–DeWitt equation not as a time-independent Schrödinger equation but as an analogue of the Klein–Gordon wave equation. In this way, this approach has QFT, and thus spacetime, undertones. Some issues with this strategy are as follows.

Superspace Problem-1. Superspace null cones are not respected by superspace trajectories, limiting the analogy.

Superspace Problem-2. There is an Inner Product Problem. Whether treated as a Tempus Post or a Tempus Ante scheme, the indefiniteness of the configuration space metric precludes such a scheme from having a Schrödinger inner product (in parallel with Klein–Gordon theory). The main issue then is whether it permits a Klein–Gordon inner product instead,⁶⁹

$$\langle \Psi_1[h] | \Psi_2[h] \rangle = \frac{1}{2i} \prod_{x \in \Sigma} \int_{\text{Riem}(\Sigma)} \mathbb{D}\Sigma_A \mathcal{M}^{AB}(h) \left\{ \Psi_1[h] \overset{\leftarrow}{\delta}_B \Psi_2[h] \right\}, \quad (702)$$

where $\mathbb{D}\Sigma_A$ is the ‘vector area’ element of Riem.

However, there is a breakdown of the analogy between geometrodynamics and stationary-spacetime Klein–Gordon theory. While there is a conformal Killing vector on superspace [397], the GR potential does not in general respect this, and so this scheme fails. Strong gravity is a toy model that gets past this point, due to its much simpler potential happening to scale consistently. It is also an issue that the positive–negative modes split of states in the usual Klein–Gordon theory arises from the presence of a privileged time; thus, without such a privileged time in the general GR case, one’s quantization scheme will not have this familiar and useful feature.

⁶⁹The capital Latin indices are obtained via the DeWitt 2-index to 1-index map. $\overset{\leftarrow}{\delta}_B$ denotes functional derivative with both backwards as well as forwards action with respect to $h^B = h_{\mu\nu}$. \mathcal{M}^{AB} is the GR configuration space metric in this notation.

Difference 31) Due to Difference 16) in definiteness of the kinetic terms, the Schrödinger scheme is available for RPM's but not for GR. Also, the Klein–Gordon-type scheme (which fails for other reasons in GR [397]) is not available in RPM's.

Rieffel induction ameliorates the Klein–Gordon type inner product problem by renormalizing out delta function contributions (see e.g. [312] for an account).

Finally, each of definiteness and indefiniteness furnishes qualitatively different mathematics: elliptic operators for RPM's versus hyperbolic ones furnishing superspace time for GR and minisuperspace.

11.6.2 Third quantization

To get round the preceding SSec's problems, it has been suggested that the solutions $\Psi[h]$ of the Wheeler–DeWitt equation be turned into operators, so that one now has an equation

$$\hat{\mathcal{H}}\hat{\Psi}\psi = 0 . \quad (703)$$

This is analogous to the second quantization of a relativistic particle whose states are described by the Klein–Gordon equation (and thus this approach also has QFT undertones, and thus also spacetime undertones). [*Second and third* quantizations are between (Hilb, Uni) type spaces. Only *first* quantization and overall quantization are formally between (Phase, Can) and (Hilb, Uni).] While Third Quantization is of interest as regards various technical issues, it was not held to provide a satisfactory approach to the Problem of Time up to the early 90's [400, 335], and I am not aware of any subsequent advances in this respect.

Difference 32) Also, the third quantization scheme makes no sense in RPM's, due to these being finite rather than field-theoretic. Second quantization is the RPM analogue (the key is that wavefunctions of the universe are themselves quantized, not how many quantizations are needed to get to that stage. This is technically like QFT, but interpretationally the wavefunction of the finite model universe has been elevated to a quantum operator (which parallels the status of the wavefunction of the universe for infinite theories in third quantization). Moreover, even second quantization is unnecessary and absurd given that the Schrodinger inner product works just fine here.

11.6.3 The Semiclassical Approach

All strategies discussed from now on in this Sec are universal.

In the Semiclassical Approach (set up by DeWitt [201], Lapchinski–Rubakov [418], Banks [74] and Halliwell–Hawking [296] and subsequently reviewed/expanded on in e.g. [144, 378, 400, 335, 104, 504, 373, 374, 375, 403, 372, 22, 40]), time is only meaningful in some semiclassical limit of the Quantum Gravity theory based on the Wheeler–DeWitt equation. As well as being an emergent time strategy, the Semiclassical Approach is furthermore important toward acquiring more solid foundations for other aspects of Quantum Cosmology (see e.g. [298, 294, 375]). In particular, the work of Halliwell–Hawking [296] goes toward that end.

In this approach, ‘heavy slow’ (h) degrees of freedom provide an approximate emergent time standard $\mathfrak{t}^{\text{em(WKB)}}$ with respect to which ‘light fast’ (l) local degrees of freedom run. Thus local physics has an emergent quantum dynamics based on what is, at least approximately, an emergent time-dependent Schrödinger equation. E.g., in GR Quantum Cosmology, the h degrees of freedom are the size of the universe and homogeneous matter modes, and the l degrees of freedom are ‘shape’ inhomogeneities in the universe's gravity and matter distribution. [In the literature, having gravitational field be h and the matter be l is quite commonplace, and there is some motivation for considering such a regime.] In the isotropic case, that is the same as scale.] For (e.g. using $h_{\mu\nu}$ as ‘ h ’ and the matter as ‘ l ’, and requiring that $|\chi\rangle$ depends nontrivially on h so that the Quantum Theory is rendered nonseparable in h, l variables). E.g. one can toy-model the above by considering the regime in which scaled RPM's have a scale variable, such as the configuration space radius playing the role of h and the pure-shape degrees of freedom playing the role of l .

Analogy 67) Expanding on Analogy 59), RPM's generalize previously-studied absolute particle models of the Semiclassical Approach [74, 141, 495, 496] by inclusion of auxiliary terms and subsequently of linear constraints. (Reduced RPM is a subcase of Datta's generalized curved configuration space mechanics work [190].) The linear constraints give RPM's more of a midisuperspace flavour, particularly in Dirac-type approach. In particular, scaled RPM is analogous to Halliwell–Hawking's scheme [296] for inhomogeneous perturbations about homogeneous semiclassical quantum GR. Moreover, it now has a rather simpler shape dynamics coupled to it. This extra simpleness is useful in investigating various features outlined below and in the Conclusion. To make this analogy, use in place of h, l , Q_{ad} , L_{inz} below I or ρ , shape S^u , \mathcal{E} , \mathcal{L}_μ for RPM's, and scale, inhomogeneity, \mathcal{H} , \mathcal{M}_μ for GR.

By ‘heavy slow’ and ‘light fast’, what one means specifically is that one uses

1) the adiabatic Born–Oppenheimer-type ansatz

$$\Psi = \psi(h^{A'})|\chi(h^{A'}, l^{A''})\rangle \quad (704)$$

and approximations of types often associated with this. [I use primed and double-primed indices for h and l species.]

2) A WKB ansatz

$$\psi(h^{A'}) = \exp(iW(h^{A'})/\hbar) . \quad (705)$$

Next, one forms the h-equation from the quadratic constraint

$$\langle \chi | \widehat{\mathcal{Q}uad} \{ |\chi \rangle \psi \} = 0 , \quad (706)$$

which, identifying $W(h^{A'})$ as Hamilton’s principal function, and under a number of simplifications, yields

$$N^{A/B'} \frac{\nabla W}{\nabla h^{A'}} \frac{\nabla W}{\nabla h^{B'}} = 2\{E - V(h^{C'})\} \quad (\text{Hamilton–Jacobi equation}) . \quad (707)$$

Here $W(h^{A'})$ is the h-part of U and $N^{A/B'}$ is the inverse of the configuration space metric for the heavy degrees of freedom, $M_{A/B'}$. One way of solving this [201, 418, 74, 296] is for an approximate emergent WKB semiclassical time $t^{\text{em(WKB)}}(h^{A'})$, via the Hamilton–Jacobi theory relation plus momentum–velocity relation combination,

$$\frac{\nabla W}{\nabla h^{A'}} = p_{A'}^h = M_{A/B'} \frac{Dh^{B'}}{Dt^{\text{em(WKB)}}} \quad (708)$$

Furthermore, rearranging the subsequent equation makes it clear that $t^{\text{em(WKB)}}$ is aligned with [22] the form of $t^{\text{em(JBB)}}$ under the same regime of approximations. (Thus it is also aligned with Newtonian time, proper time and cosmic time in the various suitable contexts). On these grounds, from now on I use the simplified notation t^{em} to denote the gestalt JBB-WKB *approximate emergent time*.

One also obtains a time-dependent Schrödinger equation for the local l-degrees of freedom with respect to a time standard (approximately) provided by the background h degrees of freedom

$$i\hbar D|\chi\rangle/Dt^{\text{em(WKB)}} = \hat{H}_l|\chi\rangle . \quad (709)$$

This is a portmanteau of a time-dependent Schrodinger equation for finite theories and a Tomonaga–Schwinger [372] type equation for infinite theories.

H_l is the remaining part of $\mathcal{Q}uad$ here, serving as a Hamiltonian for the l-subsystem; it is t^{em} -dependent. (709) originates from rearranging a kind of fluctuation equation,

$$\{1 - |\chi\rangle\langle\chi|\} \widehat{\mathcal{Q}uad} \{ |\chi\rangle \psi \} = 0 , \quad (710)$$

the emergent time dependent left-hand side term of (64) arising from

$$N^{A/B'} \frac{\nabla^2 \Psi}{\nabla h^{A'} \delta h^{B'}} \quad \text{containing the ‘crucial chroniferous cross-term’} \quad N^{A/B'} i \frac{\nabla W}{\nabla h^{A'}} \frac{\nabla |\chi\rangle}{\nabla h^{B'}} , \quad (711)$$

which, via (708), $N^{A/B'} M_{A/C'} = \delta^{B'/C'}$ and the chain-rule gives

$$i\hbar N^{A/B'} M_{A/C'} \frac{Dh^{C'}}{Dt^{\text{em}}} \frac{\nabla |\chi\rangle}{\nabla h^{B'}} = i\hbar \frac{Dh^{A'}}{Dt^{\text{em}}} \frac{\nabla |\chi\rangle}{\nabla h^{A'}} = i\hbar \frac{D|\chi\rangle}{Dt^{\text{em}}} . \quad (712)$$

$[\hat{H}_l$ is the surviving piece of $\mathcal{Q}uad$, and serves as Hamiltonian for the l-subsystem].

Note 1) I consider the Semiclassical Approach to be the most promising branch among those that amount to using the timelessness-indefiniteness cancellation hypothesis. One now does not take scale as a time, but rather considers scale to be part of some set of heavy, slow, global variables which go into providing an approximate timestandard rather than into the fast, light, local variables that deal with actually-observed subsystem physics. This then possesses a far more familiar, and probably conceptually more satisfactory positive-definite kinetic term. While it will certainly not always happen that the scale physics will be heavy and slow, this is fortunately the case in early-universe cosmology, so that one can look into the Quantum Cosmology arena from this Semiclassical Approach perspective.

Note 2) The above has omitted consideration of linear constraints \mathcal{L}_{inz} . These contribute their own h and l equations. These do not enter the specification of the timestandard (though this can acquire the presence of the corresponding auxiliary cyclic velocity \dot{c}^Z , in which case one needs to append a variational prescription that frees one of this, as per Sec 2.11). Via the properly auxiliary-corrected version of 708 \mathcal{L}_{inz} also enters (709), giving

$$i\hbar \{D/Dt^{\text{em}} - \mathcal{L}_{\text{inz}} Dc^Z/Dt^{\text{em}}\} |\chi\rangle = \hat{H}_l |\chi\rangle , \quad (713)$$

which parallels the GR version of the Tomonaga–Schwinger equation (emergent-time-dependent Tomonaga–Schwinger–Einstein–Schrödinger equation) with respect to bubble time [393],

$$i\hbar\{\delta/\delta\mathbf{t}^{\text{em}} - \mathcal{M}_\mu\delta F^\mu/\delta\mathbf{t}^{\text{em}}\}|\chi\rangle = \hat{H}_1^{\text{GR}}|\chi\rangle. \quad (714)$$

Problems with the Semiclassical Approach include the following.

- Semiclassical Problem-1. Having invoked a Wheeler–DeWitt equation results in inheriting some of its problems [400, 335].
- Semiclassical Problem-2. Making the WKB approximation requires justification [659, 144, 98, 400, 335, 77, 374, 22], without which there is a serious danger of ‘merely passing the buck’ from a time to a set of wavefronts very heavily implying a time. See Sec 12 for the various parts of this argument; for now I note that without this ansatz, the above sketch of obtaining of an emergent-time-dependent Schrödinger equation for the l-subsystem in general fails.
- Semiclassical Problem-3 This is but one of many approximations made, so it is hard to check. Other approximations include dropping non-adiabatic, averaged and back-reaction terms. I investigate these further in Sec 13.
- Semiclassical Problem-4. It is furthermore unclear how to relate the probability interpretation of the approximation with that for the underlying Wheeler–DeWitt equation itself [335, 400]. And what of less approximate schemes, which are tied to better modelling of the back-reaction of the light system on the heavy system? [400, 22, 40, 42]. (Some of these involve other than Schrödinger equations.)
- Semiclassical Problem-5. The status of the Spacetime Reconstruction Problem is unclear for the Semiclassical Approach [335].
- Semiclassical Problem-6. The Multiple Choice Problem remains present in such schemes in detail [400].
- Semiclassical Problem-7. N.B. the final probabilistic interpretation of the theory is made only *after* the identification of time. Thus the Hilbert space structure of the final theory may be related only very indirectly (if at all) to that of the quantum theory with which the construction starts. [One still has a Hilbert space, but one does not know a priori which such one will end up working with...] All in all, a principal problem in such a scheme is what to do to make sense of the Wheeler–DeWitt equation, e.g. the Hilbert space problem. The GR input is unchanged and unreformulated. A method is used to get unfrozen QM out in a certain regime. More detailed consideration of the time changes the formulation’s interpretation (order by order, there are wave equation and inner product issues).

Overall in the Semiclassical Approach, there is still an Ice Dragon but it only menaces outlandish parts. I.e. there is some chance that the realm of Quantum Cosmology is free from it, at least as regards the *practical* task of explaining the origin of the CMB inhomogeneities and the seeding of the galaxies.

11.7 What is a question? Temporal and atemporal connotations.

11.7.1 Questions concern propositions about a (sub)system’s properties

By concerning propositions, questions are closely related to *propositional logic*. That I am aware of, the idea that physics is questions which are equivalent to logic was first put forward by Mackey [437] (see also [344, 347] for suggestions of this for specific Problem of Time settings). [The relationalist will be interested here in the propositions concerning tangible physics.] I view this as an interesting observer-centred idea that is worth considering whether to elevate to a general physical principle. I term this ‘**Mackey’s Principle**’, intended as an honorative rather than a literal portrayal of the precise form of Mackey’s viewpoint; Isham and Linden [344] were already using this as a physical principle for Histories Theory; my new consideration is to speculate whether this is a *general* principle.)

From the **q**-centric perspective of Relationalism 3), this represents a distinct extension to this from that in Appendix 4.C, to Prop(**q**), the set of all possible physical propositions about the configuration space **q**. I.e. PhysProp, concerning only at least part-tangible entities. Among other things, Part III of this article considers what happens to Problem of Time strategies if these are required to comply with this principle. Some I demonstrate to be extendible to fit this principle, others are not known to be or only fit in an otherwise obviously limited way. I am not claiming my treatment of this is exhaustive... (E.g. I will take ‘property’ to be an unproblematic word, for all that is not, see e.g. p. 67 of [337] for more.) My point is that the general theory of questions involves, a priori, both nontrivial temporal content and nontrivial atemporal content. Then in strategies in which one can in fact purge questions of temporal content, one’s remaining theory should parallel the nontrivial atemporal content of questions, so in that sense the structure of the atemporal content of questions is also important for this article.

In a nutshell, **propositioning** is another way of expanding on Relationalism 3’s primality of **q**.

11.7.2 The logical structure of atemporal questions

This SSsec has value through furnishing various versions of Prop(**q**), Prop(Hist) etc. Standard logic is linked to yes/no answers (but other logics can be more imaginative as regards admissible types of answer [339]). Questions can also involve a plain answer or a probabilistic answer (a probability ascribed to each answer). In Classical Physics, probabilistic answers are due to imprecise knowledge of the system in practise. In Quantum Physics, probabilistic answers are *inherent*. Here, a

(sub)system has potential properties which may be actualized via measurement (or something like measurement, particularly in the case of Quantum Cosmology that has no external observer; in any case measurement is a conceptually difficult and unsettled issue even within ordinary Quantum Theory).

As a first layer of structure, logic involves AND, OR and NOT operations, which I denote by \vee , \wedge and \neg respectively. In *conventional propositional logic*, these obey (among other relations) the distributivities

$$P \wedge \{Q \vee R\} = \{P \wedge Q\} \vee \{P \wedge R\}, \quad P \vee \{Q \wedge R\} = \{P \vee Q\} \wedge \{P \vee R\}. \quad (715)$$

Conventional propositional logic is what one naïvely and classically uses. *Quantum logic* [336], however, differs in *not* in general being distributive (due to the effect of wave interference, see p 78 of [337]).

A second layer of structure is the *logical implication* operation, which I denote by \preceq . Mathematically, this is a partial ordering:

$$P \preceq P \vee P \quad (\text{reflexivity}) \quad (716)$$

$$\text{If } P \preceq Q \text{ and } Q \preceq R \text{ then } P \preceq R \quad (\text{transitivity}) \quad (717)$$

$$\text{If } P \preceq Q \text{ and } Q \preceq P, \text{ then } P = Q \quad (\text{antisymmetry}). \quad (718)$$

Its function in logic amounts to being a comparer of the precision of various propositions/questions ('are you six feet tall to the nearest foot' versus 'are you six feet tall to the nearest inch'). Its physical realization is as a *graining operation* (c.f. Statistical Mechanics and Histories Theory for well-known occurrences of such). The nomenclature here is that $A \preceq B$ is termed ' A is finer-grained than B ', while $C \succeq D$ is termed ' C is a coarser-grained than D '. In Statistical Mechanics, this applies to regions of phase space, and in Histories Theory to regions of the space of histories; in this article it will apply also to regions of configuration space in the case of timeless approaches and, in another sense, within Histories Theory. In each case, the *coarsest* graining of state is the whole state space itself, while the *finest* graining of state are each of the individual points that make up the space.

Note: the example given shows that a 'measure of precision'/notion of distance serves to furnish a partial ordering, however not all partial orderings will be underlied by a such.

A *lattice of propositions* is brought in as regards to how to make sense of incompatible propositions [344]. The minimal such structure is an *ortholagebra*, in which $P \preceq R$ iff $\exists S \in L$ such that $R = P \oplus S := P \vee S$ in cases in which P and S are disjoint, \vee not being defined in other cases. Passing to an orthoalgebra matters as regards supporting a satisfactory tensor product operation.

Finally, note that whilst the connection between questions and logic has long been around and obvious, the newer issue is that conceiving of physics in terms of grainings should be in direct correspondence with this.

11.7.3 The physical content of atemporal questions: being and conditional being.

Examples of **questions of being** are as follows.

Pre-requisite 1) Is a given configuration mathematically consistent?

Example 1) Whether it obeys the constraint equations is an issue in electrostatics and the GR initial value problem, and the indirectly-formulated scaled RPM example is subject to the total angular momentum of the universe being zero.

Pre-requisite 2) Is a given configuration adequately physical?

Example 2) It might solve the equations but having unacceptable behaviour at infinity, or unacceptable singularities or other mathematical peculiarities that are deemed unphysical. Or it does not in fact represent the situation that one was wishing to model, which is a constant danger in the GR initial value problem due to the difference between the input geometry and the output geometry once the Lichnerowicz equation is solved for the conformal factor.

Then the main sort of straightforward questions of being is as follows.

What is the probability that the (approximate sub)configuration has a given property?

[It is the practical fact that our knowledge of actual configurations is imperfect (or that our perfectly-known models only imperfectly model the real world) that require the word 'approximate'. And this word means that one is not dealing with individual configurations but rather with a coarse-graining description that amounts to having at least local knowledge of the structure of the (sub)configuration space, i.e. Sec 3, and, furthermore, extensions to subconfiguration space structure, see Sec 14). As examples of such questions: what is the probability that the universe is big? Flat? Homogeneous? Isotropic? These require notions of precision to quantify: big compared to what, what relative size of departures from flatness, homogeneity and isotropy. There are then conceptual thinking and Mathematical Physics implementations as regards how to quantify such properties. For the relational triangle, the probability of the model universe being big is in terms of the moment of inertia, and I have demonstrated that there are notions of uniformity (maximized by equilaterality), isoscelesness and regularity/contents homogeneity of the subclusters that one can ask such questions about.

One is furthermore to consider **questions of conditional being**. These involve two properties: given that an (approximate) (sub)configuration has property P_1 , what is the probability that it also has property P_2 ? In the case of subconfigurations,

this could involve one property of each of two distinct subconfigurations of the same instant, i.e. $\text{Prob}(S_2 \text{ has property } P_2 | S_1 \text{ has property } P_1)$.

[Conditional being questions can then involve e.g. any two of the examples of properties in the preceding square bracket, or the characterizations of inhomogeneity of any two subregions. Involving two properties is more widely useful as regards predicting and testing for one in cases for which the other is an observed given.]

11.7.4 Temporal questions

In the presence of a meaningful notion of time, one can additionally consider the following question-types to be primary. This is a natural consequence of how, in formats that do possess a notion of time, the answers to questions are in general time-dependent.

Questions of Being at a Particular Time. The general form is $\text{Prob}(S_1 \text{ has property } P_1 \text{ as the timefunction } t \text{ takes a fixed value } t_1)$ I note that without this qualification, questions of being can look very vague.

[E.g., is the universe big *today*? Was the universe homogeneous *at the time of last scattering*?]

Questions of Becoming. These involve a given particular (approximate) (sub)system state becoming some other (approximate) (sub)system state.

[E.g. do smooth universes become more inhomogeneous? Do smooth universes become populated by supermassive black holes? Note that these for now mean ‘at some time in the future’ rather than ‘at a given time in the future’ or ‘becomes permanently’.]

Moreover, the above two types of question furthermore compose: what is $\text{Prob}(\text{state } s_1 \text{ of subsystem } S_1 \text{ at } t_1 \text{ becomes state } s_2 \text{ of subsystem } S_2 \text{ at } t_2)$? And, clearly permanence is incorporable by using “at all values of the time variable that are greater than some t_2 .” [One would argue that a cut-off to avoid infinity is here physical in limiting oneself to questions answerable within a finite time. In this way, this inclusion, which allows for the scheme to be of the B-series type, does not pose a limitation on such a theoretical scheme’s capacity to answer real physical questions.]

Note: there are some control restrictions on questions of becoming. E.g. the problem need to be well-posed is well-posed, including S_1 being extensive enough and well-placed enough to be the only significant input to the process leading to S_2 . E.g. a sufficient chunk S_1 of a past Cauchy surface Σ_1 is needed to control some future chunk S_2 of Cauchy surface Σ_2 , where that sufficiency is dictated by $S_2 \subset D^+(S_1) \cup \Sigma_2$, for $D^+(X)$ the future domain of dependence of set X .

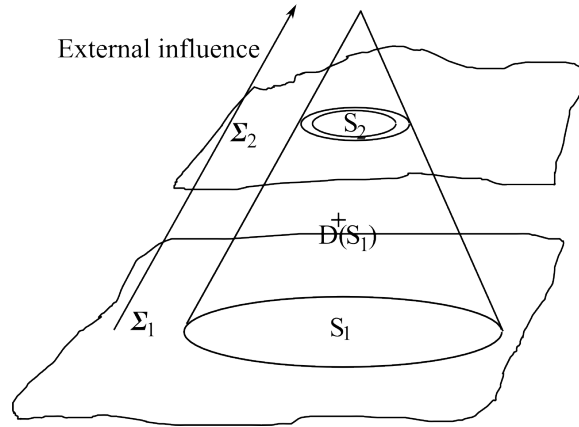


Figure 50: Propositions concerning becoming of regions. Signals from $p_1 \in \Sigma_1$ outside of S_1 cannot influence the physics of $S_2 \subset D^+(S_1) \cup \Sigma_2$.

A specific example of such a question, which is integral to the scientific enterprise, is as follows. If an experiment is set up in a particular way, what final state does its initial state become due to the active agents of the experiment?

Now, if a scheme is purely timeless, it admits a lesser range of a priori poseable questions. Secs 11.8.4 and 11.8.7 consider whether some of these *prima facie* temporally distinct question types can in fact be transmuted into each other. One punchline is that to the extent that the temporal aspect can be trivialized, whether by recasting everything timelessly or by recasting it in terms of histories, one would then expect the form of the remaining physics to strongly reflect the structure of the atemporal part of what a question is. I.e. to involve some mathematical structure along the lines of those in Sec 11.7.2 This should then (if the ‘Mackey Principle’ holds) underpin the structure of both timeless strategies (Secs 11.8, 14, 11.11) and Histories theory (Secs 11.9, 15). The devil is in the detail: exactly which logic is appropriate?

For the Semiclassical Approach, at the level of the l-time-dependent Schrödinger equation, this concerns atemporal and temporal questions about the l = shapes. For the internal time approach, atemporal and temporal questions about the true degrees of freedom?

11.8 Type 1 Tempus Nihil Est

Adopting a Tempus Nihil Est approach saves one from the thorny issue of trying to define time as outlined in Secs 11.2 and 11.2.3 However, this is supplanted by three other brier patches.

1) How to explain the semblance of dynamics if the universe is timeless as a whole. Dynamics or history are now to be *apparent notions* to be constructed from the instant [248, 292, 297, 294] by somehow implementing the following postulate.

Type 1 Nihil Postulate. One aims to supplant ‘becoming’ with ‘being’ at the primary level (see [499, 500, 80, 83, 248, 292, 298, 297, 294, 295] and also e.g. Reichenbach [526] for partial antecedents). [In this sense, these Timeless Approaches consider the instant/space as primary and spacetime/dynamics/history as secondary.]

2) **Nonstandard Interpretation of Quantum Theory.** Type 1 Tempus Nihil Est and Histories Theory both soon become entwined in general questions about the interpretation of Quantum Theory, in particular as regards whole-universe replacements for standard Quantum Theory’s Copenhagen Interpretation. I note that some criticisms of Timeless Approaches [400, 403] are subject to wishing to preserve aspects of the Copenhagen Interpretation, which may anyway not be appropriate for Quantum Cosmology and Quantum Gravity, which, of course, are both very interesting and very difficult cases.

3) **Wheeler–DeWitt equation Dilemma.** Such approaches either (horn 1) invoke the Wheeler–DeWitt equation and so inherit some of its problems, or do not, thus risking the alternative problem (horn 2) of being incompatible with the Wheeler–DeWitt equation, so that the action of the Wheeler–DeWitt operator kicks purported solutions out of the physical solution space.

11.8.1 Naïve Schrödinger Interpretation

This is an interpretation of Quantum Theory for the whole universe originally due to Hawking [314] (see also [315, 316, 616]). It is based on assuming that, for GR,

$$\text{Prob}(h \text{ has property } P) = \int_{P\text{-affirmative subset of Riem}(\Sigma)} |\Psi[\mathbf{h}]|^2 \mathbb{D}\mathbf{h} . \quad (719)$$

[This should be taken with a pinch of salt as regards the measure on $\text{Riem}(\Sigma)$ and the P-affirmative subsets happening to form a suitable region of integration; one is also dealing with *relative* probabilities here.] N.B. in the Naïve Schrödinger Interpretation, one makes no attempt to collapse all questions to questions to being, but is temporally trivial as result of discarding the other questions so as to concentrate on demonstrating that some interesting questions can be answered. It is additionally useful as a simple structural example. Set theory’s complement c , union \bigcup , intersection \bigcap and inclusion \subseteq are a realization of conventional propositional logic’s \neg , \vee , \wedge and \preceq . Continuous regions of a manifold are then one example of such a set, and so these implement conventional propositional logic.

This approach has various fairly obvious problems.

Records Problem-1. It is of limited use via not accommodating questions of being at a particular time, or of becoming [400]. Records Problem-2. This approach is menaced by horn 2 of the Wheeler–DeWitt equation Dilemma via its thus-named ‘naïve’ inner product postulation.

Records Problem-3 This logical structure for the propositions may be a questionable one to use in a quantum-mechanical context due to its *classical* form. I much prefer the next SSSEC’s projector implementation for propositions.

Records Problem-4. In the geometrodynamical case, time enters as an internal coordinate function of $h_{\mu\nu}$. Therefore it is represented by an operator. However, as pointed out in e.g. [335], there are problems with representing time as an operator.

Analogy 68) RPM’s admit toy models of the naïve Schrödinger interpretation [315, 616]. Here, simple (and “geometrically nice”) questions of being are addressed, some of which are listed in Sec 11.7.3 and examples of which are computed in Sec 14.2.

11.8.2 Proposition–projector association

The aim here is to represent propositions at the quantum level by projectors, including beyond [344] the usual context and interpretation that these are ascribed in ordinary Quantum Theory. In ordinary quantum theory, for state ρ and proposition P implemented by projector \hat{P} , $\text{Prob}(P; \rho) = \text{tr}(\hat{\rho}\hat{P})$ with Gleason’s theorem providing strong uniqueness criteria for this choice of object from the perspective of satisfying the basic axioms of probabilities (see e.g. [337]).

11.8.3 Conditional probabilities in ordinary Quantum Theory

The formula for this is (c.f. p 159-165 of [337])

$$\text{Prob}(B \in b \text{ at } \mathbf{t} = \mathbf{t}_2 | A \in a \text{ at } \mathbf{t} = \mathbf{t}_1; \rho) = \frac{\text{Tr}(\mathbf{P}_b^B(\mathbf{t}_2) \mathbf{P}_a^A(\mathbf{t}_1) \rho \mathbf{P}_a^A(\mathbf{t}_1))}{\text{Tr}(\mathbf{P}_a^A(\mathbf{t}_1) \rho)} . \quad (720)$$

[I denote the projection operator for an observable A an observable and a a subset of the values that this can take is denoted by \mathbf{P}_a^A]. N.B. that this is in the 2-time context, i.e. to be interpreted as *subsequent* measurements. It also follows that

$$\text{Prob}(B \in b \text{ at } \mathbf{t}_2 \text{ and } A \in a \text{ at } \mathbf{t}_1) = \text{Tr}(\mathbf{P}_b^B(\mathbf{t}_2) \mathbf{P}_a^A(\mathbf{t}_1) \rho \mathbf{P}_a^A(\mathbf{t}_1)) , \quad (721)$$

and this extends in the obvious way to p propositions at times \mathbf{t}_1 to \mathbf{t}_p .

11.8.4 Supplanting being-at-a-time questions

I next consider incorporating questions of being-at-a-time questions. The list of being, being at a time and becoming types of questions does not say which time is involved. Ordinary classical physics has an easy way out: there is an external time belonging to the real numbers, so that each configuration space \mathbf{q} is augmented to an extended configuration space $\mathbf{q} \times \mathbf{T}$. One key lesson from GR, however, is that there is no such external time. Questions along the lines of those above which involve time need here a specification of *which* time. Using ‘just any’ time comes with the Multiple Choice and Functional Evolution [400, 335] facets of the Problem of Time. Another way of latching onto some aspects of the above key lesson, which moreover can already be modelled at the level of nonrelativistic but temporally-relational mechanical models, is that ‘being, at a time \mathbf{t}_0 ’ is *by itself* meaningless if one’s theory is MRI in the time label. Alternatives that render particular times, whether uniquely or in families up to frame embedding variables, meaningful are specific internal, emergent or apparent time approaches. In this scheme, time is but a property that can be read off the (decorated sub)configuration. E.g. York’s internal time [652, 397, 400, 335] can be thought of in this way, as can the next SSSec’s notion of clock time. Thus, all question types involving a t are turned into the corresponding question types without one.⁷⁰ Perhaps this property concerns a particular subconfiguration lying entirely within the state space in question (‘a clock within the subsystem’). Perhaps it concerns a subconfiguration lying entirely outside the subsystem under study (‘clock within the environment’/‘background clock’). Though perhaps a clock subsystem could be part-interior and part-exterior to the subsystem under study. Indeed, one could have a universe-time to which all parts of the configuration contribute rather than a clock *subsystem*. [None of these uses of ‘clock’ necessarily carry any ‘good clock’ connotations. In each case that is to be furthermore determined.]

11.8.5 Conditional Probabilities Interpretation

This was proposed by Page and Wootters [502] (see also the comments, criticisms and variants in [400, 403, 256, 244, 522]). Conceptually, it is a refinement of the naïve Schrödinger interpretation that extends the range of questions it can answer (thus it is an improvement as regards Problem 11.8.1-1); technically and as interpretations of QM, the two schemes are highly distinct. Moreover, it specifically implements propositions at the quantum level by use of projectors. It addresses questions concerning conditioned being: conditional probabilities for the results of a pair of observables A and B , in particular concern correlations between A and B at a single instant in time. E.g. “what is the probability that the universe is almost-flat *given* that it is almost-isotropic?” One addresses such questions via *postulating* the relevance of conditional probabilities

$$\text{Prob}(B \in b | A \in a; \rho) = \frac{\text{Tr}(\mathbf{P}_b^B \mathbf{P}_a^A \rho \mathbf{P}_a^A)}{\text{Tr}(\mathbf{P}_a^A \rho)} , \quad (722)$$

for finding B in the subset b , given that A lies in the subset a for a (sub)system in state ρ . N.B. these occurring within the one instant rather than ordered in time (one measurement and *then* another measurement) places this postulation outside the conventional formalism of Quantum Theory, for all that (722) superficially resembles (720).

Moreover, the Conditional Probabilities Interpretation can be used in principle to replace questions of ‘being at a time’ by simple questions of conditioned being. This is via one subsystem A being useable as a timefunction, so that the above question about A and B can be rephrased to concern what value B takes when the timefunction-giving A indicates a particular time [502, 335].

This scheme’s traditional development did not set up a scheme of logical propositions (it came historically before awareness of that began to enter the Problem of Time community via [344, 345]). However, adding that layer of structure to the Conditional Probabilities Interpretation for first principles reasons (from Mackey’s Principle) could be seen as a cleaner successor to my chain of thought in suggesting Records theory outside of Histories theory have such a structure from the Records Theory that sits inside of Histories Theory inheriting structures from Histories Theory. In the Conditional Probabilities Interpretation scheme, then, the projectors used are a solid anchor for embodying propositions, and the structure of the atemporal part of quantum logic is then a reasonable first working guess for this newly-proposed extra level of logical

⁷⁰It is not clear which as in this setting one can have in principle different configurations take the same time value (e.g. through lying on different paths of motion).

structure, though one would need to think carefully about whether Conditional Probabilities Interpretation-Copenhagen differences affect this in any way.

CPI Problem-1. The means of replacement of ‘being at a time’ has the practical limitation that A may not happen to have characteristics that make it suitable as a sufficiently good timefunction.

CPI Problem-2. supplanting being-at-a-time by being is as far as the original Conditional Probabilities Interpretation goes; it is not a full resolution of Problem 11.8.1-1 as it does not per se address questions of becoming; for a further extension to address those too, see the ‘Records A)’ scheme by Page in the next 2 SSecs.

CPI Problem- 3. Horn 1 of the Wheeler–DeWitt equation Dilemma also applies here [400].

I also comment that some of Kuchař’s critiques [400, 403] of the Naïve Schrödinger Interpretation and the Conditional Probabilities Interpretation can be interpreted as not accepting ab initio a separate ‘being’ position rather than constituting conceptual or technical problems once one has adapted such a position.

Analogy 69) RPM’s admit toy models of the Conditional Probabilities Interpretation [502], in which questions concerning pairs of propositions of being are addressed. E.g., ‘what is the probability that the triangle model universe has a large area per unit moment of inertia given that it is approximately isosceles?’

11.8.6 Gambini–Porto–Pullin approach

This [244, 522] is built upon conditional probabilities, which are now of the form $\text{Prob}(\text{observable } O \text{ lies in interval } \Delta O \text{ provided that clock variable } \tau \text{ lies in interval } \Delta\tau) =$

$$\lim_{T \rightarrow 0} \frac{\int_0^T dt \text{Tr}(P_{\Delta O}(t) P_{\Delta\tau}(t) \rho_0 P_{\Delta\tau}(t))}{\int_0^T dt \text{Tr}(\rho_0 P_{\Delta\tau}(t))} . \quad (723)$$

The $P(t)$ ’s here are Heisenberg time evolutions of projectors P : $P(t) = \exp(iHt)P\exp(-iHt)$ These conditional probabilities allow for ‘being at a time’ to be incorporated. Another feature of this approach is that employing a non-ideal clock in itself gives rise to decoherence (though that needs to be checked case by case rather than assumed [11]). Finally, this approach possesses a modified version of the Heisenberg equations of motion of the Lindblad type,

$$i\hbar \frac{\partial \rho}{\partial \tau} = [H, \rho] + D[\rho] . \quad (724)$$

Note 1) The form of D here is $\sigma(\tau)[H, [H, \rho]]$ where $\sigma(\tau)$ is dominated by the rate of change of width of the probability distribution.

Note 2) This equation is tied to the incorporation of decoherence [and in a way distinct from that in the usual Histories Theory context].

Note 3) this equation is not unitary, which is OK insofar as it represents a system with imprecise knowledge.

Note 4) By the presence of this ‘emergent becoming’ equation, this approach looks to be more promising in practise than Page’s.

Note 5) This scheme does additionally have a number of heuristic gaps which were well-reviewed by Anastopoulos and Hu [11].

One particular such is that (723) is a new postulation rather than a part of standard QM, and, as it stands, is being interpreted outside of the axioms of what one would expect to be the standard probability theory.

Note 6) Another of [11]’s points is that some elements of Histories Theory look to be entwined into the sort of conditional probabilities involved, by which the scheme would not look to be conceptually a purely timeless one in the present article’s sense. (That per se is not a problem, but rather a reason by which this approach might well need to be reclassified as another of the growing number of *composite* Problem of Time strategies. What is a problem, then of course however is that Histories Theory comes with its own difficulties – see the next SSSec.) This should be clear by the very presence of an ‘emergent becoming’ equation rather than the supplanting of becoming by being as in Page’s approach, which *is* a purely timeless approach that does not assume any histories-theoretic structure.

The present article uses the Naïve Schrödinger Interpretation as a simple example and then looks at Records Theory, thus for now missing out on giving detailed examples of the Conditional Probabilities Interpretation. However, the Conditional Probabilities Interpretation does give tractable examples comparable to the Naïve Schrödinger Interpretation ones, albeit these do require a bit more work/space to present.

11.8.7 Supplanting becoming questions

This has been suggested by Page (e.g [501]) and also to some extent by Barbour [83]. This supplanting might, moreover, be viewed as more operationally accurate. Page’s scheme for this is as follows [501, 499, 500]. It is not the past instant that is involved, but rather this appearing as a memory/subrecord in the present instant, alongside the subsystem itself.

Thus this is in fact a correlation within the one instant. In this scheme, one does not have a sequence of events. Rather, there is one present event that contains memories or other evidence of ‘other events’. One might view such configurations as e.g. researchers with data sets who remember how they set up the experiment that the data came from (controlled initial conditions, and so on). Reasons why this might not be adopted, or might not be a complete catch-all of what one would like to be explained include the following.

Page Problem-1. Unfortunately, this is very speculative from the perspective of doing concrete calculations. Studying a subsystem S now involves studying a larger subsystem containing multiple imprints of S . Models involving memories would be particularly difficult to handle. It would be expected to be very hard even to toy-model, one would need a working Information Gathering and Utilizing System model [310].

Page Problem-2. If one wants a scheme that can additionally explain the Arrow of Time, then Page’s scheme looks to be unsatisfactory. Single instants could be used to simulate the scientific process as regards ‘becoming questions’. However, it is noteworthy that these single instants correspond to the *latest* stage of the investigation (in the ‘becoming’ interpretation), while ‘earlier instants’ will not have this complete information. Additionally, important aspects of the scientific enterprise look to be incomplete in this approach. E.g. in interpreting present correlations, one is in difficulty if one cannot affirm that one did in fact prime the measuring apparatus.

I.e. as well as the ‘last instant’ playing an important role in the interpretation, initial conditions implicit in the ‘first instant’ also look to play a role (see also [306, 292]). This could be envisaged as a progression from Bishop Berkeley’s notion of ‘time as succession of ideas in our minds’ to ‘time as an abstraction from the memories in our minds *now*’.

11.8.8 Records Theory schemes: existing and updated

The general idea here is that these are timeless schemes from which one seeks to construct a semblance of dynamics or history from the correlations within pieces of a single instant. The theoreticians involved have, however, differed somewhat both in how to make the notion of record more precise, and in how they envisage the semblance of dynamics may come about. Thus there are in fact a number of Records approaches.

Page Records The central idea here is the content of the preceding SSec.

Bell–Barbour Records. [111, 80, 83] (see [152] for further differences between these authors) reinterpret Mott’s calculation [481] of how α -particle tracks form in a bubble chamber as a “**time capsules**” paradigm for Records Theory. Barbour has then argued for Quantum Cosmology to be studied analogously, with somewhat similar arguments being made by Halliwell and others (see footnote 91).

Barbour’s own approach has a number of additional elements

- 1) reformulating classical physics in timeless terms [92, 79, 94, 83] (and adopted as starting-point in the present article). [N.B. that [83] should not be read to be literally placing emphasis on timelessness casting mystery upon why ‘ordinary physics’ works.] Of course, this leads to the emergent JBB time.
- 2) Barbour follows Leibniz in emphasizing the configuration of the universe as a whole, and how this is the only perfect clock (and then takes emergent JBB time to embody this whole-universe character).
- 3) Barbour furthermore speculates [79, 80, 83] that the asymmetry of the underlying curved stratified quotient configuration space causes concentration of probability density on “time capsules” rather than other instantaneous configurations.

Records Problem-1. I consider item 2) to render the scheme impractical, as we do not have that detailed a knowledge of the universe as a whole. I would much rather that the scheme left one free to consider subsystems. [That, of course, is entirely fine in both of the Page and Gell–Mann–Hartle–Halliwell schemes.]

Gell–Mann–Hartle–Halliwell Records [248] and Halliwell [292, 298, 294] have found and studied records contained within Histories Theory (see two SSecs down). N.B. that the records scheme sitting within a Histories theory is independent of the Gell–Mann–Hartle versus Isham–Linden distinction because these involve the single-time histories, i.e. a single projector, and then the ordinary and tensor products of a single projector obviously coincide and indeed trivially constitute a projector. Thus one can apply the Projector–Proposition Association and nicely found a propositional logic structure on this in accord with Mackey’s Principle.⁷¹

Note: viewpoints that presume and derive histories are logically distinct, with the latter requiring more work than the former.

I contend that the three pre-existing Records schemes are partially-unifiable into the following scheme.

⁷¹That Gell–Mann–Hartle–Halliwell records already come with \vee , \wedge , \neg and coarse-graining via inheriting these from the Histories Theory they sit within, which is how I first came to conceive of records needing such structures, for all that I now argue that out as from ‘Mackey’s Principle’ and the consideration of temporally-collapsed logics that are unencumbered by further specifically temporal logic features.

Records Postulate 1. Records are information-containing subconfigurations of a single instant that are localized in both space and configuration space.

[Local in space means ‘under one’s nose’. This is partly so that they are controllable, and partly so that one has more than one such to compare. It also negates signal times within conventional frameworks in which such are relevant. However, this is far from necessarily the basis for a criticism; e.g. pp. 225-226 of [635] points out that relativistic theories make use of a similar notion of locality. By this ‘local in space’ criterion, Barbour’s insistence on whole-universe configurations lies outside of the scheme and thus should be dropped. Local in configuration space concerns the imperfectness of knowledge in practise, i.e. grainings.]

Records Postulate 2 Records can be tied to atemporal propositions, which, in accord with Mackey’s Principle, form a suitable logic.

[The tying at the quantum level is preferably by the Projector–Proposition Association. The form of the logical structure remains open to debate. The notion of localization in configuration space may well furnish the graining/partial order/logical implication operation.]

Records Postulate 3. Records are furthermore required to contain useful information. I take this to mean information that is firstly and straightforwardly about correlations. Secondly, however, one would wish for such correlation information to form a basis for a semblance of dynamics or history. I have a hunch here that one will be disappointed (Possible Records Problem-2). This follows from the Mott–Bell–Barbour bubble chamber track not being the only paradigm for records; e.g. there is also the Joos–Zeh [358] dust grain being decohered by microwave background photons. The fear is then that the latter is much less contrived and thus likely to be much more common in the universe, so that one would seldom be able to deduce much from records. Some hope might return to the scheme via Halliwell’s finding usefulness even in very imperfect records. This is crucial as to the sensibleness of using few-particle RPM’s for records-theoretic investigations. These have as much or more capacity to form records as Halliwell’s model (whilst also now being understandable as whole-universe models of such a process).

Such a scheme therefore requires [28] suitable notions of locality in space and in configuration space, of propositional logic, of information, relative information and correlation.

I term Records 1) to 3) **Prerecords Theory**. Many of these are already difficult to provide for general GR. [But RPM’s are rather more amenable in these respects, making them good toy models for studying Records.]

The Projector–Proposition Association obviously carries over from the Conditional Probabilities Interpretation to the Page Records scheme, bringing it into accord with Mackey’s Principle. Upgrading the Bell–Barbour scheme likewise would first require deciding whether to adopt the Projector–Proposition Association within that context. And then whether to postulate the relevance of, and evaluate, the Conditional Probabilities Interpretation/Page Records scheme’s concrete conditional probability object is to be computed and interpreted, or to concentrate on some other kind of object. (Without such a concrete specification, the Bell–Barbour scheme remains vague from the perspective of actually doing calculations.) This includes a higher level of vagueness than in the Page Records scheme, in which *what* to compute is at least clear, for all that we have no idea how to evaluate such a thing for a nontrivially-functional model of becoming-supplantation). I also have two further qualms with the Bell–Barbour scheme.

Records Problem-2 Unfortunately, Prerecords Theory is incomplete.

The key missing step is that the scheme have a means of accounting for the **semblance of dynamics**.

Records Theory is, finally, the subsequent study of how dynamics (or history or science) is to be abstracted from correlations between such same-instant subconfiguration records.

N.B. the semblance of dynamics is the point at which the unified records approach breaks down. The Page approach proceeds via valid-in-principle reduction of becoming to being, Gambini–Porto–Pullin by their Lindblad equation, Barbour via attempting to demonstrate his conjecture, and Gell-Mann–Hartle–Halliwell by already sitting within a Histories Theory (though one might still ask if that can be attained without presupposition). Semiclassical considerations may also be invoked (or indeed a distinct three-way unification of some records scheme with some histories scheme and some semiclassical scheme, so that these cancel out each others’ deficiencies, see Sec 15.4).

Analogy 70) RPM’s are useful in the study of notion of locality in space (leading to notions of inhomogeneity and structure) and of locality in configuration space (Sec 14.4). (This leads to notions of states that are only approximately known). This is towards the timeless records approach to the Problem of Time. There are more options in RPM’s, due to i) the kinetic term positive-definiteness lending itself to the construction of notions of locality in configuration space. ii) While in GR triviality

of D_μ ties together \mathcal{M}_μ triviality and the lack of a notion of locality, in RPM's these notions are disjoint. The latter occurs even in the scaled 1- d case that has no linear constraints at all.

Analogy 71) There are also analogies useful for Records Theory at the level of notions of information/negentropy, including subsystem and mutual notions of such, though these remain very much work in progress (Sec 14.9). By QM solvability allowing one to build up a statistical mechanics and thus notions of entropy, and negentropy is a reasonable characterization of information, RPM's have tractable notions of information, subsystem information, mutual information and so on.

With the above prerecords structures in place, one would hope to be able to investigate the extent to which Records Theory can by itself produce a semblance of dynamics or history. E.g. RPM models may help provide partial reasons to (dis)believe the sequence of steps of Barbour's conjecture (see 14.10 for more).

11.9 Histories Theory

Histories postulate. Treat histories as primary.

Note 1) Adopting this entails supplanting Relationalism 3)'s 'treat configurations as primary', by a postulate that is in part very similar in having a class of object that is primary, and yet very different as regards the nature of that object and the break this change entails (at least at first sight) from the dynamical tradition of physics.

Note 2) One issue is whether histories are operationally meaningful as primary quantities: are histories something one can directly measure? Within the histories picture, measurements at one instant of the history constitute records, and the ontology allows for measurements at distinct instants of the history, so there is at least a partial sense in which the answer is affirmative: at least *constituent parts*. of histories can be determined by direct measurements.

Note 3) The above similarity means that both afford the same kind of extensions by parallel arguments, to categorization, propositioning and to a phase space version. Thus one is to consider not just Hist (the space of histories) but also (Hist, HistMorph) for HistMorph: $\text{Hist} \rightarrow \text{Hist}$, $\text{Prop}(\text{Hist})$, $\text{Context}(\text{Hist}) = \langle \text{the set of physically meaningful SubHist's} \rangle$, and such as (HistPhase, HistCan) for HistPhase consisting of histories, histories momenta and a histories bracket, which is then preserved by HistCan (or variants along the lines of this article's variants on Phase).

Note 4) It is more instructive to consider how Histories Theory developed prior to discussing these structures which only came much later in the development of Histories Theory. In any case, these structures change with developments in the mathematization of the histories themselves...

11.9.1 Hartle-type Histories Theory

1) Naïvely, at the classical level histories are sequences of configuration instants at given times, $Q^A(\mathbf{t}_i)$ $i = 1$ to n in the usual approach with discrete time-steps.

One then thinks at the quantum level level along the lines of Feynman path integrals,⁷² moreover building up much further structure. Individual histories are now built out of strings of projectors $P_{a_i}^{A_i}(\mathbf{t}_i)$, $i = 1$ to N at times \mathbf{t}_i ,

$$c_\alpha := P_{a_N}^{A_N}(\mathbf{t}_N) \dots P_{a_2}^{A_2}(\mathbf{t}_2) P_{a_1}^{A_1}(\mathbf{t}_1) . \quad (725)$$

N.B. these do not imply measurement; Histories Theory is intended to have other than the standard interpretation of QM.

Assigning probabilities to histories does not work in Quantum Theory. For, if $a(t)$ has amplitude $A[a] = \exp(i\mathbf{S}[a])$ and $b(t)$ has amplitude $B[b] = \exp(i\mathbf{S}[b])$, then these are nonadditive since in general $|A[a] + B[b]|^2 \neq |A[a]|^2 + |B[b]|^2$ in general. In fact, Histories Theory involves [248, 306] a far-reaching [335] extension of normal Quantum Theory to a form outside its conventional Copenhagen interpretation: it is a many-worlds or beables type scheme.

2) One then considers the notion of fine/coarse graining corresponding to different levels of imperfection of knowledge, by which families of histories are partitioned exhaustively and exclusively into subfamilies. I use $C_{\bar{\alpha}}$ for coarse-graining, where $\bar{\alpha}$ is a subsequence of α 's times and each projector in the new string may concern a less precise proposition. Thus, as well as the coarse-graining criteria in Sec 11.7.2, Histories Theory possesses *coarse graining by probing at less times*.

3) The final traditional ingredient of the Histories Theory approach is the **decoherence functional** between a pair of histories α, α' ,

$$\text{Dec}(c_{\alpha'}, c_\alpha) := \text{tr}(c_{\alpha'} \rho c_\alpha) . \quad (726)$$

This is useful as a 'measure' of interference between $c_{\alpha'}$ and c_α . It is zero for perfectly consistent theories. It has the following properties. i)

$$\text{Dec}(c_{\alpha'}, c_\alpha) = \text{Dec}(c_\alpha, c_{\alpha'}) \quad (\text{Hermiticity}) , \quad (727)$$

$$\text{Dec}(c_{\alpha'}, c_\alpha) \geq 0 \quad (\text{Positivity}) , \quad (728)$$

$$\sum_{c_{\alpha'}, c_\alpha} D(c_{\alpha'}, c_\alpha) = 1 \quad (\text{Normalization}) , \quad (729)$$

⁷²Doing just this for MRI mechanics is Brown and York's work [147, 148], intended to investigate the fruitfulness of the sort of transition amplitude approach that Wheeler envisaged as arising from resolving thin sandwich schemes.

$$\text{Dec}(c_{\alpha'}, c_{\alpha}) = \sum_{\alpha' \in \bar{\alpha'}, \alpha \in \bar{\alpha}} D(c_{\alpha'}, c_{\alpha}) \quad (\text{Superposition property}) . \quad (730)$$

A new probability postulate for this scheme is that

$$\text{Dec}(c_{\alpha'}, c_{\alpha}) = \delta_{\alpha', \alpha} \text{Prob}(a_N t_N, a_{N-1} t_{N-1}, \dots, a_1 t_1; \rho_0) . \quad (731)$$

Approximate consistency is held to be sufficient.

Histories Problem-1 It is not obvious whether this scheme is in correspondence with an algebra of propositions.

At least insofar as Gell-Mann–Hartle histories are the product of Heisenberg picture projection operators, and such products are usually not themselves a projection operators, so this fails to implement proposition-as-projector. They do however have a disjoint sum of histories OR and a NOT operation. What about looking to implement propositions along the lines of Hartle’s consideration of how each history intersects with a given region in configuration space? Hartle uses proper time spent in that volume for that purpose, a notion which is independent of canonical slicing. However (in parallel with my questioning the Naïve Schrödinger Interpretation’s use of classical regions to pose its quantum-mechanical questions), I do not expect this to cover all physically-relevant propositions and the inter-relations between them at the quantum level. Additionally, Kuchař has argued that it is possibly problematic that this approach takes a variable that is not dynamical and not quantized and effectively gives it meaning, thus breaking a feature of ordinary quantization without as yet providing enough justification/interpretation for this. However the next SSSec bypasses Histories Problem.1 by finding a different way in which to anchor the Projector–Proposition Association to Histories Theory.

11.9.2 Further approaches to Histories Theory

I motivate this as demonstration that a somewhat different match-up between histories and questions/propositions can in fact be made.

Isham and Linden [344, 345] established this by considering *tensor products* of the projectors

$$c_{\alpha} := P_{a_N}^{A_N}(\mathbf{t}_N) \otimes \dots \otimes P_{a_2}^{A_2}(\mathbf{t}_2) \otimes P_{a_1}^{A_1}(\mathbf{t}_1) ; \quad (732)$$

these trivially inherit the projector axioms from the individual projectors. Thus this is termed the *histories projection operator* approach. The above match-up can then readily be completed with notions of negation and disjoint sum to form an orthoalgebra/lattice of propositions, \mathcal{UP} .

This has enlarged the Hilbert space \mathcal{H} to a tensor product $\mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$, and then this approach encounters the problem of there being no natural time translation operator to move between the copies. Isham and Linden found, however, that they could extend their tensor product to an infinite-dimensional continuum limit, for which this issue of the lack of a natural time translation operator is overcome. This more technical, albeit still time-tied, issue pushed them away from Hartle’s discrete time steps to histories with continuous time.

Now one starts with a classical canonical structure in which the underlying objects are not configurations \mathbf{Q}^A forming a configuration space \mathbf{q} but histories $\mathbf{Q}^A(\mathbf{t})$ (for \mathbf{t} a continuous label) forming a space of histories, Hist. These then are to have canonical momenta $\mathbf{P}_A(\mathbf{t})$ and one is furthermore to use Poisson brackets (‘histories brackets’)

$$\{\mathbf{Q}^A(\mathbf{t}), \mathbf{P}_B(\mathbf{t}')\} = \delta^A_B \delta(\mathbf{t}, \mathbf{t}') . \quad (733)$$

Moving in another direction, HistMorph: Hist \longrightarrow Hist is the analogue of Point : $\mathbf{q} \longrightarrow \mathbf{q}$, consisting of point transformations and monotone deformations of \mathbf{t} (which are easier to envisage than in the discrete case, which was at the mercy of N varying). One can then envisage HistCan similarly in the context of HistPhase, with this additionally preserving the histories bracket. Constraints $\mathcal{C}[\mathbf{Q}^A, \mathbf{P}_B] = 0$ here become *histories constraints*,

$$\mathcal{C}^{\mathbf{t}} := \int d\mathbf{t} \mathcal{C}[\mathbf{Q}^A(\mathbf{t}), \mathbf{P}_B(\mathbf{t})] = 0 . \quad (734)$$

Note 1) In the \mathbf{g} -free case, conventionally, starting from (HistPhase, HistCan), one applies KinQuant, Assoc and $\mathbf{H}_{\mathbf{t}}$ -Solve maps to arrive at [at least formally] (HistHilb, HistUni) [meant in the enlarged sense of Isham–Linden]. This is held to be useful because 1) given the subsequent histories group, its unitary representations specifically allow one access to the orthoalgebra of projection operators as propositions about histories of the theory. 2) The histories scheme is that it involves a fresh set of structures and thus provides a second opportunity for such schemes to work out in practise. [Though the price to pay may be significant: it is less conceptually well-established that one can base a canonical quantization scheme on histories rather than on configurations.]

Note 2) In Records Theory, the histories brackets reduce to the usual Poisson brackets give rise to the *usual* canonical group with which so much exasperation was had as regards quantization [332, 342, 343]. That is a useful, if uneventful, recovery.

Note 3) In terms of the answers to physical questions, there would appear to be no difference between Isham–Linden’s scheme and Gell-Mann–Hartle’s. Isham–Linden is a technical refinement useful for establishing theorems, and also a conceptual improvement due to its manifestly fitting the Projector–Proposition association.

Note 4) As per the opening of this SSec, operational primality is not so clear here, whilst there is also will to involve canonical transformations beyond those of usual physics. On these grounds, a (HistRigPhase, HistMorph) input to the quantization procedure might be less desirable here than (RigPhase, Point) was for \mathbf{q} -first approaches, or, at least, even more discordant with current practise in the field.

Note 5) In the presence of nontrivial \mathbf{g} one adds the usual \mathbf{g} -Red moves, whether classically or quantum-mechanically or as a mixture of both. This is schematically the same as for the \mathbf{q} case. There is then also a HistPhase–HistAPhase–HistDAPhase distinction, the latter two allowing for MRI and MPI implementations of temporal relationalism. This is manifest in Appendix 2.A.5’s variants on actions.

In Isham–Linden’s scheme, the decoherence functional is regarded as a functional $\mathcal{UP} \times \mathcal{UP} \rightarrow \mathbb{C}$. Is this additional structure a significant further mathematical construct? Some partial answers to this are as follows.

- 1) The structure of this map parallels that of maps representable by matrices, so that it then makes sense to talk in terms of decoherence involving negligibility of off-diagonal elements.
- 2) Decoherence functionals are the analogues of QM states in Isham and Linden’s parallel between Histories Theory and ordinary QM.

See e.g. [12] for the specific form of the decoherence functional in Isham–Linden’s formalism for Histories Theory,

2 time notions Savvidou pointed out [557, 12], this version of Histories Theory has a distinct structure for each of two conceptually distinct notions of time:

- I) a kinematical notion of time that labels the histories as sequences of events (the ‘labelling parameter of temporal logic’ , taken by [12] to mean causal ordering, though see also [344].
- II) A dynamical notion of time that is generated by the Hamiltonian.

Savvidou has argued that having these two distinct notions of time allows for such a Histories Theory to be canonical and covariant at once, which is of obvious interest in understanding, and reconciling various viewpoints in, QG.

Kouletsis and Kuchař [389, 388]’s classical work on GR, and on the bosonic string as a toy model, involves a *space map* as well as a *time map* for how the family of geometries along each history embed into spacetime. This work also makes substantial contact with the Internal Time Approach and with the Problem of Observables; here one goes about quantization (only carried out to date for simpler models: ordinary mechanics [345, 346], relativistic mechanics [559], minisuperspace [12] and quantum field theories [10]) via replacing the canonical group by the histories group. Taking time to be continuous in such approaches means that one obtains a 1-*d* QFT in time even for finite toy model examples.

Histories Problem-2/**Question*** incompleteness of development: the Kouletsis–Kuchař approach remains unfinished at the quantum level. Does the histories group approach really succeed where the canonical group approach got stuck?

For completeness of review, I mention two further variants of Histories Theory type approaches.

Marolf’s alternative [445] involves a distinct way of obtaining history brackets: here the Hamiltonian is as an extra structure by which the Poisson bracket is extended from being a Lie bracket on phase space to a Lie bracket on the space of histories. In the more usual approach above, however, one puts the equal-time formalism aside here and introduces a new phase space in which the Poisson bracket is defined over the space of histories from the beginning.

Sorkin’s Causal Sets Approach [588] may be viewed as much like a Histories Theory in which less structure is assumed. Here, causal ordering as occurs in classical spacetime is taken to be the fundamental notion to be kept in Quantum Gravity. However, Spacetime Reconstruction difficulties follow from this approach’s insistence on very sparse structure (for recent advances here, see e.g. [528]).

Analogy 72) Histories theory schemes of Hartle and Isham–Linden type can be set up for RPM’s with reasonable parallel to in the case of GR (see Sec 15; this includes Histories Algebras).

Histories Problems -3 to 6. Using a path integral formulation avoids having to encounter the frozen Wheeler–DeWitt equation and its Inner Product Problem. However the measures involved in gravitational path-integrals are difficult to deal with (and otherwise hard to define), and looks prone to Multiple Choice and Functional Evolution issues [400]). Finally, one might be wary that path integrals are the quantum equivalent of thick sandwiches, which are already unpleasant at the level of classical geometrodynamics.

Histories Problem-7 How is Diff(M) invariance is dealt with in practical terms in Histories Theory [335]?

Histories Problem-8 Histories theory is usually done in gauge-dependent form, and each gauge needs an internal time and so many internal time problems come back (e.g. multiple-choice and global problems). Histories Problem-9. GR time’s many-fingeredness brings in foliation-dependence issues.

Histories Problem-10. Which degrees of freedom decohere which is unclear in the GR context [294].

Possible Problem Histories-11. Is the generalization of QM involved in Histories Theory self-consistent and meaningful? Does it reduce to ordinary QM in cases testable by experiment? (Kuchař has argued [403] that it does not, in the case of a relativistic particle.) Elsewise, it has been commented on by Kent and Dowker [213, 370] that future and past can easily fall apart in this scheme, possibly compromising the capacity to do science in such a universe!

11.10 Suggested collates of Semiclassical, Records and Histories ideas

11.10.1 Motivations for this unification

Pro 1) There is the common ground of a logical structure underlies both histories and timeless approaches.

Pro 2) There is a Records Theory within Histories Theory [248, 292, 294], so Histories Theory supports Records Theory by provides guidance as to the form a working Records Theory would take. This will also allow for these two to be jointly cast as a mathematically-coherent package (as already illustrated in preceding subsections. As Gell-Mann and Hartle say [248],

Records are “*somewhere in the universe where information is stored when histories decohere*”. (735)

Anti 1) As regards the opposite passage: records from first principles from which histories are to be derived, there is however a problem that the decoherence functional collapses out of Records Theory via the path integrals in it ceasing to be defined as the history collapses to a single instant.

Pro 3) Histories decohereing is a leading (but as yet unproven) way by which a semiclassical regime’s WKB approximation could be legitimately obtained in the first place. Thus Histories Theory could support the Semiclassical Approach by freeing it of a major weakness.

Pro 4) The Semiclassical Approach and/or Histories Theory could plausibly support Records Theory by providing a mechanism for the semblance of dynamics (though the possibility of a practically useable such occurring within pure Records Theory has not been discarded). Such would go a long way towards Records Theory being complete. I note that emergent semiclassical time amounts to an approximate semiclassical recovery [22] of Barbour’s classical emergent time [79], which is an encouraging result as regards making such a Semiclassical–Timeless Records combination...

Pro 5) The elusive question of which degrees of freedom decohere which should be answerable through where in the universe the information is actually stored, i.e. where the records thus formed are [248, 294]. In this way, Records Theory could in turn support Histories Theory.

Pro 6) The Semiclassical approach aids in the computation of timeless probabilities of histories entering given configuration space regions. This is by the WKB assumption giving a wavefunction flux into each region [295] in terms of W and the Wigner function (see Sec 15.4.2). Such schemes go beyond the standard Semiclassical Approach, and as such there may be some chance that further objections to the Semiclassical Approach (problems inherited from the Wheeler–DeWitt equation and Spacetime Reconstruction Problems) would be absent from the new unified strategy.

Caveat. For now I know of no way in which this combined scheme can ameliorate the History Problems-2 to 5 that it inherits.

11.10.2 Halliwell’s investigations so far

In 1999, Halliwell investigated Pro 2) and 5) in [292] for a very simple toy model: a heavy particle moving through a medium consisting of a few light particles. The heavy particle disturbs these into motion. Subsequent instants consist of the particles’ positions and momenta. It is these instants which are the records, and the motion or history of the large particle can then be reconstructed (perhaps to some approximation) from them. A particular feature of this calculation is that indeed a very small environment of l -particles suffices in order to have a nontrivial notion of imperfect record. This points to the possibility of using finite toy models with additional desirable theoretical parallels, which is central to the present article. Halliwell [292] also makes (and reasonably establishes, in the context of that paper’s very simple models) the information-theoretic and Histories–Records abridging conjecture that the number of bits required to describe a set of decoherent histories is approximately equal to number of bits thrown away to environment (this ties in well with the aforementioned use of notions of information).

Halliwell investigated Pro 6) in 2003 [294] for a free particle, addressing, for an energy eigenstate, what is the probability of finding the system in a series of regions of configuration space without reference to time. Halliwell [295] has recently (2009) investigated decohereing histories as a possible means of constructing the probability distribution for the Wheeler–DeWitt equation. (This also uses Semiclassical Approach techniques and could be useful for avoiding problems involving how to interpret the Semiclassical Approach’s Wheeler–DeWitt equation). Some intermediate/supporting steps in this program were co-authored with his students Dodd, Thorwart, Wallden and Yearsley [297, 298, 299, 300, 650].

I note that this approach also taps into the bubble chamber paradigm.

11.10.3 RPM extension

Analogy 73) scaled RPM is well-suited for investigating the combination of Histories, Records and Semiclassical Approaches. This is because it is a simple enough toy model to get far in its study, whilst nevertheless possessing sufficient features to do a reasonable job of toy-modelling midisuperspace. I want to consider these together because there are indications that these are able to prop each other up toward finally providing a resolution of the Problem of Time and associated issues in Quantum Cosmology [294]. I elaborate further in Sec 15.4.

11.10.4 Removing a sometimes-given motivation for Records Theory

One of the arguments used toward Records Theory is that this considers the entities that are actually operationally meaningful. For the study of records is how one does science (and history) in practise, whether or not one ascribes reality to whatever secondary frameworks one reconstructs from this (such as histories, spacetimes or the local semblance of dynamics)? However, I argue that Histories Theory being primarily a reconstruction from records is *not a motivation for* Records Theory but, rather, *an assertion to be demonstrated*. For, the records present in nature may not in general be of sufficient quality (localization, accessibility, sufficiently high retrieveable information content of the right kind) to be able to reconstruct history. Thus one should seek to

A) pin down where the “somewhere” in (735) is (which is the central motivation in some of Halliwell’s papers [292, 297]).
 B) Determine whether the record at this location is *useful*. E.g. Gell-Mann and Hartle assert that what they call records “*may not represent records in the usual sense of being constructed from quasiclassical variables accessible to us*” (p 3353 of [248]), which, in my parlance, amounts to insightfully allowing for the possibility that some of nature’s records may not be very useful to us.

Another issue is that the α -particle track in the Mott–Bell–Barbour bubble chamber example may well be atypical in its neatness and localization. For, bubble chambers are carefully selected environments for revealing tracks – much human trial and error has gone into finding a piece of apparatus that does just that. α -tracks being useful records could then hinge on this careful pre-selection, records in general then being expected to be poorer, perhaps far poorer, as suggested e.g. by the Joos–Zeh paradigm of a dust particle decohering due to the microwave background photons [358]. In this situation, records are exceedingly diffuse as the information is spread around by the CMB photons and ends up “*in the vastness of cosmological space*”. It is then a fair point that bubble chambers are atypical in being specifically constructed as specialized pieces of equipment for seeing clear-cut tracks, so that the dust-grain paradigm should be more typical of what is found in nature. It is a further fair point that what happens depends furthermore on the magnitude of the object. Dust grains are bigger than α -particles and, likewise, macroscopic bodies are bigger than dust grains; These typically self-decohere, and on an extremely short timescale, which is a further paradigm and presumably dominant over both the preceding as regards cosmological structure formation. There is then a spectre that records could be too problematic to access and/or of unsuitable information content in the case of quantum gravity (e.g. via high nonlocality, any useful cites for programs in which quantum gravity is highly nonlocal). Though my own goals are for now the more modest origin of quantum-cosmological structure and assessing which of the recoveries of late-universe subsystem physics stand up to careful scrutiny.

C) I note that, most well-known old arguments about the physics being in the correlations in fact specifically do not refer to timeless approaches. E.g. what Wigner said on the subject is, in detail, ([638] p 145, see also [356]) “*quantum mechanics only furnishes us with correlations between SUBSEQUENT observations*” (my caps), and Wheeler wholly agreed with this ([633] p 295). Were one to wish to draw motivation from these statements, they would need to be rephrased and the original context would be lost.

11.11 Type 2 ‘Rovelli’ Tempus Nihil Est

These approaches began with [536, 537, 538] (though one might view [200, 646, 502] as forerunners in some ways, see also [158] and the reviews [545, 599]). They involve various types of observables and related concepts, as defined below.

Observables alias **constants of the motion** alias *perennials* are any function(al)s of the canonical variables $\mathbf{O}[Q, P]$ of the canonical variables such that, at the classical level, their Poisson brackets with all the constraint functions vanish (perhaps weakly [335]). Thus, for geometrodynamics

$$\{O, \mathcal{M}_\mu(x)\} = 0 \ , \quad (736)$$

$$\{O, \mathcal{H}(x)\} = 0 \ . \quad (737)$$

This concept is well-known to date back to Dirac and Bergmann. Justification of the name ‘constants of the motion’ follows e.g. in the GR case from the Hamiltonian taking the form $H[\alpha, \beta^\mu] := \int_\Sigma d^3x (\alpha \mathcal{H} + \beta^\mu \mathcal{M}_\mu)$ so that (736,737) imply that

$$\frac{dO}{dt}[h(t), \pi(t)] = 0 \ . \quad (738)$$

Thus, observables are automatically constants of the motion with respect to evolution along the foliation associated with any choice of α and β^μ . The quantum counterpart of these then straightforwardly involves some operator form for the canonical variables and commutators in place of Poisson brackets.

Alternative Frozen Formalism Facet: The operator-and-commutator counterparts of the above are then another manifestation of the Frozen Formalism Problem of classical canonical GR. [This is some sort of ‘Heisenberg’ counterpart of the ‘Schrödinger’ Wheeler–DeWitt equation being frozen.]

Kuchař observables are as above except that only their brackets with the linear constraints need vanish. Thus, in the case of geometrodynamics, only (736) and not (737) need hold.

It has been argued (see e.g. [400, 93]) that this less stringent condition may suffice.

True observables (Rovelli 1991) alias **complete observables** (Rovelli 2002) (which at least Thiemann [607] also calls evolving constant of the motion) classically involve operations on a system each of which produces a number that can be predicted if the state of the system is known.

Partial observables (Rovelli 1991) are, on the other hand, classically involve operation on the system that produces a number that is possibly totally unpredictable even if the state is perfectly known.

Quantum-mechanically, each of the above two definitions are likewise except that the entities whose predictabilities enter the definitions are now probability distributions (and the states are now taken to be specifically a Heisenberg state).

While the above definitions were more or less in place by 1991, the early 90's and 2000's forms of the Problem of Time strategies that use these do themselves in part differ. Since these approaches will largely not play a further role in the present article, I refer to [39] for their further characterization and remaining difficulties. What I do use in the present article are that three attitudes to the Problem of Observables are as follows.

Attitude 1) Kuchař observables are all. N.B. it is clear that finding these is a timeless pursuit: it involves configuration space or at most phase space but no Hamiltonian and thus no dynamics. The downside now is that there is still a frozen quadratic energy-type quantum constraint on the wavefunctions, so that one has to concoct some kind of Tempus Post Quantum or Tempus Nihil Est manoeuvre to deal with this. To fit my mythological mnemonic, the Ice Dragon is here rendered flightless by disarmament treaty: it concedes not to have/use its wings in exchange for a number of one's strategies ceasing to be useable.

Attitude 2) Complete observables are necessary as a concept and one needs to know them in order to fully unlock QG. This is what Kuchař considers to be like having the good fortune of possessing a Unicorn [401]. To fit my mythological mnemonic, this makes most sense if one imagines one's Unicorns, like She Ra's 'Swift Wind' [573], to be winged, so that in this approach to dragon-slaying, one is nullifying the Ice Dragon having wings by oneself having a winged unicorn to ride.

Attitude 3) In fact it is but partial observables that are necessary. Here the Ice Dragon's ever having possessed wings, and the subsequent need for flighted Unicorns to compensate for this, are held to have always been a misunderstanding of the true nature of observables, which are in fact commonplace but meaningless other than as regards correlations between more than one such considered at once.

See also Sec 16.11 for RPM versions of this SSec with further comments.

11.12 Appendix A: strategies for the Problem of Time in Affine Geometrodynamics

In Affine Geometrodynamics [340, 341, 381], one has a distinct form for the unreduced Hilbert space and for the detailed structure of the Wheeler–DeWitt equation. Tempus Ante Quantum approaches have no plain–affine distinction at the classical level at which the timefunctions are found. There may however still be some scope for different commutation relations and operator orderings subsequently arising at the quantum level. The affine approach still has an analogue of superspace time (as the signature of the wave equation is unaltered by passing to the affine approach). The subsequent inner product issue involving the potential not respecting the conformal Killing vector is at the classical level, so that the non-useability of a Klein–Gordon inner product carries through to the affine case. I am not aware of Third Quantization having been tried in the affine case, but a number of the reasons for it not appearing to be very promising as a Problem of Time resolution do look to carry over to the affine case.

The Naïve Schrödinger Interpretation should not care about such a change, since this approach does not make use of the Wheeler–DeWitt equation. As the Conditional Probabilities Interpretation does work in a Wheeler–DeWitt framework [400], this will change in detail. The start of Records Theory involves notions of distance and of information at the classical level, which are unaffected by this change, though subsequent changes would then be expected at the quantum level. A Histories Theory approach to Affine Geometrodynamics was considered by Kessari [371]. Observables approaches are, in any case, geared toward the Loop Quantum Gravity setting (though it would still be interesting to see how these approaches do in the geometrodynamical setting). [335] indicates that affine version of evolving constants of the motion exists.

All in all, one seldom gets far enough [338] in non-formal detail with Problem of Time approaches to Geometrodynamics for the ordinary versus affine geometrodynamics distinction to give notable changes in behaviour.

12 Tempus ante Quantum

12.1 Emergent JBB time candidate

$\tau^{\text{em(JBB)}}$ is a candidate time that already emerges at the classical level as per Sec 2. Moreover, it does not unfreeze the frozen formalism, nor does it in any other way directly give quantum equations different from the usual ones. One can view emergent JBB time, rather, as an object that is already present at the classical level that is subsequently to be recovered by more bottom-up work at quantum level (since it matches up with the emergent semiclassical time of Sec 13). As I showed in Sec 2, emergent JBB time additionally amounts to the relational recovery of a number of well-known notions of time in various contexts (Newtonian time, GR proper time, cosmic time), which have the overall property of casting the classical equations in a particularly simple form. As I show in Sec 13, approximate emergent JBB time is furthermore significant as the actual object which is aligned with the Semiclassical Approach's hitherto far more widely used approximate emergent WKB time. Via this, the present critical SSec gains considerable extra value.

12.1.1 Is it globally defined? Monotonic?

If $\tau^{\text{em(JBB)}}$ the emergent JBB time candidate exists for (a given portion of) a given motion, its monotonicity is guaranteed: $W > 0$, so $dI \geq 0$, so by (79) $d\tau^{\text{em(JBB)}} \geq 0$.

In terms of the action it emerges from priorly needing to exist, it is not in general globally defined, by Sec 2.12.5's problem of 'zeros, poles and nonsmoothness' (JBB Problem-1). Then at the level of the emergent JBB time formula itself, sufficiently benign blow-ups in ds/\sqrt{W} (i.e. those retaining integrability so that the emergent JBB time candidate does exist) correspond to the $\tau^{\text{em(JBB)}}$ graph becoming infinite in slope. There may also be frozenness: at points for which the graph is horizontal, i.e. $ds = 0$ or W infinite. Both zero and infinite slope may compromise use of $\tau^{\text{em(JBB)}}$ itself to keep track for some ranges of mechanical motion. However, at least in some cases, redefined timestandards may permit motions to be followed through such points.

12.1.2 Is it operationally meaningful?

There are also difficulties on the one hand with observing the emergent JBB time candidate itself, and on the other hand with using more readily observable approximations to it as detailed in Sec 12.1.3. As regards the former, perhaps this is not a problem via belief in Mach's 'time is abstracted from change' [Relationalism 6]] and via finding a suitable clock-hand (see Sec 12.1.8).

12.1.3 Subsystem-wise unreduced h-l split of RPM's

To set up approximate emergent JBB time, some details of the h-l split are needed. I present this here for the simple case of a set of h-particles and a set of l-particles, though many of the points I make in this simple setting transcend to the scale-dominates-shape situation that more closely parallels Quantum Cosmology.

Assume the q^I can be split into heavy coordinates $h_{i'}$ with $i' = 1$ to p and masses $M_{i'}$, and light coordinates $l_{i''}$ with $i'' = p + 1$ to N and masses $m_{i''}$ such that⁷³

$$m_{i''}/M_{i'} = \epsilon_{\text{hier}} \ll 1 \quad (\text{h-l mass hierarchy}) . \quad (739)$$

In making such a split I assume 'sharply peaked hierarchy' conditions

$$\max_{i', j'} |M_{i'} - M_{j'}|/M_{i'} =: \epsilon_{\Delta M} \ll 1 , \quad \max_{i'', j''} |m_{i''} - m_{j''}|/m_{i''} =: \epsilon_{\Delta m} \ll 1 . \quad (740)$$

$$\frac{m_{i''}}{M_{i'}} = \frac{\frac{m_{i''}-m}{m}m + m}{\frac{M_{i'}-M}{M}M + M} = \frac{\{\frac{m_{i''}-m}{m} + 1\}m}{\{\frac{M_{i'}-M}{M} + 1\}M} \sim \frac{m}{M} \left\{ 1 + \frac{m_{i''}-m}{m} - \frac{M_{i'}-M}{M} \right\} \sim \epsilon_{\text{hier}} \{1 + O(\epsilon_{\Delta M}, \epsilon_{\Delta m})\} \quad (741)$$

(the second equality by the binomial expansion) allow for only one h-l mass ratio to feature in subsequent approximations.

Then the classical scaled RPM action is⁷⁴

$$\mathbf{S}_{\text{JBB}}^{\text{ERPM}} = \sqrt{2} \int \sqrt{E_{\text{Uni}} - V_h - V_l - J_{hl}} \sqrt{||d_{\underline{A}, \underline{B}} \mathbf{h}||_{\mathbf{M}_h}^2 + ||d_{\underline{A}, \underline{B}} \mathbf{l}||_{\mathbf{M}_l}^2} , \quad (742)$$

$$\text{for } V_h = V_h(h_{j'} \cdot h_{k'} \text{ alone}) , \quad V_l = V_l(l_{j''} \cdot l_{k''} \text{ alone}) , \quad J_{hl} = J_{hl}(h_{j'} \cdot h_{k'}, l_{j''} \cdot l_{k''}, h_{l'} \cdot l_{l''} \text{ alone}), \quad (743)$$

(the 'interaction potential' or 'forcing term'). The conjugate momenta are now

$$P_{i'\mu}^h = M_{i'} *_{\underline{A}, \underline{B}} h^{i'\mu} , \quad P_{i''\mu}^l = m_{i''} *_{\underline{A}, \underline{B}} l^{i''\mu} . \quad (744)$$

⁷³It is to be understood that the ϵ 's in this article are small quantities; I use these instead of ' \ll ' to keep a more precise account of requisite inter-relations and rankings among these small quantities in the approximate approaches under investigation.

⁷⁴I denote the complex conjugates by $P_{i'}^h$ and $P_{i''}^l$, and use M_h, N_h, M_l, N_l as configuration space metrics and inverses for each of the h and l parts (assumed block-separable as in ordinary mechanics and relative mechanics in Jacobi coordinates). I use E_h because an energy-like separation constant E_l will arise further on in the working. Then the fixed universe $E_{\text{uni}} = E_h + E_l$.

The classical energy constraint now splits into

$$E = \mathcal{E}_h + \mathcal{E}_{hl} = E_h, \quad \text{for } \mathcal{E}_{hl} := \mathcal{E}_l + J_{hl}, \quad \mathcal{E}_h := \|P_h\|_{N_h}^2/2 + V_h, \quad \mathcal{E}_l := \|P_l\|_{N_l}^2/2 + V_l. \quad (745)$$

The classical zero angular momentum constraint likewise splits into

$$\underline{\mathcal{L}} = \underline{\mathcal{L}}^h + \underline{\mathcal{L}}^l = 0, \quad \text{for } \underline{\mathcal{L}}^h = \sum_{i'=1}^p \underline{h}^{i'} \times \underline{P}_{i'}^h, \quad \underline{\mathcal{L}}^l = \sum_{i''=p+1}^n \underline{l}^{i''} \times \underline{P}_{i''}^l \quad (746)$$

the h- and l-subsystems' angular momenta respectively. The evolution equations are

$$*_{\underline{A}, \underline{B}} P_{i'\mu}^h = -\partial\{V_h + J_{hl}\}/\partial h^{i'\mu}, \quad *_{\underline{A}, \underline{B}} P_{i''\mu}^l = -\partial\{V_l + J_{hl}\}/\partial l^{i''\mu}. \quad (747)$$

The quantum versions of the above constraints are

$$\widehat{\mathcal{E}}_h \Psi_h := -\hbar^2 \|\partial_h\|_{N_h}^2 \Psi_h/2 + V_h \Psi_h = E_h \Psi_h, \quad \widehat{\mathcal{E}}_l \Psi_l := -\hbar^2 \|\partial_l\|_{N_l}^2 \Psi_l/2 + V_l \Psi_l = E_l \Psi_l \quad \text{and} \quad (748)$$

$$\widehat{\underline{\mathcal{L}}}_h \Psi_h = \sum_{i'=1}^p \underline{h}_{i'} \times \{h/i\} \partial_h^{i'} \Psi_h = 0, \quad \widehat{\underline{\mathcal{L}}}_l \Psi_l = \sum_{i''=p+1}^n \underline{l}_{i''} \times \{h/i\} \partial_l^{i''} \Psi_l = 0. \quad (749)$$

though one may well wish to solve the h-part classically prior to quantization, in the style of Sec 13.

12.1.4 Analogy 74) Approximate emergent JBB time candidate and its operational significance

The expression (79) for emergent JBB time candidate is now

$$t^{\text{em(JBB)}} - t^{\text{em(JBB)}}(0) = \underset{\text{of } \mathbf{S}_{\text{JBB}}^{\text{ERPM}}}{\text{extremum } \underline{A}, \underline{B} \text{ of Eucl}(d)} \left(\int \sqrt{\frac{\|\underline{d}_{\underline{A}, \underline{B}} \mathbf{h}\|_{\mathbf{M}}^2 + \|\underline{d}_{\underline{A}, \underline{B}} \mathbf{l}\|_{\mathbf{M}}^2}{2\{E_h - V_h - V_l - J_{hl}\}}} \right). \quad (750)$$

In contrast, the approximate emergent JBB time candidate is

$$t_h^{\text{em(JBB)}} - t_h^{\text{em(JBB)}} = \underset{\text{of } \mathbf{S}_{\text{JBB}(h)}^{\text{ERPM}}}{\text{extremum } \underline{A}, \underline{B} \text{ of Eucl}(d)} \left(\int \|\underline{d}_{\underline{A}, \underline{B}} \mathbf{h}\|_{\mathbf{M}} / \sqrt{2\{E_h - V_h\}} \right), \quad (751)$$

for

$$\mathbf{S}_{\text{JBB}(h)}^{\text{ERPM}} = \sqrt{2} \int \|\underline{d}_{\underline{A}, \underline{B}} \mathbf{h}\|_{\mathbf{M}_h} \sqrt{\{E_h - V_h\}}. \quad (752)$$

I define $*^h = \partial/\partial t_h^{\text{em(JBB)}}$ as an approximation to use in the preceding SSec's equations:

$$P_{i''\mu}^l = m^{i''j''\mu\nu} *_{\underline{B}}^h l_{j''\nu}, \quad *_{\underline{A}, \underline{B}}^h P_{i''\mu}^l = -\partial\{V_l + J_{hl}\}/\partial l^{i''\mu}. \quad (753)$$

Moreover, one is driven into a curious indirect procedure in making such an approximation. I.e., one can not simply compare the sizes of the various energy terms, but must rather assess this at the level of the resulting force terms upon variation. That these two procedures are very different and that one must do the latter are well brought out by the example of the star-planet-neighbouring galaxy 3-body problem ('Earth-Sun-Andromeda'). In the potential, Andromeda being far is rather offset by Andromeda being massive (one is to compare $m_A/|q_A - q_S|$ and $m_S/|q_S - q_E|$, i.e. a ratio of m/r terms). However, at the level of the forces/equations of motion, the the equation for the earth-sun separation vector $\rho = q_S - q_E$ gives tidal terms for the Earth-Sun-Andromeda system:

$$\rho_{\text{ES}}^{\mu**} =: \rho_1^{\mu**} = -\delta_1^j \delta^{\mu\nu} \frac{\partial \mathcal{V}}{\partial \rho^{j\nu}} = \frac{\partial}{\partial \rho_1^\mu} \approx -\frac{m_E m_S}{\rho_{\text{ES}}^3} \rho_{\text{ES}}^\mu \left\{ 1 + \left\{ C_1 \frac{m_A}{m_E} + C_2 \frac{m_A}{m_S} \right\} \left\{ \frac{\rho_{\text{ES}}}{\rho_{\text{EA}}} \right\}^3 \right\} \quad (754)$$

(for some unimportant constants C_1 and C_2), so that the effect of Andromeda on the solar system comes in as a ratio of m/r^3 terms, so the fact that Andromeda is relatively very far away now very heavily outweighs the fact that Andromeda is relatively very massive.⁷⁵ And it is indeed neglecting Andromeda that fits right into our successful modelling of the solar system. I label this by JBB Limitation-1.

⁷⁵Physically, the potential due to Andromeda is felt by solar system objects, but it is felt *extremely evenly* by all of these objects; i.e. it is the relative 'tidal' effect between objects (ratio of m/r^3 terms) that is physically relevant, and this is extremely small for the case of Andromeda acting on the solar system.

12.1.5 Scale–shape h–l split of RPM’s

Later workings in this article require the scale is heavy and shape is light split version of the above. Then the approximations in 5.3.1 hold with the scale ρ playing the h role and the shapes S^a playing the l role.

Note 1) One can see in this case that the effect of h = size of the universe is not negligible like for h = Andromeda, since h = scale enters the force law in a distinct homogeneous way.

Note 2) The h = scale equation for the Jacobi vector formulation does not have any B -corrections. Thus the approximate time is now defined without ambiguity or need to successfully reduce. (This feature carries over to GR too, via $h_{\mu\nu} = a^2 u_{\mu\nu}$ leading to $\{d - \mathcal{L}_{dF}\}\{a^2 u_{\mu\nu}\} = a^2\{\{da/a\}u_{\mu\nu} + du_{\mu\nu} - D_\mu(\mu dF_\nu) + 0\} = a^2\{d - \mathcal{L}_{dF}\}u_{\mu\nu}$, the 0 arising from the constancy in space of the scalefactor-as-conformal-factor killing off the extra conformal connection; here D_μ is the covariant derivative associated with $u_{\mu\nu}$). By this observation, scale–shape split approximate JBB time (and approximate WKB time which coincides with it) avoids the Sandwich/Best Matching Problem.

12.1.6 A global problem with h–l approximations themselves

JBB Problem-2. It should be clear that in passing from the h–l approximation’s familiar Celestial and Molecular Physics domain’s flat mass metric to a curved configuration space metric that the h and l notions can become merely local in configuration space and thus well capable of breaking down over the course of a given motion. Thus use of h–l approximations in general entails a global-in-space problem; this holding equally for RPM’s and GR, it constitutes Analogy 75).

12.1.7 Extent to which emergent JBB time is provided by all of the contents of the universe

This is relevant as regards Mach’s ‘time is abstracted from change’ [Relationalism 6)] and Leibniz and Barbour’s [extreme form of] Relationalism 7): ‘the whole universe is a perfect clock – the only perfect clock’. However, again in practise in the subsystem-oriented h–l split formulation one neglects those terms which fail to contribute relative ‘tidal’ forces between parts of the subsystem under study. Thus the formula for emergent JBB time should *not* in this context be held to be embodying any way in which the ‘distant stars’ are non-negligibly contributing to the timestandard of a local quasi-isolated subsystem of interest such as the solar system. On the other hand, Note 1) above amounts to the scale–shape oriented h–l split being different in this regard. For mechanical models, this scale is the total moment of inertia of the universe, to which all constituent parts of the universe contribute. For GR, the situation is more delicate: the scalefactor is indeed now a single gravitational–geometric variable, but on the other hand the form it takes is determined by solving field equations that contain averaged matter density terms to which all of the contents of the universe contribute. In each case, the scale in question then gives rise to the approximate emergent JBB time.

As a separate critique of the extreme form of Relationalism 7) being implemented by $t^{\text{em(JBB)}}$, linear species’ kinetic terms do *not* enter the emergent JBB time (Sec 2.6.5). By this, the whole contents of the universe are *not* contributing (or, at least, not all contributing in the same way: quadratic species contribute to the potential and kinetic factors within the emergent time, whilst linear ones contribute only to the potential factor). This is significant at the level of GR coupled to fundamental matter fields, since it is precisely the fermions that are the principal constituents of the universe’s matter contents that have the linear kinetic terms that are excluded from entering the formula for $t^{\text{em(JBB)}}$ [15].

Relationalism 7M) [where the M stands for ‘middling’ as opposed to the weak form in the Conclusion, of LMB-A.] On the basis of the above, I advocate a less strong form of LMB Relationalism 7): that **careful modelling of one’s surroundings does produce a better timefunction to the extent of meeting all realistic expectations of accuracy in e.g. an island universe setting such as the solar system.**

Barbour then goes on to state that (see e.g. [88]) emergent JBB time is the ephemeris time. Discussing this requires first a general discussion of astronomical timestandards, of which ephemeris time is a particular concrete procedural example (or set of examples). This nonuniqueness in itself establishes that Barbour’s statement is at most qualitative/conceptual as opposed to precise/computational.

12.1.8 Astronomical timestandards and relationalism

Sidereal time is kept by the rotation of the Earth relative to the background of stars. This served as a timestandard for around 2000 years (its use dates back to Ptolemy), due to happening to be

1) stable enough by the Earth continuing to rotate with only small fluctuations (and the background of stars only changing very slowly).

2) Elsewise found to be a reliable timestandard in terms of which to make predictions about other celestial bodies (unlike various other long-standing astronomical timestandards).

Ephemeris time was an improvement on this dating from 1952 [177] necessitated by greater accuracy. It was an improvement by concerning a greater plurality of celestial mechanics motions, by which it is more Machian in an LMB sense, or, more precisely still, within the scope of LMB-A Relationalism 7M).

Note 1) This still did not consider relativistic effects, but more accurate successors to it do.

Note 2) Astronomical timestandards are rather less well-known nowadays because they have been superceded in one practical sense by **atomic clock based timestandards** in practise; however, astronomical timestandards *remain* important as per the argument below.

The main limitation with astronomical timestandards comes from the limited knowledge of accuracy of the contents of the solar system (e.g. the internal constitution of the Earth) causing various fluctuations depending on complicated and only partly-known specifics of the physics.

Atomic clocks are, in contrast, selected to have

- 1) well-known physics,
- 2) very stable physics,
- 3) to be fairly small so as to be easily shielded from disturbances and convenient to get a quick reading from (this is a far more convenient clock-hand than the relative position of the moon).

However, the use of atomic clocks is not a simple story; they are not a fully independent paradigm of timekeeping. Firstly, observe that marching in step with astronomical phenomena is a useful property for clocks to have for ‘everyday purposes’ (and astronomical predictions); this was much of why sidereal time was selected and retained as a good timestandard for two millennia. Secondly, the early atomic clocks were

- 4) found to march in step well with the astronomical ephemeris [444].

Without this there would have been a lot more difficulties with adopting such as alternative timestandards (very stable but ‘out of step’ with the quotidianity of solar system physics would at best constantly require use of conversion tables, and at worst render the proposal untenable). Thirdly, for all that one usually reads time off atomic clocks nowadays, one still has to recalibrate them, not only to allow for SR and GR effects concerning moving the clock around and its relation to other atomic clocks elsewhere, but also to maintain the desired correlation with solar system physics (see e.g. p4 of [565], p11 of [238], [177, 635]). This is e.g. illustrated by one of the two reasons behind leap seconds.

A) Less interestingly, the Earth day that is central for everyday convenience is not itself quite physically constant, by a combination of composite-body, frictional and tidal effects.

B) More interestingly, the atomic time does not quite march in step with (modernized versions of) astronomical times, and so needs to occasionally be recalibrated so as to match these.

In the event of a deviation, one can tell which of the clocks being out or the solar system not being understood is the case. For, if the clocks are out, they are similarly out as regards a wide range of different astronomical motions, so one can infer that it is far more likely the clocks’ time is slightly out of control rather than there being lots of unexpected internal composition changes etc in the solar system objects. Thus one is still ‘living’, conceptually and in terms of some small corrections in the recalibration within an astronomical type time scheme, within an astronomical timestandard of the LMB-A relational ‘time is abstracted from a sizeable amount of locally relevant change’ type. [For all that the method of most convenience for reading off the time itself has progressed from keeping track of the position of the moon to taking readings of an atomic clock.]

Now, making approximations at the level of the terms in the formula (750) will clearly fail to reproduce the particular details of the elaborate iteration process so as to fit the positions of the defining bodies roughly within the error bounds on their positions [167, 177, 634]. One could of course take the relational action for the solar system plus a large sum of external bodies, pass to the equations of motion, argue there for the negligibility of massive but distant objects, and then perform the usual ephemeris-time-type analysis for the remaining bodies. Note however that in this procedure the relational underpinning is doing nothing new for the actual physical calculation of the ephemeris time, whereas the *practical* knowledge of how to handle the solar system physics iteratively to set up the ephemeris, which was developed without ever any reference to relationalism, is fully used. Thus I do not feel that relationalism contributes anything *practical* to the astronomers’ ephemeris time (or any other more updated astronomical timestandard by iteration on positions of solar system bodies).

What relationalism does do, however, is provide an alternative philosophical underpinning by which the calculation can be said to rest on a relational rather than absolute view of the universe). This is also an opportune place to discuss a point recently raised by Lawrie [419]: “*Ordinarily, there seems to be a clear sense in which a well-constructed clock reads ‘10s’ seven seconds after it reads ‘3s’, and this does not appear merely to result from a conspiracy amongst the manufacturers of time-pieces*”.

The Newtonian answer to this would be that all good clocks march in step with absolute time, though this is not a practically-implementable selection procedure. A stronger answer to this would involve how these manufacturers’ professional antecedents separately agreed with *natural standards* rather than merely with each other. I.e. they had to produce *useful devices* where the measure of that usefulness was marching reasonably well in step with solar system phenomena such as night and day, where the sun appears in the sky from earth, the length of the seasons... Clock designs failing to march in step with these things would be deemed to be poor and selected against by people wishing to be able to keep successively more accurate appointments (from ‘meet you at nightfall’ to ‘meet you at 10:25 am Greenwich Mean Time’). Thus the agreements to reasonable approximation between all surviving types of clocks have a clear mechanism behind them.

Example 0) Even if the clockmaking observers had come from different civilizations which did not agree on the unit of tick-length ('second'), their clocks as selected and calibrated by the above procedures would have marched in proportion to each other, much as different civilizations' calendars allotted approximately the same number of days to each year whilst differing in choice of calendar year zero. In other words, even for 2 different isolated civilizations on Earth, one would expect reasonably advanced clockmaking and calendar design to come up with times of the form $Tt + C$ where the t marches in step with solar system physics and the T and C are the personal choices of each civilization. It is to this extent that clocks would march in step. C has no effect on clocks, and T is a number to be specified just once.

In the Newtonian set-up this might be tied to the preceding by some kind of postulation that the celestial bodies move in very close accord with the absolute time. In the LMB-A relational set-up, however, the Newtonian answer is supplanted by all good clocks marching in step with the emergent JBB time, which then, in the solar system, is to very good approximation the time by which the solar system's motions are simplest. The LMB-A relational set-up is then superior through not involving an extra metaphysical entity or a postulate about how the celestial spheres then happen to have motions aligned with that metaphysical entity. This is why the LMB-A relational set-up (and its idealized LMB relational antecedent) are conceptually important to the understanding of timekeeping, for all that they do not enter into most of the practicalities (and enter no differently from the Newtonian or SR-and-GR-corrected Newtonian type scheme's recalibrations to ensure non-departure from marching reasonably in step with astronomical timestandards).

12.1.9 The extent to which '3-geometry is the carrier of information about time' [71]

Having said that the JBB formula for the ephemeris time in practise does not contribute any faster-computation or improved-computation method for the actual practical evaluation of any ephemeris time like quantity, I remind the reader that a computation along those lines in the GR case for the lapse was also given as the last stage of BSW's formulation of geometrodynamics. Thus the above critique also suggests a number of corresponding limitations on the practical use of the 3-geometry as the 'carrier of information about time'. Additionally, this GR counterpart is in practise in general also hampered by a previous stage in that formulation requiring the solution of the thin-sandwich problem. Finally, the use of this formulation as the basis for an approach to quantum gravity is subjected to the same limitations as for the RPM case (by this Sec's opening paragraph's non-unfreezing point, for all that it does represent a classical derivation of what can be later recovered as an emergent WKB time).

12.1.10 'Marching in step' conventions for multiple observers

Barbour has suggested that using emergent JBB time (which he terms ephemeris time, but I argued above for that to be just a qualitative characterisation ensures that 'clocks march in step' [88]) for all observers in the universe. Barbour posits this criterion as one by which **independent observers would be able to 'keep appointments' with each other**; he argued furthermore that this criterion applies uniquely to this timefunction. He sketched an argument for this, however I question part of that argument, and this leaves me arguing partly that the posited result is true under certain circumstances to some level of precision, and partly that the posited result cannot be precisely true in general.

Firstly, model a universe containing two quasi-isolated island subsystems. Then Sec 12.1.4's approximations applied to each will ensure that the details of the other's contents will contribute negligibly. Thus each's timestandard constructed in an ephemeris-type fashion would be independent of the other's. Thus mechanical ephemeris time type constructions do not in practise by themselves appear to provide a common time standard.

There is a JBB timefunction that interpolates between the two (Fig 51). However, neither observer would know enough about the other's local solar system so as to be able to at all accurately compute that interpolation. So whilst it is not a practically constructible entity, the underlying interpolatory JBB time for the union of the two subsystems provides a cause of marching in step? The problem here is a lack of demonstrated mechanism as to how. (Barbour himself suggested [88] that T increasing would be offset by $E - V$ increasing due to $E = T + V$. However, detailed inspection of $E = T + V$ reveals that the T within is in terms of d/dt^{em} . Thus it so happens that the expression for t^{em} itself is a rearrangement of $E = T + V$, and would thus appear to be a formula within which $E = T + V$ itself cannot be re-used since that is not an independent equation but rather just another form of the self-same equation.) Thus I am left having to consider a number of examples where the posited result works, and a number of examples in which the result does not appear to work.

Example 1) Let us first consider ourselves using our moon as a clock whilst a Jovian uses their moons as a clock. Each could do that without knowing much at all about the other's moons. Each of us would reach this conclusion by noticing our moons to furnish reasonable timestandards with respect to which other physics is simple and reasonably accurately done. And in this setting, these moon clocks (allowing for other major solar system bodies, excluding each others' moons) march in step to very good accuracy! But, unlike in the case of 2 independent civilizations on Earth, this is lacking a why, as t_E^{em} and t_J^{em} are a priori not naturally the same (and in more ways than just tick-length and calendar year zero conventions that were the sole differences to good approximation for Example 0)'s two independent civilizations on Earth). One could in this case argue instead for universality in Keplerian physics. It is also far from clear whether independent observers within the solar system would hit upon exactly the same procedure or required accuracy standard, so Barbour's 'the ephemeris' does have

to be taken with a pinch of salt as regards precision and nonuniqueness. This ambiguity can be taken into account by more comprehensive rephrasings: “similar kinds of astronomical timestandards” in place of “the ephemeris”, alongside(my caps added) **being able to keep appointments TO REASONABLE ACCURACY.**

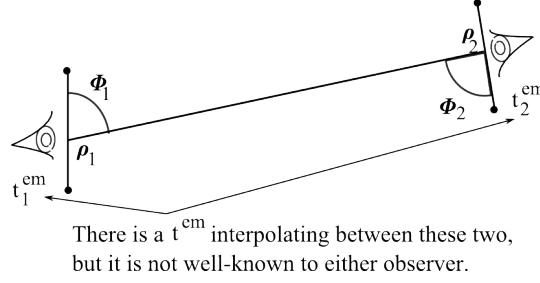


Figure 51: Approximate emergent time in 2-island universe model. In each system, the other is far away enough to play but an Andromeda-like role and thus be ignorable.

Example 2) **Coping with non-celestial mechanics users.** One needs to be more general in e.g. setting up appointments e.g. with our Venerian sisters who have never observed any celestial mechanics due to Venus always being immersed in thick cloud. Note that the ephemeris-type procedure would still apply as regards what set of most reliable things were available (eg rest-pulses and pendulums and the fairly regular variations over time in the surface temperature of a cloud-covered planet), in the sense of an iterative procedure to find a widely-applicable simplifying time.

Example 3) Why should iterated times that are each simplest for most of the motions arising in 2 separate places march in step with each other. That one might come across an abandoned warehouse populated by a batch of commonly-defective clocks suffices to doubt that general agreement among phenomena in one location will produce a timestandard marching in step with that obtained by applying the same kind of iterated ‘time is abstracted from change’ procedure elsewhere.

Example 4) Might it be that what is locally simplest or locally common may differ in some parts of the universe? (A chaotic planetary system around a stellar cluster gives non-Keplerian motion; can we be sure of what kind of timestandard is locally simplest then?)

Example 5) In the case of the **two observers partly sharing observed subsystems**, each’s clock will keep roughly in step with the shared subsystems and thus roughly in step with each other. Pulsars could well often furnish shared observable subsystems.

Example 6) One possible path out of not having any sharedly-observable is that one would however expect observers living within two different such quasi-isolated subsystems to be able to make similar large-scale cosmological observations, on which basis they may well develop very similar notions of cosmic time. [This does partly rely on the Copernican principle holding, at least to good approximation. It is also tied to the practical appropriateness of the scale = h identification in the kind of universe that we live in, and Note 1) of 12.1.3’s observation of the difference between scale = h and ‘some subsystem = h’.] And emergent JBB time for GR in the cosmological setting is aligned with cosmic time, so JBB time per se continues to be the right concept, it is just that one needs to take care as to what one designates to be h and l. [This is not a shift from Newtonian gravity to GR on scales a few orders of magnitude larger than planetary systems? Rather, it is a shift to scale–shape alignment, regardless of whether one’s cosmological model is Newtonian, an RPM or generally-relativistic. After all, by Appendix 5.C.3 these are all good approximate descriptions of the overall late universe in the absence of such as microlensing issues; in particular, each possesses a satisfactory notion of cosmic time. Though it should then be said that, with current-Earth technology, cosmic time cannot be set up to be used as an *accurate* timestandard! It is also not guaranteed that 2 independent civilizations will single out cosmic time as the most natural time variable for cosmology and inter-stellar appointment keeping. For these reasons, I do not in the end view this alternative as being as promising as might appear at first sight.

A different line of thought [413] is that **‘marching in step’ is at least very closely akin to a synchronization procedure** (or, even more close to the mark, a maintenance of synchronization for spatially separated observers in relative motion with respect to each other). Unfortunately then relativity tells us that clocks in different places in relative motion with respect to each other do not stay synchronized. This counts as an argument particularly against the automatic maintenance of ‘marching in step’, and, as a corollary, of the ‘marching in step’ not being perfectly realizable. However, there are counterarguments at least against most such effects. Firstly, many SR and GR effects are local to the clock and would thus be factored into the sufficiently accurate ephemeris-like procedure by which each observer’s timestandard is established and occasionally recalibrated to suit. Secondly, if there is a shared observed subsystem, then each observer’s ephemeris-type procedure would remain (recalibrating if necessary) roughly in step with this, and thus roughly in step with each other. This argument suggests capacity for even mutual relativistic effects to be factored in by each observer applying an ephemeris-type procedure, though again this depends on there being a shared observed subsystem. Aside from this sharing not always holding, another hole in the argument is that keeping in time with something depends on the position and motion of the observer in GR (compare comoving with a particle falling into a black hole with watching this happening from infinity: one process looks to occur in finite time and the other to take infinite time). All in all, this suggests that ephemeris-like procedures’ tendency to produce

marching in step should be checked in individual detail against a long list of specific Newtonian, SR and GR effects.

I continue this SSsec's discussion in Sec 16.3 by abstracting a notion of generalized local ephemeris procedure from it.

12.1.11 Some consequences of there being a PPSCT-related family of emergent times

I consider this for the case of finite degrees of freedom and trivially spatially relational case (i.e. mechanics with temporal relationalism only, encompassing e.g. fully reduced 1- or 2- d relational particle mechanics and minisuperspace).

Analogy 76) I take the below to comprise this.

Here, the evolution equations are the parageodesic equation

$$D_{\text{abs}}^2 Q^A / D\vec{t}^2 := Q^{A**} + \Gamma^A_{BC} Q^{B*} Q^{C*} = \partial^A W, \quad (755)$$

which would be the geodesic equation were the right-hand-side term 0. This is 0 (previously termed Simplification A) if:

Case 1: if W is prescribed as a constant. This is the case in mechanics if V is constant and in (for the moment) minisuperspace GR if $\text{Ric}(h)$ is constant.

Case 2: if not, use the PPSCT with $\Omega^2 = kW$ for k constant so $T \rightarrow \bar{T} = kWT$ and $W \rightarrow \bar{W} = W/kW = 1/k$. This corresponds to obtaining an action $\mathbf{S} \propto \int d\tilde{s}$ and passing from $t^{\text{em(JBB)}}$ to $t^{\text{em(aff-geo)}}$. Moreover, Case 2 has range of validity caveats (Sec 2.12.5) for regions containing zeros, poles or non-smoothnesses of W as the conformal transformation's definition and invertibility preclude these. In the minisuperspace case, this SSsec's contents were spelled out by Misner [464] following more partial mention in earlier work of DeWitt [203]. The mechanics counterpart of this is in e.g. [415, 79].

Individually, both conformal transformation and non-affine parametrization complicate the equations of motion. How exactly then does it come to work out that the equations of motion are *preserved* when *both* of these transformations are applied together? This is answered by looking at a different, longer proof of Appendix 2.B.2's point 5) at the level of the equations of motion themselves. By

$$\bar{\Gamma}^A_{BC} = \Gamma^A_{BC} + \{2\delta^A_{(B}\partial_{C)}\Omega - M_{BC}\partial^A\Omega\}/\Omega,$$

symmetry and the definition of $T_{\vec{t}}$ in terms of velocities with respect to \vec{t} ,

$$\bar{\Gamma}^A_{BC} Q^{B*} Q^{C*} = \Omega^{-4} \Gamma^A_{BC} Q^{B*} Q^{C*} + 2\Omega^{-5} \{\partial_B \Omega Q^{B*} Q^{C*} - T_{\vec{t}} \partial^A \Omega\},$$

which, then, alongside using obvious product rule expressions for $\{\Omega^{-2} Q^{A*}\}^*$ and $\partial^A \{\Omega^{-2} W\}$ gives

$$0 = Q^{A**} + \bar{\Gamma}^A_{CB} Q^{B*} Q^{C*} - \bar{\partial}^A \bar{W} = \Omega^{-4} \{Q^{A**} + \Gamma^A_{CB} Q^{B*} Q^{C*} - \partial^A W\} + 2\Omega^{-5} \{\partial_B \Omega Q^{B*} Q^{A*} - \Omega^* Q^{A*} + \{W - T_{\vec{t}}\} \partial^A \Omega\}. \quad (756)$$

Then the second big bracket vanishes by the second term plus use of the chain-rule cancelling with the first term, whilst $W - T_{\vec{t}} = 0$ is a universal 'energy statement', e.g. for mechanics it is $E - V - T_{\vec{t}} = 0$ (conservation of energy), or, likewise, for (for the moment minisuperspace) GR, $W - T_{\text{GR}} = \text{Ric}(h) - 2\Lambda - T_{\text{GR}} = 0$ (Lagrangian form of the Hamiltonian constraint).

It is worthwhile to further analyze the above in terms of non-affine parametrization and conformal transformation sub-workings. The second term in the second large bracket is the result of non-affine parametrization. It cancels with the first term, which is the first of two complicating terms from the conformal transformation. The second such is the $T_{\vec{t}}$, which itself cancels with the PPSCT's compensatory conformal scaling of W by the 'energy statement'.

Note 1) The above working is thus an interesting configuration space generalization of the result by which null geodesics conformally map to null geodesics (see e.g. [622]). There, the first conformal complication is balanced by a change in what constitutes a suitable affine parametrization, while the second one vanishes by the geodesic being null with respect to the indefinite spacetime metric. For us, the nullness condition is replaced by the positive-definite kinetic term, which then cancels off with the PPSCT's compensatory scaling of the potential factor W via the 'energy statement'. Thus 'in indefinite spaces null geodesics conformal-map to null geodesics' becomes 'in configuration spaces of whatever signature, paths of motion PPSCT-map to paths of motion'.

Note 2) To cover also RPM's with non-trivial spatial relationalism, hang g's on the above D and $*$'s and the and then the preceding analysis carries through. To the extent that the previous paths of motion were geodesics, the current paths are so to, provided that the constituent configurations are suitably aligned by auxiliary \mathfrak{g} -transformations.

Consequence 1) Solely non-affinely parametrizing or solely rescaling the kinetic metric complicate the equations of motion away from the simple geodesic equation form that using the naïve (PPSCT-unaware) emergent JBB time places them in. However, performing both of these operations alongside the compensating W rescaling preserves this simple form, choice of emergent time indeed being nonunique up to this PPSCT freedom. Thus, if one's problem requires rescaling or non-affinely parametrizing, one's problem may be sufficiently unrestricted so as to permit one to 'complete' the required transformation to a 3-part conformal transformation. By this, the effect of solely rescaling or solely non-affinely parametrizing causing

departure from the simple geodesic equation form is circumvented. Thus emergent time derivative's being a PPST-covector provides a *robustness* result for the property of providing simple equations of motion.

Consequence 2) Since PPST's are very natural from the relational product-type parageodesic action principle as per Appendix 2.B, in the relational context these might be viewed as primary. Thus the relationalist may view nonaffine reparametrization as a consequence of a very simple property of the form of the relational action.

Note 3) Preservation under 3-part conformal transformation means that $\tilde{\tau}$ corresponding to *any* Ω carries out simplification A). One can then pick $\Omega^2 = kW$ so that $\tilde{\nabla}^A \tilde{W} = \tilde{\nabla}^A k = 0$, and then $\tilde{*} = \Omega^{-2*} = \{kW^{-1}\}*$, i.e. so that simplification B) *also* holds. Thus one has obtained a passage from physics with a restricted class of affine parameters under which the equations of motion take the simple geodesic equation form to physics with a restricted class of PPST factors under which the equations of motion take geodesic form.

Note 4) The construct in Note 3) is nonunique up to the usual calendar year zero and tick-duration freedoms.

Consequence 3) Affine transformations send \mathbf{t}_{old} to $\mathbf{t}_{\text{new}}(\mathbf{t}_{\text{old}})$ subject to

I) nonfreezing and monotonicity, so $d\mathbf{t}_{\text{new}}/d\mathbf{t}_{\text{old}} > 0$ which can be encoded by having it be a square of a quantity F with no zeros in the region of use, and

II) this derivative and hence F being a physically-reasonable function (to stop the transition damaging the equations of motion). But this can be recast as $d/d\mathbf{t}_{\text{new}} = F^{-2}d/d\mathbf{t}_{\text{old}}$, by which (and other properties matching⁷⁶) one is free to identify this F with Ω , so any affine transformation is of a form that extends to a PPST. If one then chooses to 'complete' it to a 3-part conformal transformation, the above calculation can be interpreted as the extra non-affine term being traded for a T term. This is by having an accompanying conformal transformation of the kinetic metric, and then this being traded for $\nabla^A W$ by energy conservation and the compensating transformation of W . Thus the freedom to affinely-transform the geodesic equation on configuration space can be viewed instead as the freedom to PPST a system's equation of motion. *Thus the relational approach's simplicity notion for equations of motion has the same mathematical content as prescribing an affine rather than non-affine parameter for the geodesic equation on configuration space.* Thus the PPST-related $\tilde{\tau}$ corresponds to 'the set of (generally) nonaffine parameters for the geodesic-like equation of motion on configuration space'. (However, each is paired with a different, conformally-related \mathbf{M} and W . On, the other hand, $\mathbf{t}^{\text{em}(\text{aff-geo})}$ indeed remains identified with the much more restricted set (unique up to a multiplicative constant time-scale and an additive constant calendar year zero) of affine parameters for the geodesic equation on configuration space.

Consequence 4) Thus part of the argument for emergent time being fixed by the universe's contents" [79] is lost as it is revealed to contain an arbitrary factor. The interpretation of this is as follows. The true physics is in PPST-invariant

$\{\mathbf{M}, W, \mathbf{t}\}$ triples. These are equivalence classes of systems: a conformally-transformed system has a conformally-transformed time for which the one representative path is obtained. On the other hand, PPST-invariant $\{\mathbf{M}, W\}$ pairs already do this without having to involve a \mathbf{t} , so the uniqueness claim in [79] is in fact OK from this perspective? But each $\{\mathbf{M}, W\}$ choice is the same system... Also, the resultant path is a $\{\mathbf{M}, W, \mathbf{t}\}$ triple which itself is PPST-invariant, so the \mathbf{t} 's are indeed nonunique, but only ever are interpreted within a $\{\mathbf{M}, W, \mathbf{t}\}$ triple which is unique.

Consequence 5) (A rephrasing of Consequence 4) As originally presented in the literature, emergent JBB time looks unique up to choice of scale and of time origin, in contrast to Newtonian time's additional freedom of affine reparametrization. However, emergent JBB time can readily be interpreted as nonunique under PPST's, and, after some algebra, this amounts to the same freedoms as affine reparametrization.

Consequence 6) It is then a useful probe for any purported arguments for the uniqueness of emergent JBB time as to whether PPST-related versions of that argument in fact also hold, thus sidestepping that uniqueness. This is of some concern since much of the usual Newtonian picture's freedom of choice of time variable (non-affine parametrizations) was then hidden for many years, via the above small (but not so easy to spot) piece of algebra, within the less familiar guise of the PPST's/3-part conformal transformations.

12.2 Hidden dilational time candidates in scaled RPM

12.2.1 More on candidate scale and dilational times

One place to seek for an internal time for RPM's is among the theories' natural scalars such as the moment of inertia scale I or the dilational quantity \mathcal{D} [20]. This of course entails the usual belief in canonical transformations. Before proceeding with this discussion, I note the following ambiguity.

Ambiguity: there is a whole family of scale variables and therefore of dilational conjugates to them. This not the obvious fact that positive functions of a scale are also scales. It is rather that, once the scales themselves are dismissed as candidate times due to their non-monotonicity in recollapsing or bouncing models and one passes on to consideration of their conjugates as time candidates, then the original choice of scale variable turns out to nontrivially affects the internal time approach's procedure for solving the Hamiltonian constraint/energy equation. (This will be illustrated below by an example, and can be envisaged as a further aspect of the Multiple Choice Problem).

⁷⁶Though conventional affine transformations have rather less differentiability than is usually assumed of conformal transformations (C^1 to C^∞).

Note: this relies on accepting a nontrivial (i.e. \mathbf{Q}^A , \mathbf{P}_A mixing) canonical transformation. Thus the approach requires use of (Phase, Can) rather than (\mathbf{q} , Point) or (RigPhase, Point).

In detail, the passage to using the dilational quantity conjugate to one's scale as a coordinate that is then promoted to a candidate timefunction is underlied by a simple canonical transformation by which the dilational quantity conjugate to the scale becomes a coordinate to be identified as a candidate time \mathbf{t}^{dil} , whilst the scale itself becomes $-P_{\mathbf{t}^{\text{dil}}}$. Now a multiplicity of possibilities for the scale variable itself reflects itself as a multiplicity of such dilational quantities conjugate to each of these scales.

In RPM's, the in some ways simplest such dilational quantity is \mathcal{D} , which in this context I termed the *Euler time candidate*, t^{Euler} . This is conjugate to $\ln \rho$,

$$\{\ln \rho, \mathcal{D}\} = 1 . \quad (757)$$

which reveals that the aforementioned analogy between the Euler time candidate and GR's York time candidate ([20, 22] Secs 1 and 2) is somewhat loose, by virtue of the above ambiguity. Moreover, this observation paves the way to substantially more accurate RPM-GR analogies (Sec 12.2.3).

More generally, consider $f(\rho)$ as a scale variable. Then

$$\{f, \mathcal{D}/L_D f\} = 1 , \quad (758)$$

for L_D the linear dilational operator $\rho \partial_\rho$. Likewise (useful in Sec 12.2.2) for $\mathcal{F}(\mathbf{I})$ as a scale variable,

$$\{\mathcal{F}, \mathcal{D}/L_D f\} = 1 , \quad (759)$$

for L_D now written in the form $2\mathbf{I} \partial_{\mathbf{I}}$.

Some simple examples then are as follows.

Example 1) $f'(\rho)\rho = 1$ gives back $f = \ln \rho$ being conjugate to \mathcal{D} itself. By this simpleness and that in the next SSSec, I term $\ln \rho$ the *subsequently simplest scale*.

Example 2) $f = \rho$ (configuration space radius scale), the conjugate of which is \mathcal{D}/ρ i.e. indeed just p_ρ : radial dilational time candidate t_ρ .

Example 3) $f = \mathbf{I}$ (moment of inertia scale), the conjugate of which is $\mathcal{D}/2$: MOI dilational time candidate $t_{\mathbf{I}}$.

Example 4) $f = 1/\rho := v$ (*reciprocal radius*), the conjugate of which is $-\rho \mathcal{D}$: reciprocal radius dilational time candidate, t_v .

Logarithmic impasse with Euler time candidate itself. This case's $\ln \rho$ case gives a part-linear form with the structure $-P_{t^{\text{Euler}}} = \ln(F(P_a^S, S^a, t^{\text{Euler}}))$ and this logarithm in the 'true Hamiltonian' then furthers the operator ordering ambiguities and troublesomeness in making rigorous the functional analysis underpinning the form of the candidate true Hamiltonian operator at the quantum level.

This is apparent in the workings in [22] and Example 2) of Sec 12.2.4 but is very readily bypassable by the above ambiguity.

12.2.2 Analysis of monotonicity: Lagrange-Jacobi equation and a generalization

For a time candidate to be satisfactory, it is important that this is monotonic. For the Euler time candidate $\mathcal{D} = t^{\text{Euler}}$, this follows [20] for a number of substantial cases (see below) from the Lagrange-Jacobi identity [550] (viewing $*$ as d/dt^{Newton} in this context)

$$t^{\text{Euler}*} = \mathbf{I}^{**}/2 = 2T - nV = 2E - \{n + 2\}V , \quad (760)$$

for V homogeneous of degree n . Sums of homogeneous potentials all of which obey a common index inequality also satisfy monotonicity. Interpret n in that way from now on.

This provides a fairly strong guarantee that t^{Euler} is monotonic: it is so in a number substantial sectors.

Sector 1 = $\{E \geq 0, V \geq 0, n \leq -2\}$ and Sector 2 = $\{E \geq 0, V \leq 0, n \geq -2\}$ give

$$t^{\text{Euler}*} \geq 0 . \quad (761)$$

Sector 3 = $\{E \leq 0, V \leq 0, n \leq -2\}$ and Sector 4 = $\{E \leq 0, V \geq 0, n \geq -2\}$ give, using $-t^{\text{Euler}}$ as timefunction instead,

$$-t^{\text{Euler}*} \geq 0 . \quad (762)$$

One can immediately check against Secs 4 and 5 for which models lie within the above sectors. Free models do, whilst HO's do not (not unexpected for oscillators).

The question then is whether other dilational time candidates are anyway near as good in this respect (particularly due to the above 'logarithmic impasse' with the Euler time candidate that is absent from dilational candidate times that

are conjugate to whichever power of ρ . I proceed by generalizing the Lagrange–Jacobi equation viewed as the Euler time candidate propagation equation to the following general dilational time candidate propagation equation:

$$t_{\mathcal{F}}^* = \{GI^*\}^* = I^{**}G(I) + G'\{I^*\}^2 = 2\{2E - \{n + 2\}V\}G + 4\mathcal{D}^2G' , \quad (763)$$

for $G(I) = 1/2L_D\mathcal{F}$. Thus if G and G' are the same sign, there is monotonicity in Sectors 1 and 2. On the other hand, if G and G' are of opposite signs, there is monotonicity in sectors 3 and 4. The first of these happens for $\mathcal{F} = I^\alpha$ for $\alpha < 0$ and the second for $\alpha > 0$. This makes $v = 1/\rho$, and minus its conjugate t^ν as a candidate time, useful below.

12.2.3 GR counterpart

Analogy 77) The subsequently simplest scale $2\ln a = \text{Mis}$, a Misner variable. This has the simplest dilational conjugate,

$$\{2\ln a, \pi\} = 1 . \quad (764)$$

Then canonically transform such that

$$t^{\text{sss}} = \pi , \quad p_{t^{\text{sss}}} = -\text{Mis} . \quad (765)$$

Analogy 78) This has the simplest t -propagation equation,

$$\circ_F t^{\text{sss}} = 2\sqrt{h}\{\dot{I}\text{Ric}(h) - \Delta\dot{I}\} . \quad (766)$$

To maximally align this with Sec 12.2.2, rewrite the above as

$$*_F t^{\text{sss}} = 2\sqrt{h}\{\text{Ric}(h) - \{\Delta dI\}/dI\} \quad (767)$$

(noting that a derivative-of-the-instant term has become trapped inside derivatives by an integration by parts move). This is then an equation in a^{**} just as the Lagrange–Jacobi equation is in terms of I^{**} : both are equations for double derivatives of a scale variable. Moreover (766) is the trace of the GR evolution equation, so the cleanest identification of the analogy is between Raychaudhuri and Lagrange–Jacobi equations. [These are the dilational time propagation equations corresponding to each theory’s subsequently simplest scale.]

Analogy 79) Each then constitutes a guarantee of monotonicity in certain cases. A simple such case for GR is closed minisuperspace (for which scale variables themselves fail to be monotonic):

$$\circ_F t^{\text{sss}} = 2\sqrt{h}\dot{I}\text{Ric} > 0 \quad (768)$$

(since $\dot{I} > 0$ by the definition of the velocity of the instant, $\sqrt{h} > 0$ by nondegeneracy and $\text{Ric} > 0$ for such closed models.

Analogy 80) Then for other scales $f(h)$,

$$\{f(h), 2G(h)\pi\} = 1 \quad (769)$$

for $G(h) = 1/L_D f(h)$ and linear dilational operator $L_D = a\partial_a = 6h\partial_h$.

Analogy 81) Then upon passing by canonical transformation to $t^{\text{dil}} = 2G(h)\pi$, $p_{t^{\text{dil}}} = -f(h)$, one has the generalized dilational time candidate propagation equation

$$\circ_F t^{\text{dil}} = 2\sqrt{h}\{\dot{I}\text{Ric}(h) - \Delta\dot{I}\}G - G'\dot{I}\pi^2/\sqrt{h} . \quad (770)$$

This is analogous to the generalized Lagrange–Jacobi equation of mechanics, as is brought out most clearly by casting it in the form

$$*_F t^{\text{dil}} = 2\sqrt{h}\{\text{Ric}(h) - \{\Delta dI\}/dI\}G - G'\pi^2/\sqrt{h} . \quad (771)$$

It retains monotonicity in the above closed minisuperspace context if G and G' are of opposite signs (i.e. like in Sectors 3 and 4 of the mechanical counterpart). For $f(h) = h^k$, $k > 0$ guarantees this.

A notable example among these is then $f(h) = h^{1/2}$. Then $t_{\text{dil}} = 2\pi/3\sqrt{h} = t_{\text{York}}$, for which

$$\circ_F t^{\text{York}} = \frac{4}{3} \left\{ \dot{I}\text{Ric}(h) - \Delta\dot{I} + \frac{\dot{I}\pi^2}{4\sqrt{h}} \right\} , \quad (772)$$

which is recognizable as the CMC VOTIFE analogue of the CMC LFE (163). The most relational form for this is the CMC DOTIFE

$$*_F t^{\text{York}} = \frac{4}{3} \left\{ \text{Ric}(h) - \frac{\Delta dI}{dI} + \frac{\pi^2}{4\sqrt{h}} \right\} . \quad (773)$$

Then t^{York} is known to have better monotonicity guarantees than in just the above closed minisuperspace example [652]. Moreover, for GR this particularly good monotonicity case is unhampered by the particular obstruction of the logarithmic

impasse (which falls, rather, upon the conjugate of the above-mentioned Misner variable. Note that this does *not* in general march in step with t^{em} .

Question. Is the York time uniquely privileged among the family of times in question as regards having particularly good monotonicity properties?

Analogy 81) The closest parallel to GR's York time is the conjugate of the pseudo-volume scale. This has the misfortune of being particle number-and-dimension dependent (though that is no worse than n-dimensional GR having a different expression for York time for each dimension). Here $f(\rho) = \rho^{\dim(\mathcal{R}(N,d))}$, so the conjugate pseudo-York time is $\mathcal{D}/\dim(\mathcal{R}(N,d))\rho^{\dim(\mathcal{R}(N,d))} = P_\rho/\dim(\mathcal{R}(N,d))\rho^{\dim(\mathcal{R}(N,d))-1}$.

The idea is then that $\text{Quad} = 0$ is to be interpreted as equation for P_t .

Difference 33) For RPM's, one has a simpler equation to solve than in GR to obtain a true Hamiltonian – an algebraic equation of the form

$$[\text{rational polynomial}](\text{scale}) = 0 \quad (774)$$

[e.g. eq. 777 and explicitly excluding the subsequently problematic logarithmic case] as opposed to the form

$$[\text{rational polynomial}](\text{scale}) = \Delta \text{scale}. \quad (775)$$

of the than the quasilinear elliptic Lichnerowicz equation (161) of GR. In each case, scale is interpreted as $-P_t$, so that solving for it is indeed now linearly isolating one of the momenta as per Sec 11.5.3. This simpleness permits progress past where the general GR case gets stuck. Which scale variable produces the most palatable powers for algebraic solution. However, plenty of other difficulties and absurdities then become apparent, casting further doubt over the sensibleness of internal time programs (see the next SSSec).

12.2.4 3-stop metroland examples of difficult and likely inequivalent Schrödinger equations

These suffice to establish the desired features.

Example 1) I use the conjugate to reciprocal radius v as a time since this suits the combination of physical relevance and monotonicity. [The free case at the classical level requires $E > 0$ and thus a same-sign monotonicity sector, which, in the case of the scale being a power of the configuration space radius, requires that power to be negative. Then we have

$$t_v^2 p_{t_v}^4 + p_\varphi^2 p_{t_v}^2 - 2E = 0, \quad (776)$$

which is solved by

$$p_{t_v} = \pm \sqrt{\frac{-p_\varphi^2 \pm \sqrt{p_\varphi^2 + 8Et_v^2}}{2t_v^2}}. \quad (777)$$

The inner sign needs to be '+' for classical consistency, and I argue in Sec 12.2.5 for the outer sign to be '-' as regards approximate recovery of a close-to-conventional QM. (777) is then promoted to the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t_v} = -\sqrt{\frac{\hbar^2 \partial^2 / \partial \varphi^2 + \sqrt{\hbar^4 \partial^4 / \partial \varphi^4 + 8Et_v^2}}{2t_v^2}} \Psi \quad (778)$$

unambiguously orderwise since the sole constituents p_φ and t_v commute.

Example 2) On the other hand, working with Euler time,

$$t_{\text{Euler}}^2 + p_\varphi^2 = 2E \exp(-2p_{t_{\text{Euler}}}) \quad (779)$$

which is solved by

$$p_{t_{\text{Euler}}} = -\frac{1}{2} \ln \left(\frac{t_{\text{Euler}}^2 + p_\varphi^2}{2E} \right) \quad (780)$$

This is then likewise unambiguously promoted to the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t_{\text{Euler}}} = \frac{1}{2} \ln \left(\frac{t_{\text{Euler}}^2 - \hbar^2 \partial^2 / \partial \varphi^2}{2E} \right) \Psi. \quad (781)$$

Having this pair of approaches leading to such unusual and disparate equations illustrates that it does appear to make a big difference what one chooses as scale (and neither looks anything like the exactly soluble form (or the semiclassical form, c.f. the next SSSec). Thus, as claimed in Sec 12.2.1 the ambiguity in defining scales is nontrivial.

12.2.5 An approximate approach to internal time

One can apply the method of approximating in a series at the classical level and only then promoting the outcome of that to quantum operators (and Sec 12), which are then rather better defined and less ambiguous. This has parallels to the treatment of relativistic wave equations (done before in the semiclassical approach but not as far as I know for the hidden time approach). Suppose then that we expand in powers of p_φ prior to promotion to operators (which is close to expanding in powers of \hbar and thus is semiclassical-like and likely to benefit from lessons from approximations made in the study of relativistic QM). Then, for the given choice of sign,

$$i\hbar \frac{\partial \Psi}{\partial t_v} = -\frac{\hbar^2}{2} \frac{\partial^2 \Psi}{\partial \varphi^2} - \frac{\{2E\}^{1/4}}{\sqrt{t_v}} \Psi \quad (782)$$

[+ $O(\hbar^4)$ times a 4th order derivative] which is an ordinary time-dependent Schrödinger equation with a particular t -dependent potential and, moreover, soluble to give

$$\Psi \propto \exp(i \, d \, \varphi) \exp \left(i \hbar t_v \left\{ \frac{d^2}{2} + \frac{2}{\hbar^2 \sqrt{t}} \{2E\}^{1/4} \right\} \right) \quad (783)$$

which for t big takes the form of a free wave, with some distortion for t small.

The Euler time one gives, with respect to a *rectified time* (i.e. a redefinition by which the time-dependent Schrödinger equation takes on a simplified form by the time absorbing prefactors of the configuration variables' derivatives) $\mathcal{T} = \int dt/2Et_{\text{Euler}}^2 = -1/2Et_{\text{Euler}}$

$$i\hbar \frac{\partial \Psi}{\partial \mathcal{T}} = -\frac{\hbar^2}{2} \frac{\partial^2 \Psi}{\partial \varphi^2} - \frac{1}{4E \mathcal{T}^2} \ln(8E^3 \mathcal{T}^2), \quad (784)$$

which I can also write a solution for:

$$\Psi \propto \exp(i \, d \, \varphi) \exp \left(i \left\{ \frac{\hbar d^2}{4Et_{\text{Euler}}} + \frac{t_{\text{Euler}}}{2\hbar} \{ \ln(8E^3) + 2 - 2 \ln(-2 E t_{\text{Euler}}) \} \right\} \right). \quad (785)$$

By examining the approximate approach, somewhat more standard mathematics appears.

Time-dependent potentials are hard to interpret here; such are usually held to correspond to non-conservative physics, but, unlike in that setting, the model is a whole-universe so there is nothing for energy to dissipate into.

12.2.6 RPM examples of operator-ordering and well definedness problems with internal times

These examples need the presence of shape potentials to confer operator-ordering ambiguity. An upside-down HO example will do (so $A < 0$). $E > 0$, $k = -2 < 2$, $V < 0$. v -time then does nicely. Then

$$p_{t_v} = \pm \sqrt{\frac{p_\varphi^2 + 2\{A + B \cos 2\varphi\} \pm \sqrt{\{p_\varphi^2 + 2\{A + B \cos 2\varphi\}\}^2 + 8Et_v^2}}{2E}}, \quad (786)$$

which clearly has $p_\varphi f(\varphi)$ products [specifically $p_\varphi \cos 2\varphi$ ones] causing nontrivial operator-ordering ambiguities. Also in this example not everything under square roots is positive, so one is away from nice guarantees of being able to promote such to an operator. Note: there is a spectral theorem result for well-definedness of positive combinations under square root signs, but in the case of Example 2) above, I do not know of a guarantee that this Hamiltonian is well-defined.

Thus, well-definedness and operator-ordering issues are present in the dilational time approach.

12.2.7 End-check against hidden time's problems

Hidden Problem-1 is avoided since for RPM's the canonical transformation in question is simple.

Hidden Problem-2: the Multiple Choice Problem is bolstered by the significant ambiguity arising from the choice of scale variable.

Hidden Problem-3: some aspects of the Global Problem of Time are present. E.g. monotonicity can fail to be global in time [though there are a number of significant sectors in which one is protected from that by as per Sec 12.2.2]. But, being a finite theory, the 'global in space' issues are absent. Finally,

Difference 34) The Torre impasse is absent for this article's 1- and 2- d RPM's. This is due to these not having any configuration space stratification.

Analogy 83) However, this argument is then clearly bypassed for 3- d RPM's.

Analogy 84) [A further Hidden Problem-6]. Having looked at a number of toy model examples, it is questionable whether the above conceptually-standard quantization procedure is can be done in practise. For, even in the absence of the above constructibility impasse for York time (for toy models such as RPM's, strong gravity (see this Sec's Appendix) and some

minisuperspace models, for which the Lichnerowicz–York equation is replaced by a solvable algebraic equation) at least some cases would appear to have [372, 22] well-definedness, negative-probability issues, all of which is tangled up with operator-ordering ambiguities too. This was already noted by Blyth and Isham [128] in the minisuperspace arena. Additionally, from my work above, I also note that the internal approach’s equations do not look anything like the equations encountered in the various conventional approaches to the same problem which are available for comparison for these RPM’s.

Since they are specifically spacetime issues, Problems Hidden Problem-4 and 5 are absent for RPM’s.

There is no a priori time in accord with Relationalism 4) , but a time is found be rearrangement. Not clear if doing so is much of an abstraction from things it is in the sense that it is among the configuration variables and those are material things (compliance with Relationalism 6) but it certainly is not from the strongest Leibniz–Barbour perspective since it is but a small amount of the universe degrees of freedom that go into it (Relationalism 7 non-compliance) and it does not have the more sensible Relationalism 7M compliance of getting better for larger subsystems even if one does not need to take this to the extreme of including the whole universe. It is also an interesting counterpoint to Leibniz for there to exist formulations with an extraneous time that are related to some with no such thing by canonical transformation. So rooting out an apparent extraneity may not always be the ‘right’ answer (particularly if canonical transformations are, as standardly, held to be allowed).

12.3 An analogue of matter time (Analogy 85)

Consider for instance scaled triangleland in the subsystem-centric parabolic coordinates for a fixed relative angular momentum situation/ Φ -independent potential. Then one can attempt to consider Φ as a time variable, leading to an elliptic (Klein–Gordon analogue) equation

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial \Phi^2} = \frac{4I_1 I_2}{I_1 + I_2} \left\{ \hbar^2 \left\{ \frac{\partial}{\partial I_1} \left\{ I_1 \frac{\partial \Psi}{\partial I_1} \right\} \frac{\partial}{\partial I_2} \left\{ I_2 \frac{\partial \Psi}{\partial I_2} \right\} \right\} + 2\{E - V\} \Psi \right\} \quad (787)$$

or the time-independent Schrödinger equation

$$-i\hbar \frac{\partial \Psi}{\partial \Phi} = \sqrt{\frac{4I_1 I_2}{I_1 + I_2} \left\{ \hbar^2 \left\{ \frac{\partial}{\partial I_1} \left\{ I_1 \frac{\partial}{\partial I_1} \right\} \frac{\partial}{\partial I_2} \left\{ I_2 \frac{\partial}{\partial I_2} \right\} \right\} + 2\{E - V\} \right\}} \Psi \quad (788)$$

with the extra benefit that the Φ -independence of the potential and kinetic terms helps with equivalence. Monotonicity is assured in this case by (341) with \mathcal{J} being fixed and partial moments of inertia being positive. One problem is why this angle and not another one, which grows in ambiguity as the model increases in size. Choosing I_1 or I_2 as a time turns out to lack a number of these benefits.

12.4 Reference particle time?

Analogy 86) [52] It is conceivable that there may be notions of reference particles within RPM’s, associated with gauge-fixing $\text{Rot}(d)$ mathematics much as GR’s reference fluids are associated with gauge-fixing the diffeomorphisms.

Question. Can some particles in an RPM be considered as time-defining reference particles for the other particles? Does some form of this usefully capture some further elements of the reference fluid approaches to matter time in geometrodynamics?

A Machian critique of reference fluid matter in GR would be to misbelieve all cases with matter in insufficient quantities to be tangible. One would also suspect use of matter with unphysical equation of state. Unfortunately these go against much of what has been proposed in this area.

12.5 No unimodular gravity counterpart for RPM’s

Difference 35) Is it conceivable that an analogue of Λ provides a hidden time from Analogy 21) or the rather probably more correct Analogy 49)? No! This is because this scheme relies on the quadratic constraint being an integrability of the linear constraint (as N or \dot{I} is not varied). However, firstly, whilst \mathcal{M}_μ has $\mathcal{H}_{,\mu}$ as an integrability, which, upon integrating, gives $\mathcal{H} + 2\Lambda = 0$ with \mathcal{H} the vacuum expression and Λ now interpreted as a constant of integration, \mathcal{L}_μ does not in any way involve \mathcal{D} [so this Difference follows from the no criss-cross Difference 8)].

Secondly, in relational approaches, variation with respect to the lapse N (or with respect to the instant I) is replaced by a primary constraint i.e. arising purely from the form of the action with no variation done. Thus this way of obtaining a quadratic constraint with a part of it interpreted as a constant of integration is not an available option open if one takes a relational approach, even to GR [via a BFO-A type action (149)].

Finally, I note that Gryb [286] showed that unimodular gravity amounts to the insertion of a background time. My above work shows, conversely that unimodular gravity is not possible within purely relational thinking.

13 Semiclassical Approach

This scheme starts from the h–l split as already detailed in Sec 12. I choose to use reduced scale = h shape = l models for most detailed work, though some explanations and comparisons also require the Dirac version or the subdivision of h and l by particle subsystems instead. Thus I usually start from the reduced scale–shape split Schrödinger equation (541). For some comparison and GR analogue purposes, the Dirac scale–shape split Schrödinger equation (512) comes alongside the quantum zero total angular momentum equation (513), which, being pure-shape as per Sec 2.5, is now pure-l. The alternative Dirac quantization scheme with p heavy particles and q light particles has Schrödinger equation (748) alongside the quantum zero total angular momentum constraint (749); it parallels the first example in Sec 12.1.

13.1 Born–Oppenheimer (BO) scheme and its quantum-cosmological analogue

By this, I mean ansatz (704) and the following associated approximations. Define $\hat{h} := \hat{H}_l + J_{hl} + V_h$ and its $|\chi\rangle$ -wavefunction expectation value (integrated over the l degrees of freedom, i.e. ‘l-averaged’) by

$$o := \langle \chi | \hat{h} | \chi \rangle . \quad (789)$$

The latter may also be regarded as the ‘ h_i ’ parameter dependent eigenvalue’ via

$$\hat{h}(h_{i'}, l_{i''}, \underline{P}_1^{i''}) |\Phi(h_{j'}, l_{j''})\rangle = o(h_{i'}) |\Phi(h_{j'}, l_{j''})\rangle . \quad (790)$$

The $|\chi\rangle$ sometimes requires suffixing by its quantum numbers, which I take to be multi-indexed by a single straight letter: $|\chi_j\rangle$. Thus the above o is, strictly, o_{jj} and there is an obvious off-diagonal equivalent

$$o_{jk} := \langle \chi_j | \hat{h} | \chi_k \rangle . \quad (791)$$

The Born–Oppenheimer approximation alias ‘diagonal dominance condition’ is that

$$\text{for } j \neq k, |o_{jk}/o_{jj}| =: \epsilon_{BO} << 1 . \quad (792)$$

Assuming this, one then looks at $\langle \chi | \times$ the quantum equation with the Born–Oppenheimer (BO) ansatz substituted in.

Additionally, because the ansatz is a product in which both factors depend on h, the h-derivatives acting upon it produce multiple terms by the product rule. In particular, both BO’s own atomic-and-molecular-physics case and the quantum-cosmological case contain second derivatives in h, for which the product rule produces three terms, schematically

$$\chi \partial_h^2 \phi, \quad \partial_h \phi \partial_h \chi, \quad \phi \partial_h^2 \chi . \quad (793)$$

The first is always kept. BO discard the next two as being far smaller than the first (this is due to a first kind of **adiabatic assumption**, by which h-changes in χ are considered to be much smaller than h-changes in ϕ). However, as explained in Sec 13.2.4, the emergent semiclassical time approach to the Problem of Time requires the cross-term to be kept. (This is a case in which qualitative importance for reasons given in Sec 13.2.4 overrides smallness; see Sec 13.8.1 for further discussion of types of adiabatic assumption). Likewise, if there is a linear derivative term, the product rule produces two terms, schematically

$$\chi \partial_h \phi, \quad \phi \partial_h \chi, \quad (794)$$

and the second of these is discarded likewise due to the above kind of adiabatic assumption.

13.1.1 Consequences of using a single h degree of freedom

Berry phase [123, 175] (as explained in Appendix A) is a further consideration that is in general necessary in setting up the preceding SSsec on geometrical grounds and with physical consequences.

Brout–Venturi Preclusion [144]. However, if the h-part is 1-d as is the case in this article (the scale part of the scale–shape split) there is no Berry phase issue due to the reparametrizability of 1-d geometry.

Note 1) Some motivations for considering one h degree of freedom are that $M_{\text{Planck}} \gg M_{\text{inflaton}}$, or that the single scale factor dominates over the anisotropic, inhomogeneous modes in GR cosmology. [The latter argument does have limitations, e.g. one often uses scalefactor *and* homogeneous matter mode(s) as the h-part.]

Note 2) Datta’s challenge (Appendix 12.B) may get round the Brout–Venturi Preclusion.

Note 3) Berry phase is in any case generally a relevant consideration in the case of N' heavy particles and N'' light particles since Brout preclusion does not apply for sufficient heavy particle number and dimension.

13.2 The WKB scheme

I take this to consist of the subsequent ansatz (705) for the h-wavefunction alongside associated approximations. The contentiousness of this ansatz in the quantum-cosmological context is addressed in Sec 13.7. For physical interpretation, I rewrite the principal function W by isolating a heavy mass M , $W(h) = MF(h)$. [For 1 h degree of freedom, this is trivial; for more than 1 h degree of freedom it still makes sense if the sharply-peaked hierarchy condition $\epsilon_{\Delta M}$ holds.] The associated **WKB approximation** is the negligibility of second derivatives,

$$|\hbar \partial_h^2 F / M |\partial_h F|^2| =: \epsilon_{\text{WKB}} \ll 1. \quad (795)$$

The associated dimensional expression is

$$\hbar / MF =: \epsilon_{\text{WKB}'} \ll 1. \quad (796)$$

This is to be interpreted as (quantum of action) \ll (classical action) via the reinterpretability of W as classical action (see e.g. [415]), which has clear semiclassical connotations.

As further incentives for the 1 h degree of freedom scheme, 1) Having a single h degree of freedom additionally trivially gets round the problem of nonseparable Hamilton–Jacobi equations which plague the case of > 1 h degrees of freedom [262, 108]. 2) Having the scale be h additionally precludes the need for any sandwich solving if done for non-reduced case; this is because rotational best matching is pure-shape.

13.2.1 The BO–WKB scheme’s scale–shape split RPM h- and l-equations

I present this for the usual as opposed to spherical presentation of triangle land case; for the latter, use instead barred quantities and I in place of ρ .

The h-equation $\langle \chi | \times$ (time-independent Schrödinger equation), with the associated integration being over the l degrees of freedom and thus in the present context over shape space,

$$\langle \chi | \hat{\mathcal{O}} | \chi \rangle = \int_{\mathfrak{S}(N,d)} \chi^*(\sigma, S) \hat{\mathcal{O}} \chi(\sigma, S) \mathbb{D}S \quad (797)$$

(for $\mathbb{D}S$ the measure over the shape space $\mathfrak{S}(N, d)$), and with the BO and WKB ansätze substituted in, is

$$\begin{aligned} \{\partial_\rho W\}^2 - i\hbar \partial_\rho^2 W - 2i\hbar \partial_\rho W \langle \chi | \partial_\rho | \chi \rangle - \hbar^2 \{\langle \chi | \partial_\rho^2 | \chi \rangle + \{nd - 1 - d\{d - 1\}/2\} \rho^{-1} \langle \chi | \partial_\rho | \chi \rangle\} - i\hbar \rho^{-1} \{nd - 1 - d\{d - 1\}/2\} \partial_\rho W \\ + \hbar^2 \rho^{-2} k(\xi) + 2V_\rho(\rho) + 2\langle \chi | J(\rho, S^u) | \chi \rangle = 2E. \end{aligned} \quad (798)$$

I have discarded an additional $\hbar^2 \langle \chi | D_S^2 | \chi \rangle$ term by integration by parts and the shape spaces in question being compact without boundary.

The l-equation is given by (time-independent Schrödinger equation) $- | \chi \rangle \times$ (h equation), and takes the form

$$\{1 - P_\chi\} \{-2i\hbar \partial_\rho | \chi \rangle \partial_\rho W - \hbar^2 \{\partial_\rho^2 | \chi \rangle + \partial_\rho | \chi \rangle \{nd - 1 - d\{d - 1\}/2\} \rho^{-1} + \rho^{-2} \Delta_{\mathfrak{S}(N,2)}\} | \chi \rangle + J(\rho, S^u) | \chi \rangle\} = 0. \quad (799)$$

For now, this takes the form of a fluctuation equation.

13.2.2 Emergent WKB time is aligned with emergent JBB time

We now use that $\partial_\rho^2 W$ is negligible by the WKB approximation to remove the second term from the h-equation, and apply

$$\partial_\rho W = p_\rho = *^{\{0\}} \rho \quad (800)$$

by identifying W as Hamilton’s function, and using the expression for momentum in the Hamilton–Jacobi formulation, the momentum-velocity relation, and the chain-rule to recast ∂_ρ as $\partial_\rho t *^{\{0\}}$. I referred to this $*^{\{0\}}$ as $*^h$ in Sec 12.

N.B. this is indeed aligned with the previous use of $*^{\{0\}} = d/dt^{\text{em(JBB)}}_{\{0\}}$:

WKB–JBB Time-Alignment Lemma [Continuation of Analogy 74]⁷⁷ $t_{\{0\}}^{\text{em(WKB)}} = t_h^{\text{em(JBB)}} = t^{\text{em(JBB)}}$, the last equality up to the accuracy notified in Sec 12.1. I.e., the Semiclassical Approach’s zeroth-order approximation for emergent time is the previously-encountered h-physics approximation of the emergent time at the classical level. I celebrate this by subsequently denoting this approximate emergent JBB–WKB time gestalt entity simply by t^{em} [with $\{0\}$ subscript when I need to make clear that it is the 0th order approximation.] It then follows from this identification and Sec 2 that the approximate emergent WKB time is aligned with Newtonian time, proper time and cosmic time in the various contexts but can additionally be regarded on a relational footing, and that Sec 12.1’s properties and critiques extend to approximate emergent WKB time. That it acts in an emergent semiclassical way follows from (eq in Sec 11) (see also 2 SSsecs down).

⁷⁷That these coincide is an expansion [22] on comments by Barbour [86], Kiefer [373] and Datta [190].

13.2.3 Full h-equation and its usual Hamilton–Jacobi equation approximand

Then

$$\begin{aligned} & \{*\rho\}^2 - 2i\hbar*\rho\langle\chi|\partial_\rho t*|\chi\rangle - \hbar^2\{\langle\chi|\{\partial_\rho t*\}^2|\chi\rangle + \{nd - 1d\{d-1\}/2\}\rho^{-1}\langle\chi|\partial_\rho t*|\chi\rangle\} \\ & - i\hbar\rho^{-1}\{nd - 1 - d\{d-1\}/2\}*\rho + \hbar^2\rho^{-2}k(\xi) + 2V_\rho(\rho) + 2\langle\chi^{\{0\}}|J(\rho, S^u)|\chi^{\{0\}}\rangle = 2E . \end{aligned} \quad (801)$$

Neglecting the second, third, fourth, fifth, sixth and eighth terms (see Sec 13.5 for various possible justifications) the h-equation takes the HJE form

$$\{\partial_\rho W\}^2 = 2\{E - V_\rho\} , \quad (802)$$

or, equivalently, the energy equation form

$$*\{0\}\rho^2 = 2\{E - V_\rho\} . \quad (803)$$

A reformulation of the latter is of use in further discussions in this article is the analogue Friedmann equation,

$$\left\{ \frac{*\{0\}\rho}{\rho} \right\}^2 = \frac{2E}{\rho^2} - \frac{2V_\rho}{\rho^2} . \quad (804)$$

This equation is also later explicitly required for the spherical presentation of triangleland:

$$\left\{ \frac{\check{*\{0\}}_I}{I} \right\}^2 = \frac{E}{2I^3} - \frac{V_I}{2I^3} . \quad (805)$$

The energy equation form is then solved by

$$\tilde{t}^{\text{em}} - \tilde{t}^{\text{em}}(0) = \frac{1}{\sqrt{2}} \int \frac{d\rho}{\sqrt{E - V_\rho}} \quad \text{or} \quad \bar{t}^{\text{em}} - \bar{t}^{\text{em}}(0) = \sqrt{2} \int \frac{\sqrt{I}dI}{\sqrt{E - V_I}} . \quad (806)$$

13.2.4 Recasting of the l-equation as a time-dependent Schrödinger equation

I first discuss the crucial chroniferous cross-term move by which the missing piece to make a time-dependent Schrödinger equation arises. The first term in (l-equation)/2 is $i\hbar\partial|\chi_{\{0\}}\rangle/\partial t_{\{0\}}^{\text{em}}$ by (800) and the chain-rule in reverse:

$$N^{\rho\rho}i\hbar\frac{\partial W}{\partial\rho}\frac{\partial|\chi\rangle}{\partial\rho} = i\hbar N^{\rho\rho}\pi\frac{\partial|\chi\rangle}{\partial\rho} = i\hbar N^{\rho\rho}M_{\rho\rho}*\{0\}\rho\frac{\partial|\chi\rangle}{\partial\rho} = i\hbar\frac{\partial\rho}{\partial t_{\{0\}}^{\text{em}}}\frac{\partial|\chi\rangle}{\partial\rho} = i\hbar\frac{\partial|\chi\rangle}{\partial t_{\{0\}}^{\text{em}}} . \quad (807)$$

Thus the fluctuation l-equation (799) can be rearranged to obtain a time-dependent Schrödinger equation with respect to an emergent time that is ‘provided by the h-subsystem’:

$$\begin{aligned} \{1 - P_\chi\}i\hbar\frac{\partial|\chi\rangle}{\partial t_{\{0\}}^{\text{em}}} &= \{1 - P_\chi\}\left\{-\frac{\hbar^2}{2}\left\{\frac{1}{\rho(t_{\{0\}}^{\text{em}})^2}\Delta_{\mathfrak{s}(N,d)}|\chi_{\{0\}}\rangle + \frac{\partial t_{\{0\}}^{\text{em}}}{\partial\rho}\frac{\partial}{\partial t_{\{0\}}^{\text{em}}}\left\{\frac{\partial t_{\{0\}}^{\text{em}}}{\partial\rho}\frac{\partial}{\partial t_{\{0\}}^{\text{em}}}\right\}|\chi\rangle + \right. \right. \\ & \quad \left. \left. \frac{nd - 1 - d\{d-1\}/2}{\rho}\frac{\partial t_{\{0\}}^{\text{em}}}{\partial\rho}\frac{\partial}{\partial t_{\{0\}}^{\text{em}}}|\chi\rangle\right\} + J|\chi\rangle\right\} . \end{aligned} \quad (808)$$

Then use (802), (803) or (806) to express ρ as a function of t^{em} . N.B. it is logical and consistent for the other ρ -derivatives to also be expressed as $t_{\{0\}}^{\text{em}}$ -derivatives, giving in full:

$$\begin{aligned} \{1 - P_\chi\}i\hbar\frac{\partial|\chi_{\{0\}}\rangle}{\partial t_{\{0\}}^{\text{em}}} &= \{1 - P_\chi\}\left\{-\frac{\hbar^2}{2}\left\{\frac{1}{\rho^2(t_{\{0\}}^{\text{em}})}\Delta_{\mathfrak{s}(N,d)}|\chi_{\{0\}}\rangle + \frac{1}{\sqrt{2\{E - V_\rho(\rho(t_{\{0\}}^{\text{em}}))\}}}\right. \right. \\ & \quad \left. \left. \left\{\frac{\partial}{\partial t_{\{0\}}^{\text{em}}}\left\{\frac{1}{\sqrt{2\{E - V_\rho(\rho(t_{\{0\}}^{\text{em}}))\}}}\frac{\partial|\chi\rangle}{\partial t_{\{0\}}^{\text{em}}}\right\} + \frac{nd - 1 - d\{d-1\}/2}{\rho(t_{\{0\}}^{\text{em}})}\frac{1}{\sqrt{2\{E - V_\rho(\rho(t_{\{0\}}^{\text{em}}))\}}}\frac{\partial|\chi_{\{0\}}\rangle}{\partial t_{\{0\}}^{\text{em}}}\right\}\right\} + J|\chi\rangle\right\} . \end{aligned} \quad (809)$$

Now consider (801) and (808) as a pair of equations to solve for the unknowns $t_{\{0\}}^{\text{em}}$ and $|\chi\rangle$.

Note 1) We need invertibility (at least on some intervals of the mechanical motion) in order to set up the t^{em} -dependent perturbation equation and more generally have a time provider equation followed by an explicit time-dependent rather than heavy degree of freedom dependent equation,

$$h = h(t_{\{0\}}^{\text{em}}) . \quad (810)$$

This is not in general guaranteed, but the examples in question do possess it. Invertibility needed (and that the examples in question do possess it) The inversion can also be used to convert h-derivatives to t^{em} -derivatives, so one has a bona fide l-equation.

13.2.5 Various forms of proposed approximate l-equations.

The time-dependent Schrödinger equation thus constructed is, modulo the h-l coupling term, ‘ordinary relational l-physics’. In turn, this is ‘ordinary l-physics’ modulo the effect of the angular momentum correction term. [This is itself absent in 1-d or if one repeats the above working in a spatially nonrelational setting]. Thus the purported simple situation has ‘the scene set’ by the h-subsystem for the l-subsystem to have dynamics. This dynamics is furthermore slightly perturbed by the h-subsystem, while neglecting the back-reaction of the l-subsystem on the h-subsystem. One might even argue for the interaction term to be quantitatively negligible as regards the observed l-physics.

$$i\hbar \frac{\partial |\chi\rangle}{\partial t^{\text{em}}} = -\frac{\hbar^2}{2} \frac{\Delta_{\mathbf{s}(N,d)}}{\rho^2(t_{\{0\}}^{\text{em}})} |\chi\rangle + J |\chi\rangle . \quad (811)$$

This amounts to neglecting the averaged terms and the unaveraged first and second terms of (808) or (809) (see Sec 13.5 for various possible justifications).

13.2.6 Use of rectified time, and that this amounts to working on shape space

The time-dependent Schrödinger equation core (811) simplifies if one furthermore chooses the rectified time given by

$$\rho^2 \partial / \partial t_{\{0\}}^{\text{em}} = \partial / \partial t^{\text{rec}} , \quad (812)$$

i.e.

$$t^{\text{rec}} - t^{\text{rec}}(0) = \int dt_{\{0\}}^{\text{em}} / \rho^2(t_{\{0\}}^{\text{em}}) . \quad (813)$$

We can get to this all in one go by combining its definition with the energy equation (so as to cancel out the emergent time):

$$t^{\text{rec}} - t^{\text{rec}}(0) = \frac{1}{\sqrt{2}} \int \frac{d\rho}{\rho^2 \sqrt{E - V(\rho)}} . \quad (814)$$

In the case of the spherical presentation of triangleland, I use instead the ‘rectified time’ [35]

$$t^{\text{rec}} - t^{\text{rec}}(0) = \int d\check{t}_{\{0\}}^{\text{em}} / I^2(\check{t}_{\{0\}}^{\text{em}})^2 . \quad (815)$$

Then, all in one go,

$$t^{\text{rec}} - t^{\text{rec}}(0) = \sqrt{2} \int \frac{dI}{I^2} \sqrt{\frac{I}{E - V(I)}} . \quad (816)$$

As regards interpreting the rectified timefunction, in each case using t^{rec} amounts to working on the shape space itself, which amounts to using the geometrically natural presentation. Note: to keep formulae later on in the paper tidy, I use t for t^{em} and \mathcal{T} for t^{rec} up to a constant [always including $t^{\text{rec}}(0)$ and sometimes including the constant of integration from the other side of (814)]. Finally, approximately isotropic GR has an analogue of rectification too, amounting to absorption of extra factors of the scalefactor a viewed as a function of t^{em} .

Note: t_{rec} is indeed t_{geo} referring to the shape geometry, which is the natural geometry for the l-physics, since this Sec’s study has l aligned with shapes.

Lemma: suppose $t_{\{0\}}^{\text{em}}$ is monotonic. Then the rectified time \mathcal{T} is also monotonic.

Proof

$$\frac{d\mathcal{T}}{d\rho} = \frac{d\mathcal{T}}{dt_{\{0\}}^{\text{em}}} \frac{dt_{\{0\}}^{\text{em}}}{d\rho} = \frac{1}{\rho(t_{\{0\}}^{\text{em}})^2} \frac{dt_{\{0\}}^{\text{em}}}{d\rho} \geq 0 \quad (817)$$

by the chain-rule in step 1, (814) in step 2 and the positivity of squares and the assumed monotonicity of $t_{\{0\}}^{\text{em}}$ in step 3. For the spherical presentation of triangleland, the $\rho \rightarrow I$, barred counterpart of this argumentation holds also. \square

In terms of this, one can view the l -equation as (perhaps perturbations about) a time-dependent Schrödinger equation on the shape space,

$$i\hbar \frac{\partial |\chi\rangle}{\partial \mathcal{T}} = \frac{-\hbar^2}{2} \Delta_{\mathbf{s}(N,d)} |\chi\rangle + \hat{J} |\chi\rangle \quad \{+ \text{further perturbation terms}\} \quad (818)$$

(this article’s specific examples of which are, mathematically, familiar equations). \hat{J} denotes $\rho^2(t_{\{0\}}^{\text{em}}(\mathcal{T}))J$.

The rectified time’s simplification of the emergent-time-dependent Schrödinger equation can be envisaged as passing from the emergent time that is natural to the whole relational space to a time that is natural on the shape space of the l -degrees of freedom themselves, i.e. to working on the shape space of the l -physics itself.

13.3 The negligible-back-reaction regime

Parametrize the smallness of the interaction term between the h and l subsystems by splitting out a factor ϵ : $J \longrightarrow \epsilon J$. Then apply $t = t_{\{0\}} + \epsilon t_{\{1\}}$ and $|\chi\rangle = |\chi_{\{0\}}\rangle + \epsilon |\chi_{\{1\}}\rangle$. Usually the first J is kept, since otherwise the l -subsystem's energy changes without the h -system responding, violating conservation of energy. But if this is just looked at for a “short time” (few transitions, the drift may not be great, and lie within the uncertainty to which an internal observer would be expected to know their universe's energy. Then the system of equations becomes

$$\left\{ \frac{\partial \rho}{\partial t_{\{0\}}} \right\}^2 = 2\{E - V\} , \quad (819)$$

$$i\hbar \frac{\partial |\chi\rangle}{\partial \mathcal{T}_{\{0\}}} = -\frac{\hbar^2}{2} \Delta_{\mathbf{s}(N,d)} |\chi\rangle + \epsilon \dot{J} |\chi\rangle . \quad (820)$$

Here, I do not explicitly perturbatively expand the last equation as it is a decoupled problem of a standard form: a \mathcal{T} -dependent perturbation of a simple and well-known \mathcal{T} -dependent perturbation equation.

Note 1) The J renders the l-equation nonseparable as is required for the Semiclassical Approach to work; it also constitutes an imprint other than the time provision itself unto the l-subsystem (which in more accurate cosmological models one might hope to be detectable).

Note 2) that there are no averaged terms in the h-equation, i.e. no back-reaction and the system is decoupled: one can solve the h-equation first as no $|\chi\rangle$'s are present in it.

I next provide solutions in some specific RPM cases of these equations.

13.3.1 First approximation for time-dependent wave equation for 3-stop metroland

(820) can be viewed as a \mathcal{T} -dependent perturbation of what is, in the N -stop metroland case, a well-known \mathcal{T} -dependent Schrödinger equation (the usual one on the circle/sphere/hypersphere).

For the 3-stop metroland HO, the emergent time-dependent Schrödinger equation is

$$i\hbar \frac{\partial |\chi_{\{1\}}\rangle}{\partial \mathcal{T}} = -\frac{\hbar^2}{2} \frac{\partial^2 |\chi_{\{1\}}\rangle}{\partial \varphi^2} + \frac{B E^2 \cos 2\varphi}{A^2 \{1 + 2E^2 \mathcal{T}^2 / A\}^2} |\chi_{\{1\}}\rangle , \quad (821)$$

which, for B small in relation to e.g. the A -term or the ∂_φ^2 term and corresponding to ϵ small in this example, i.e. $K_1 \approx K_2$, poses, about a very simple quantum equation, a (fairly analytically tractable) \mathcal{T} -dependent perturbation problem. The reason for studying the above ‘negative curvature balanced by negative cosmological constant’ type scenario is that it is exactly soluble by means not usually available in Semiclassical Approach studies, allowing for a number of further checks (work in progress). Take $|\chi_{\{0\}}\rangle$ as the solution of the unperturbed problem and $|\chi_{\{1\}}\rangle$ as the solution of the perturbed problem to first order.

Note that many of the other examples listed in [35] can be taken as far as an analytic counterpart of eq. (821) time-dependent Schrödinger equation. However, I omit these further examples due to complications in subsequent stages of the working.

13.3.2 Perturbation theory integrals for 3-stop metroland

The unperturbed equation is solved by

$$\left| \chi_d^{c\{0\}} \right\rangle = \exp(iE_d \mathcal{T} / \hbar) \cos(d\varphi) / 2\pi , \quad \left| \chi_d^{s\{0\}} \right\rangle = \exp(iE_d \mathcal{T} / \hbar) \sin(d\varphi) / 2\pi \quad (822)$$

for $E_d = \hbar^2 d^2 / 2$ by Sec 7.1.3.

The standard approach to time-dependent perturbation theory gives, to first order [$\{1\}$ -superscripts]

$$\left\langle \chi_{d'}^{t\{1\}} \right| \chi_d^{t\{0\}} \rangle = \delta_{dd'} - \frac{iBE^2}{\hbar A^2} I^t(d, d') I(\beta, \mathcal{T}) \quad (823)$$

for the following split-up integrals (the superscript t stands for ‘trig function’ and takes the values c for the cosine solution and s for the sine solution).

$$I^c(d, d') := \int \frac{\cos d'\varphi}{\sqrt{2\pi}} \cos 2\varphi \frac{\cos d\varphi}{\sqrt{2\pi}} d\varphi = \frac{1}{4} \{ \delta_{d'+2,d} + \delta_{d'-2,d} \} = \int \frac{\sin d'\varphi}{\sqrt{2\pi}} \cos 2\varphi \frac{\sin d\varphi}{\sqrt{2\pi}} d\varphi =: I^s(d, d') \quad (824)$$

for $d, d' > 2$, the other cases including the following nonzero exceptions:

$$I^c(0, 2) = I^c(2, 0) = 1/2, \quad I^c(1, 1) = 1/4 \quad \text{and} \quad I^s(1, 1) = -1/4. \quad (825)$$

Also,

$$I(\beta, \mathcal{T}) := \int_0^{\mathcal{T}} \exp(i\beta \mathcal{T}') d\mathcal{T}' / \{1 + 2E^2 \mathcal{T}'^2 / A\}^2, \quad (826)$$

$$\text{where } \beta := \hbar\{d'^2 - d^2\}/2. \quad (827)$$

This integral is simple in the $\beta = 0$ ($d = d'$) case (the only survivor of which by the ϕ -integral is $d = d' = 1$) for which the complex numerator collapses to 1:

$$I(0, \mathcal{T}) = \frac{1}{2E\sqrt{2/A}} \left\{ \frac{E\sqrt{2/A}\mathcal{T}}{2E^2\mathcal{T}^2/A + 1} + \arctan(ET\sqrt{2/A}) \right\}. \quad (828)$$

For all the other cases, the integral is complex, and comes out as a complicated combination of elementary, Si and Ci functions that I do not provide.

The first-order perturbed wavefunctions then comes out as

$$\left| \chi_d^{\{1\}}(\mathcal{T}) \right\rangle = \left| \chi_d^{\{0\}} \right\rangle - \frac{iBE^2}{4\hbar A^2} \{I(2\hbar\{1+d\}, \mathcal{T}) \left| \chi_{d+2}^{\{0\}} \right\rangle + \{I(2\hbar\{1-d\}, \mathcal{T}) \left| \chi_{d-2}^{\{0\}} \right\rangle\} \quad d > 2, \quad (829)$$

$$\left| \chi_2^{\{1\}}(\mathcal{T}) \right\rangle = \left| \chi_2^{\{0\}} \right\rangle - \frac{iBE^2}{4\hbar A^2} \{I(6\hbar, \mathcal{T}) \left| \chi_4^{\{0\}} \right\rangle + 2I(-2\hbar, \mathcal{T}) \left| \chi_0^{\{0\}} \right\rangle\}, \quad (830)$$

$$\left| \chi_1^{\{1\}}(\mathcal{T}) \right\rangle = \left| \chi_1^{\{0\}} \right\rangle - \frac{iBE^2}{4\hbar A^2} I(0, \mathcal{T}) \left| \chi_1^{\{0\}} \right\rangle, \quad (831)$$

$$\left| \chi_1^{\{s\}}(\mathcal{T}) \right\rangle = \left| \chi_1^{\{0\}} \right\rangle + \frac{iBE^2}{4\hbar A^2} I(0, \mathcal{T}) \left| \chi_1^{\{0\}} \right\rangle, \quad \text{and} \quad (832)$$

$$\left| \chi_0^{\{1\}}(\mathcal{T}) \right\rangle = \left| \chi_1^{\{0\}} \right\rangle - \frac{iBE^2}{8\hbar A^2} I(4\hbar, \mathcal{T}) \left| \chi_2^{\{0\}} \right\rangle. \quad (833)$$

Using the dimensionless $\mathcal{T}_d = ET/\sqrt{A}$, the transition probability goes as

$$\left\{ \frac{B}{A} \frac{E}{\hbar\sqrt{A}} \mathcal{T}_d \right\}^2 \quad (834)$$

or, using frequencies $\omega_1 = \sqrt{K_1}$ and $\Delta\omega = \sqrt{K_2} - \sqrt{K_1}$, as

$$\left\{ \frac{\Delta\omega}{\omega_1} \frac{E_{\text{Free}}}{E_{\text{QM-HO}}} \mathcal{T}_d \right\}^2 \quad (835)$$

i.e. as frequency contrast squared multiplied by the free to QM HO energy ratio squared.

13.3.3 Time-dependent wave equations for triangleland

$$i\hbar\partial_{\mathcal{T}}|\chi^{\{1\}}\rangle = -\{\hbar^2/2\}\Delta_{\mathfrak{S}(3,2)}|\chi^{\{1\}}\rangle + \dot{J}(\mathcal{T}, S^u)|\chi^{\{1\}}\rangle \quad \text{where now } \dot{J} := I^2(t^{\text{em}}(\mathcal{T}))^2 J, \quad (836)$$

is also a \mathcal{T} -dependent perturbation of a well-known \mathcal{T} -dependent Schrödinger equation (the usual one on the sphere). For the $A < 0, E = 0$ model's potential, one has the time-dependent Schrödinger equation

$$i\hbar\frac{\partial|\chi^{\{1\}}\rangle}{\partial\mathcal{T}_{\{0\}}} = -\frac{\hbar^2}{2} \left\{ \frac{1}{\sin\Theta} \frac{\partial}{\partial\Theta} \left\{ \sin\Theta \frac{\partial}{\partial\Theta} \right\} + \frac{\partial^2}{\partial\varphi^2} \right\} |\chi^{\{1\}}\rangle - \frac{B \cos\Theta}{2A\{\mathcal{T} - \mathcal{T}_c\}^2} |\chi^{\{1\}}\rangle. \quad (837)$$

13.3.4 Perturbation theory integrals for this example

The unperturbed equation is solved by

$$\left| \chi_d^{\{0\}} \right\rangle = \exp(iE_S\mathcal{T}/\hbar) Y_{Sj}(\Theta, \Phi) \quad (838)$$

for (see Sec 8.3.1) $E_S = \hbar^2 S\{S+1\}/2$ (the factor of 4 has been absorbed. and \mathcal{Y}_{Sj} the usual spherical harmonics, albeit with this article's unusual triangleland meanings.

The standard approach to time-dependent perturbation theory gives, to first order [$\{1\}$ -superscripts];

$$\langle \chi_{Sj}(\mathcal{T}) | \chi_{S'j'} \rangle = \delta_{S'S} \delta_{j'j} + \frac{iB}{2\hbar A} I(\gamma, \mathcal{T}) I_{\Theta\Phi}(S, S', j, j') \quad (839)$$

for

$$\gamma := \hbar\{S'\{S' + 1\} - S\{S + 1\}\}/2 \quad (840)$$

and for the following split-up integrals. Firstly,

$$I_{\Theta\Phi}(S, S', j, j') := \langle \chi_{S'j'}^{\{0\}} | \cos \Theta | \chi_{Sj}^{\{0\}} \rangle \quad (841)$$

which has the same mathematical form as the integral that appears in the derivation of the selection rules for electric dipole transitions [50, 629, 458]; more generally, it is a simple example of a 3- \mathcal{Y} integral [468]. Its nonzero cases are given by

$$\langle \chi_{S+1,j}^{\{0\}} | \cos \Theta | \chi_{Sj}^{\{0\}} \rangle = \sqrt{\frac{\{S+1\}^2 - j^2}{\{2S+1\}\{2S+3\}}} , \quad (842)$$

and

$$\langle \chi_{S-1,j}^{\{0\}} | \cos \Theta | \chi_{Sj}^{\{0\}} \rangle = \sqrt{\frac{S^2 - j^2}{\{2S-1\}\{2S+1\}}} . \quad (843)$$

Secondly,

$$I(\gamma, \mathcal{T}) := \int_0^{\mathcal{T}} d\mathcal{T}' \exp(i\gamma\mathcal{T}') / \{\mathcal{T}' - \mathcal{T}_c\}^2 = \frac{\exp(i\gamma\mathcal{T})}{\mathcal{T}_c - \mathcal{T}} - \frac{1}{\mathcal{T}_c} + \gamma \{ \sin \gamma \mathcal{T}_c \{ \text{Ei}(i\gamma\{\mathcal{T}_c - \mathcal{T}\}) - \text{Ei}(i\gamma\mathcal{T}_c) - i \cos \gamma \mathcal{T}_c \{ \text{Ei}(i\gamma\{\mathcal{T} - \mathcal{T}_c\}) - \text{Ei}(-i\gamma\mathcal{T}_c) \} \} . \quad (844)$$

The final answer for the first-order perturbed wavefunctions then takes the form

$$\begin{aligned} \left| \chi_{Sj}^{\{1\}}(\mathcal{T}) \right\rangle &\propto \left| \chi_{Sj}^{\{0\}} \right\rangle + \frac{i}{2\hbar} \frac{B}{A} \left\{ \left\{ \frac{\exp(i\hbar\{S+1\}\mathcal{T})}{\mathcal{T}_c - \mathcal{T}} - \frac{1}{\mathcal{T}_c} + \hbar\{S+1\} \{ \sin(\hbar\{S+1\}\mathcal{T}_c) \{ \text{Ei}(i\hbar\{S+1\}\{\mathcal{T}_c - \mathcal{T}\}) \right. \right. \\ &\quad \left. \left. - \text{Ei}(i\hbar\{S+1\}\mathcal{T}_c) \} - i \cos(\hbar\{S+1\}\mathcal{T}_c) \{ \text{Ei}(i\hbar\{S+1\}\{\mathcal{T} - \mathcal{T}_c\}) - \text{Ei}(-i\hbar\{S+1\}\mathcal{T}_c) \} \right\} \right\} \sqrt{\frac{\{S+1\}^2 - j^2}{\{2S+1\}\{2S+3\}}} \left| \chi_{S+1,j}^{\{0\}} \right\rangle \\ &\quad + \left\{ \frac{\exp(-i\hbar S\mathcal{T})}{\mathcal{T}_c - \mathcal{T}} - \frac{1}{\mathcal{T}_c} - \hbar R \{ -\sin(\hbar S\mathcal{T}_c) \{ \text{Ei}(i\hbar S\{\mathcal{T} - \mathcal{T}_c\}) - \text{Ei}(-i\hbar S\mathcal{T}_c) \} \right. \\ &\quad \left. \left. - i \cos(\hbar R\mathcal{T}_c) \{ \text{Ei}(i\hbar S\{\mathcal{T}_c - \mathcal{T}\}) - \text{Ei}(i\hbar S\mathcal{T}_c) \} \right\} \right\} \sqrt{\frac{S^2 - j^2}{\{2S-1\}\{2R+1\}}} \left| \chi_{S-1,j}^{\{0\}} \right\rangle \} . \end{aligned} \quad (845)$$

Finally, I consider the small \mathcal{T} regime (small time evolution makes sense for an early-universe application, unlike for an atomic application in which one is considering a long-lasting final state; after that further transitions could occur, corresponding to second and higher order terms, which is beyond the scope of this paper). Also the motivation for noninteracting initial and final states of atomic physics (involving thus large values of the time) is very much inappropriate in the current analogue-cosmological setting.

$$\left| \chi_{Sj}^{\{1\}}(I, \Theta, \Phi) \right\rangle \approx \left| \chi_{Sj}^{\{0\}}(\Theta, \Phi) \right\rangle + \frac{i}{2} \frac{B}{A} \frac{\{I - I_0\}}{I} \frac{I_0}{I_{\text{HO}}} \left\{ \sqrt{\frac{\{S+1\}^2 - j^2}{\{2S+1\}\{2S+3\}}} \left| \chi_{S+1,j}^{\{0\}} \right\rangle + \sqrt{\frac{S^2 - j^2}{\{2S-1\}\{2S+1\}}} \left| \chi_{S-1,j}^{\{0\}} \right\rangle \right\} \quad (846)$$

for $I_{\text{HO}} := \hbar\sqrt{-A/2}$ the characteristic quantum moment of inertia scale for this system (see Sec 7.6).

This regime is of course computable even in cases for which the full integral cannot be analytically calculated.

13.4 Small but non-negligible back-reaction regime: motivation

As one has seen, the full h and l equations of the Semiclassical Approach contain numerous terms. Thus investigating a number of approximate regimes for these is a healthy attitude toward whether Halliwell–Hawking’s treatment is robust and therefore to be substantially trusted as a framework in which to make classical cosmological predictions about CMB inhomogeneities and galaxies.

One interesting feature is that the l -subsystem can back-react on the h -subsystem rather than just merely receive a time-standard from it. That this occurs in RPM’s as well as in GR is Analogy 87). One’s l -equation is now coupled to a less approximate chroniferous h -equation, which takes the form

$$\text{Hamilton–Jacobi equation plus small quantum-mechanical expectation-type perturbations} . \quad (847)$$

Some early literature on back-reaction in Quantum Cosmology is by Brout and collaborators (see e.g. [144]) and by Kiefer (see e.g. [372]). The previously-suggested simple procedure of solving the h -Hamilton–Jacobi equation first is insufficient by

itself to capture this level of detail. This makes sense, as W is for conservative systems while if the h-subsystem interacts (weakly as that may be) with the l-subsystem, one expects the h-subsystem then to be more general than conservative [415]. Moreover, this feature looks to be both technically and conceptually important for the following reasons.

Book-keeping argument Above, I considered the l-physics to have an h-imprint via the interaction term that I treated perturbatively. However, it then becomes an issue that the interaction term induced transitions of the energy levels of the l-subsystem are not being compensated by changes to the energy levels in the h system (this would be particularly relevant if they secularly build up). Thus it is desirable from a book-keeping perspective for the h system itself to contain hl interaction terms too. Without this, there is no built-in way of maintaining the underlying fact that one's model is supposed to have the meaning of 'life in a fixed energy eigenstate.'

Whether the energy in the h-equation should be interpreted as the energy of the universe E or as an E_h that is only approximately equal to this is a clearly related issue. One approach to this would be to give a distinct t^{em} per E_h . Thus the h-problem now amounts to solving a 1-parameter family of Hamilton–Jacobi equations and permitting the trajectory to slip between these in response to the l-physics. Then l transition by $+\Delta E$ shifts E_h by $-\Delta E$. Then one has a slightly different standard with respect to which further transition rates are to be calculated. This happens for each transition in the h–l model, but only occasionally within a hll model with a large frequency hierarchy. Thus most of the transitions are l–l, and the interaction with the large-scale mode of the universe produces only a tiny correction to the motion as one would intuitively expect (see also [20]). Note that in absence of interaction terms, not only does the timefunction contain a zero factor but also the l-equation is then as frozen as the whole hl-system is, in which case unfrozenness fails.

Back-reaction as inducing a correction on the emergent time. This is of clear interest to the central theme of this article, both as regards the rough order of magnitude of the correction and qualitatively as regards how now the whole universe contributes to the timestandard ($t_{\{0\}}^{\text{em}}$ in some ways lacks in Machian/ephemeris character, but $t_{\{1\}}^{\text{em}}$ makes up for this).

Back-reaction is conceptually central to GR. Matter back-reacting on the geometry is a conceptually important part of GR (both at the level of what the Einstein field equations mean and in GR's aspect as supplanter of absolute structure). Thus toy models that have this feature are conceptually desirable.

Relational motivation If one goes up one iteration, is the time-provision more properly Machian? Whilst the still having something to dissipate into and out of point remains fine. Incidentally, counting suffices to establish semiclassical-hidden approach inequivalence

13.5 Interlude: the 14 often-omitted terms

There are 14 terms that are often neglected in the reduced system, and most of them have multiple significances. h and l in the enumeration denote which equation these terms feature in.

- h1) $\hbar\partial_\rho W\langle\partial_\rho\rangle$ is a back-reaction, $O(\hbar)$, a time-derivative, an average and a \mathcal{L} -non-adiabatic term.⁷⁸
- h2) $\hbar^2\langle\partial_\rho^2\rangle$ is a back-reaction, $O(\hbar^2)$, a *higher* time-derivative, an average and a \mathcal{L} -non-adiabatic term.
- h3) $\hbar^2\rho^{-1}\langle\partial_\rho\rangle$ is a back-reaction, $O(\hbar^2)$, a time-derivative, an average and a \mathcal{L} -non-adiabatic term.
- h4) $\langle J\rangle$ is a back-reaction, an average and small for J perturbatively small (compared to V_ρ).
- h5) $\hbar^2\rho^{-2}k(\xi)$ is $O(\hbar^2)$.
- h6) $\hbar\partial_\rho^2 W$ is $O(\hbar)$ and the term usually neglected by the WKB approximation itself.
- l1) $\hbar\partial_\rho W$ is $O(\hbar)$, a time-derivative, an average and a p -non-adiabatic term.
- l2) $\hbar^2\langle\partial_\rho^2\rangle$ is $O(\hbar^2)$, a *higher* time-derivative, an average and a p -non-adiabatic term.
- l3) $\hbar^2\rho^{-1}\langle\partial_\rho\rangle$ is $O(\hbar^2)$, a time-derivative, an average and a p -non-adiabatic term.
- l4) $\langle J\rangle$ is an average and small for J perturbatively small (compared to V_ρ).
- l7) $J|\chi\rangle$ is small for J perturbatively small.
- l8) $\hbar^2\langle D_S^2\rangle$ is $O(\hbar^2)$ and an average.
- l9) $\hbar^2\partial_\rho^2|\chi\rangle$ is $O(\hbar^2)$, a *higher* time-derivative and a p -non-adiabatic term.
- l10) $\hbar^2\rho^{-1}\partial_\rho|\chi\rangle$ is $O(\hbar^2)$, a time-derivative and a p -non-adiabatic term.

The preceding Secs considered a regime for which the fairly common case of l7) being less negligible was considered, followed by h4) often being the least-negligible back-reaction term. Then consistency dictates l4) be kept also.

Then to get the approximate h-equation, one needs to argue e.g. for $O(\hbar)$ neglect (and $J \ll V_\rho$ and/or averages being small to eliminate this potential interaction back-reaction term). One then finds that the same assumptions on the l-subsystem remove all the other usually-removed terms, but that one would then lose the chroniferous term and the $D_S^2|\chi\rangle$ term. Thus one needs to expand the argument to include p -adiabaticity and a qualitative change due to the chroniferous term and

⁷⁸ \mathcal{L} and p adiabaticities are explained in Sec 13.8.1.

that $D_S^2|\chi\rangle$ is much larger than its average. The qualitative change then has the problem of there being a number of other qualitatively changing terms (all other time derivatives), so one needs to do something such as neglecting averages and insisting that $O(\hbar^2)$ overrides inclusion on grounds of qualitative change. I agree that this last argument at least has the potential to be dubious; I consider my RPM scheme here as an adequate area to investigate that effect. For detailed discussion of each such assumption, see Sec 13.5. Some of these approximations can be made prior to knowing the explicit form of $|\chi\rangle$, however not all can be, since dimensional arguments cannot be used to argue that ‘averages are small’.

Next, I “justify the consistency” of the previously-made omissions for the specific models I am investigating. The \hbar -equation’s neglected pieces are as follows. Some of these neglects can be made prior to knowing the explicit form of $|\chi\rangle$, however not all can be, since dimensional arguments cannot be used to argue that ‘averages are small’. For 3-stop metroland,

$$\hbar^2 \partial^2 W \approx O(\hbar) \partial^2 W, \{ \hbar^2 / \rho \} \langle \partial_\rho \rangle \approx O(\hbar^3) \{ B/A \}^2, \text{ and } \{ \hbar^2 \} \langle \partial_\rho^2 \rangle \approx O(\hbar^4) \{ B/A \}^2. \quad (848)$$

The $\rho \rightarrow I$, barred counterpart of the above holds for triangleland.

For 3-stop metroland, $B \langle \cos 2\varphi \rangle$ in the case that survive the selection rule, goes as $O(\hbar)O(B)$. The triangleland counterpart of this is $B \langle \cos \Theta \rangle$, which is a 3- Y integral, with ± 1 selection rule, and goes like $O(\hbar)O(B)$. Now, $O(B) > O(\hbar)$ for the time-dependent Schrödinger equation to make sense, but clearly whether this is true or not is down to what regime is selected; these equations have regimes other than the one in which a time-dependent Schrödinger equation is a good approximation.

13.6 Small but non-negligible back-reaction: working

Some previously proposed more accurate \hbar -equation types are the habitual one including $\langle J \rangle$, Datta’s $\langle \chi | \{ 1 - P_\chi | \chi \rangle$ or use of $\langle \partial_t \rangle$, or use of two terms smaller than this by $O(\hbar)$. Finally, one could have not just one such term but combinations of them.

My particular RPM proposal that is a simple modelling of back-reaction is as follows (the first of the above). Look to solve

$$\frac{1}{2} \left\{ \frac{\partial \rho}{\partial t} \right\}^2 + \epsilon \langle \chi | J | \chi \rangle = E - V, \quad (849)$$

$$i\hbar \frac{\partial |\chi\rangle}{\partial t} = -\frac{\hbar^2}{2} \frac{\Delta_{\mathbf{s}(N,d)} |\chi\rangle}{\rho^2(t)} + \epsilon J |\chi\rangle. \quad (850)$$

[Note that if there were a separate V_S , rectification leads to this becoming part of \dot{J} , and, in any case, I am considering V ’s that are homogeneous in scale variable, thus I do not write down a separate V_S : the isotropic V_ρ and the direction-dependent interaction term J contain everything that ends up to be of relevance.]

Then in the small but non-negligible-back-reaction regime, there is a $t_{\{1\}}$ equation to solve, and the $|\chi_{\{1\}}\rangle$ stage then gives the following set of equations

$$d\rho^2 = 2\{E - V(\rho)\} dt_{\{0\}}^2, \quad (851)$$

$$i\hbar \frac{\partial |\chi_{\{0\}}\rangle}{\partial \mathcal{T}_{\{0\}}} = -\frac{\hbar^2}{2} D_S^2 |\chi_{\{0\}}\rangle, \quad (852)$$

$$dt_{\{1\}}^{\text{em}} \{E - V(\rho)\} = \langle \chi_{\{0\}} | J | \chi_{\{0\}} \rangle dt_{\{0\}}^{\text{em}}, \quad (853)$$

$$i\hbar \frac{\partial |\chi_{\{1\}}\rangle}{\partial \mathcal{T}_{\{0\}}} = -\frac{\hbar^2}{2} \Delta_{\mathbf{s}(N,d)} |\chi_{\{1\}}\rangle + \left\{ \dot{J} - \frac{\hbar^2}{2} \frac{d\mathcal{T}_{\{1\}}}{d\mathcal{T}_{\{0\}}} \Delta_{\mathbf{s}(N,d)} \right\} |\chi_{\{0\}}\rangle. \quad (854)$$

One can then solve (853) for $t_{\{1\}}$ using knowledge of the solutions of the first two decoupled equations,

$$t_{\{1\}}^{\text{em}} - t_{\{1\}}^{\text{em}}(0) = \frac{1}{2} \int_{t_{\{0\}}(0)}^{t_{\{0\}}} \left\{ \frac{1}{E - V(\rho_{\{0\}}(t'_{\{0\}}))} \int \chi_{\{0\}}^*(t_{\{0\}}, S^u) J(t_{\{0\}}, S^u) \chi_{\{0\}}(t_{\{0\}}, S^u) D S dt'_{\{0\}} \right\} \quad (855)$$

and then also using

$$\frac{d\mathcal{T}_{\{1\}}}{d\mathcal{T}_{\{0\}}} = \frac{\partial t_{\{1\}}}{\partial t_{\{0\}}} = \frac{1}{4\{E - V\}} \int \chi_{\{0\}}^* J \chi_{\{0\}} D S \quad (856)$$

to render the fourth equation into the form

$$i\hbar \frac{\partial |\chi_{\{1\}}\rangle}{\partial \mathcal{T}_{\{0\}}} = -\frac{\hbar^2}{2} \Delta_{\mathbf{s}(N,d)} |\chi_{\{1\}}\rangle + \left\{ \dot{J} - \frac{\hbar^2 \langle \chi_{\{0\}} | J | \chi_{\{0\}} \rangle \Delta_{\mathbf{s}(N,d)}}{4\{E - V(\rho(\mathcal{T}_{\{0\}}))\}} \right\} |\chi_{\{0\}}\rangle. \quad (857)$$

I provide partial solutions to this negligible-back-reaction scheme in some specific cases in Sec 6. [Modulo leaving the solution of (857) in a formal form involving the below use of Green’s functions.]

Note: the last term with the big bracket is by this stage a known, so this is just an inhomogeneous version of the second equation and therefore amenable to the method of Green's functions.

$$\begin{aligned} |\chi_{\{1\}}\rangle &= \int_{\mathcal{T}'=0}^{\mathcal{T}} \int_{\mathfrak{S}(N,d)} G(\mathcal{T}_{\{0\}}, S^u; \mathcal{T}'_{\{0\}}, S^{u'}) \left\{ \dot{J} - \frac{\hbar^2 \langle \chi_{\{0\}} | J | \chi_{\{0\}} \rangle \Delta_{\mathfrak{S}(N,d)}}{4\{E - V(\rho(t'_{\{0\}}(\mathcal{T}'_{\{0\}}))\}} \right\} (S^{u'}, \mathcal{T}'_{\{0\}}) |\chi_{\{0\}}(S^{u'}, \mathcal{T}'_{\{0\}})\rangle \text{DS}' d\mathcal{T}' \\ &=: \int_{\mathcal{T}'=0}^{\mathcal{T}} \int_{\mathfrak{S}(N,d)} G(\mathcal{T}_{\{0\}}, S^u; \mathcal{T}'_{\{0\}}, S^{u'}) f(\mathcal{T}', S^{u'}) \text{DS}' d\mathcal{T}' \end{aligned} \quad (858)$$

modulo additional boundary terms/complementary function terms. Now, this is a very standard linear operator for this article's simple RPM examples (time-dependent 1- d and 2- d rotors). However, i) the region in question is less standard (an annulus or spherical shell with the time variable playing the role of radial thickness). ii) Nor is it clear what prescription to apply at the boundaries. On these grounds, I do not for now provide explicit expressions for these Green's functions, though this should be straightforward enough once ii) is accounted for.

13.6.1 3-stop metroland example

$$t_{\{1\}} - t_{\{1\}}(0) = \frac{1}{2} \int_0^{\mathcal{T}_{\{0\}}} \left\{ \frac{1}{\{E - A\rho^2(\mathcal{T}_{\{0\}})\}} \int_{\varphi=0}^{2\pi} \exp\left(\frac{i\{E_{d'} - E_d\}\mathcal{T}}{\hbar}\right) \frac{BE^2 \langle \psi_{d'} | \cos 2\varphi | \psi_d \rangle}{A^2 \{1 + 2E^2 \mathcal{T}^2 / A\}^2} d\varphi d\mathcal{T} \right\} = \frac{B}{4AE} I_{\varphi}^t(d', d) J(\beta, \mathcal{T}) \quad (859)$$

for

$$J(\beta, \mathcal{T}) := \int_0^{\mathcal{T}_0} \frac{\exp(i\beta\mathcal{T}) d\mathcal{T}}{\{1 + 2E^2 \mathcal{T}^2 / A\} \mathcal{T}^2} . \quad (860)$$

It again has a much simpler case for $\beta = 0$. For this, the integral looks to diverge as $1/\mathcal{T}$ as $\mathcal{T} \rightarrow 0$. However, \mathcal{T} goes like $\cot(t^{\text{em}}) \approx \cos(t^{\text{em}})/\sin(t^{\text{em}}) \approx 1/t^{\text{em}}$ for t^{em} small, so that the time transformation blows up if one reaches exactly $\mathcal{T} = 0$, for which $t^{\text{em}} \rightarrow \infty$, or $t^{\text{em}} = 0$. Noting that t^{em} is this toy model's analogue of t^{cos} and infinite cosmic time is unphysically far to the future in Big Bang type universes, we are physically justified in practise to employ some cutoff procedure before $\mathcal{T} = 0$ is attained so that the $1/\mathcal{T}$ divergence is in fact averted.

Finally, the inhomogeneous term is

$$f(\varphi, \mathcal{T}) = \frac{B}{A} \left\{ \frac{E^2 \cos 2\varphi}{A\{1 + 2E^2 \mathcal{T}^2 / A\}^2} \pm \frac{\hbar^2 A \delta_{1d}}{32 E^2 \mathcal{T}^2} \right\} , \quad (861)$$

with the plus sign holding for the cosine solutions and the minus sign holding for the sine solutions.

The general answer is then the linear combination of the complementary functions for the linear operator in question,

$$L_{3\text{-stop}} := i\hbar \frac{\partial}{\partial \mathcal{T}} + \frac{\hbar^2}{2} \frac{\partial^2}{\partial \varphi^2} \quad (862)$$

plus the integral (858) with the Green's function corresponding to this $L_{3\text{-stop}}$ and the above f inserted into it. This particular example illustrates a back-reaction occurring to first order.

13.6.2 Triangleland example

Now, the $t_{\{1\}}^{\text{em}}$ integral has a $I_{\Theta\Phi}(\mathbf{R}, \mathbf{R})$ factor which is always zero. Thus this example gives no back-reaction to first order; I take second order to be beyond the scope of this article. The corrected time-dependent Schrödinger equation then has no expectation term, but still does have a particular integral from its first term,

$$\tilde{J}|\chi^{\{0\}}\rangle \propto \frac{\cos \Theta}{\{\mathcal{T} - \mathcal{T}_c\}^2} \mathcal{Y}_{\text{Sj}}(\Theta, \Phi) = f(\mathcal{T}, \Theta, \Phi) . \quad (863)$$

The general answer is then the linear combination of the complementary functions for the linear operator in question,

$$L_{\Delta} := i\hbar \frac{\partial}{\partial \mathcal{T}} + \frac{\hbar^2}{2} \left\{ \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left\{ \sin \Theta \frac{\partial}{\partial \Theta} \right\} \right\} \quad (864)$$

plus the integral (858) with the Green's function corresponding to this L_{Δ} and the above f inserted into it.

13.7 Commentary: the WKB procedure is crucial but unjustified (Analogy 88)

It is easy to argue out the applicability of the WKB scheme to Quantum Cosmology to general physics audiences in a way which sounds plausible and familiar and thus not requiring careful scrutiny. Unfortunately, the familiarity stems from the commonplace occurrence of the WKB approximation in basic ordinary QM, where it rests safely on the Copenhagen interpretation and on controlled lab conditions, neither of which are meaningful in Quantum Cosmology [633]. In ordinary QM, one often assumes the WKB ansatz as a consequence of the pre-existence of a surrounding classical large system [416], which is no longer applicable for the whole universe. In ordinary QM, the WKB ansatz can be justified by the lab set-up being a “pure incoming wave”. But if one assumes a pure wave in the quantum cosmological context, its wavefronts orthogonally pick out a direction which serves as timefunction, so this amounts to ‘supposing time’ rather than a ‘bona fide emergence of time’ as required to resolve the Problem of Time. Another argument rests on the constructive interference which underlies classicality [632, 633], but this amounts to imposing, rather than *deducing* (semi)classicality, as does using (semi)classicality as a ‘final condition’ restriction on quantum-cosmological solutions.⁷⁹ And not being able to justify the WKB ansatz is a particular problem for the Semiclassical Approach since its trick by which the chroniferous cross-term becomes the time-derivative part of a time-dependent Schrödinger equation is exclusive to wavefunctions obeying the WKB ansatz [658, 659, 98, 400, 77, 335, 80, 83].

Moreover, the WKB ansatz is not general or a priori natural. The W-function will solve an h-equation that is mathematically a Hamilton–Jacobi equation (or at least approximated by a such). However, Hamilton–Jacobi equations have 2 solutions W^\pm . (For the case in which the velocities feature solely homogeneous-quadratically in the kinetic term, these are \pm the same thing, but more generally the 2 solutions are \pm in the sense of being a complex conjugate pair.) Thus in general one would expect not $\exp(iW)$ but [659, 98, 77, 335, 400, 260, 660]

$$\phi(h) = A_+ \exp(iW_+(h)/\hbar) + A_- \exp(iW_-(h)/\hbar) . \quad (865)$$

Is it justifiable for the ansatz for the wavefunction to contain a single piece? Suppose it does not: take (865) instead of the WKB ansatz.⁸⁰ E.g. for $A_+ = A_-$, $W := W^- = -W^+$, one gets not a time-dependent Schrödinger equation but a real equation that is somewhat reminiscent of a diffusion equation,

$$\frac{\hbar}{h^2} |\chi\rangle = 4 \frac{M}{h} \tan\left(\frac{MW}{h}\right) \left\{ \frac{\partial}{\partial t^{\text{em}}} - \left\langle \chi \left| \frac{\partial}{\partial t^{\text{em}}} \right| \chi \right\rangle \right\} |\chi\rangle \quad (866)$$

([22], which somewhat generalizes a result from [77]). For this, the chroniferous rearrangement ceases to work: one *does not* obtain the emergent time term $i\hbar\partial/\partial t^{\text{em}}$, but rather a cumbersome complex function that contains both W_+ and W_- .

If a single WKB piece, why the particular sign? Zeh and Barbour [658, 659, 77, 80] have built up an argument pointing out that it is meaningless to assert that a + sign in the WKB exponent corresponds to an expanding universe and a – sign corresponds to a contracting one. This often plays a role in Arrow of Time arguments (rightly or wrongly). Scaled RPM is a reasonable arena to investigate this (with increases and decreases in moment of inertia deputizing for the increases and decreases of the volume of the universe in GR).

13.7.1 Global problem: WKB in general only holds in certain regions

The WKB regime cannot be expected to hold everywhere. Thus the Semiclassical Approach is not a complete resolution of the Problem of Time, nor even of the ‘paradox’ of the timeless appearance of the quantum theory. (Though there may not necessarily be any need to resolve this in regions outside ‘quotidian experience’ – one cannot testify that there is a semblance of dynamics in regions in which semiclassical Quantum Cosmology does not apply).

JBB-WKB alignment extends the relevance of the issues raised with JBB time in Sec 12.1. Thus WKB time is also globally limited by $E - V$ having zeros. Often one will have oscillatory behaviour on the one side of these and decaying behaviour on the other. The WKB procedure is then invalid at each zero and a very poor approximation near each zero. A distinct approximation is required around each zero (as per the theory of connection functions in the ordinary differential equation case). Thus a time arising from a WKB procedure cannot be claimed to be generically applicable over configuration space. Rather, one should expect a number of patches in configuration space where a different regime applies, for which emergent WKB time is not a valid answer to the Problem of Time. Additionally, if the zeros are sufficiently near to each other, there is no room for a WKB regime in the region between them, so applicability of a WKB procedure is scarce in that region of configuration space. From a physical perspective, (Semi)classicality in the sense of WKB does need not occur everywhere or ‘everywhen’ in a mechanical motion. I note that the tessellation by the physical interpretation method helps with identifying such regions for RPM’s.

The Problem of Zeros has well-known connection to the breakdown of the WKB approximation.

⁷⁹LQC can also be argued to not address this point either, there aggravated by the number of rejected solutions becoming larger than usual. This is also akin to how Griffiths and Omnès remove by hand the superposition states that they term “grotesque universes” [275] due to their behaviour being other than we experience today.

⁸⁰Moreover, there may be further scope for superposition of terms due to ‘multiple saddles contributing’.

Question. What type of equation arises in the connecting regions, and is there any resolution of the problem of time for this? If so, is it patchable to the emergent WKB time resolution in the other regions?

Question. Does some version of the connection formula procedure for patching together regions remain suitable at the level of prolonging QM evolution?

For the 3-stop metroland example provided, (428) is stable to small angular disturbances about φ for some cases of harmonic oscillator/cosmological constant, but it is unstable to small angular disturbances for the gravity/dust sign of inverse-power potential. Positive-power potentials are finite-minimum wells, but cease to be exactly soluble. For Newtonian Gravity/dust models (or negative power potentials more generally), near the corresponding lines of double collision, the potential has abysses or infinite peaks. Thus the scale-dominates-shape approximation, which represents a standard part of the approximations made in setting up the Semiclassical Approach to Quantum Cosmology, fails in the region around these lines. It is also then possible that dynamics that is set up to originally run in such a region leaves it, so a more detailed stability analysis is required to ascertain whether semiclassicality is stable in such models. This issue can be interpreted as a conflict between the procedure used in the Semiclassical Approach and the example of trying to approximate a 3-body problem by a 2-body one (see 5.3.7 for further comments on this and c.f. Fig 39). The difference in the analogy between Cosmology and the spherical presentation of triangle-land causes the potentials for that which are studied in this article (and the wider range of these considered in [37]) to be purely positive powers for which the preceding problem does not occur.

There is also the issue of whether one should apply higher-order WKB techniques [378] (which is relevant when the associated small quantity $\epsilon_{\text{em(WKB)}}$ is insufficiently small for $\epsilon_{\text{em(WKB)}}^2$ to be entirely negligible). Moreover, WKB techniques are but one of a family of techniques in semiclassical QM [448, 163]. Does the chroniferous role and the difficulty in justifying the semiclassical regime pervade all of this?

13.7.2 The next level of arguments for why the WKB ansatz may apply

1) If the many-worlds interpretation of QM is adopted, each piece of the typical superposition is realized in a different branch. However, one may be able to object that this is not in accord with actually experiencing such a superposition, or undermine the many-worlds interpretation on grounds of unnecessary richness or, possibly, of conceptual difficulties. This is another case of trading off time problems for interpretation of QM problems.

2) Alternatively, it has been suggested (see e.g. [375, 372] and references therein) that the WKB ansatz could be justified by *decoherence*. Though some reservations about this have also been expressed (e.g. [400, 335, 96, 260, 373, 298, 294] are between far from and not entirely optimistic about this). This includes getting the decoherence from the machinery of Histories Theory (see Sec 15.4).

13.7.3 Testing approximations by ulterior exact solvability

A different perspective to postulating unproven suggestions is that the ansatz can be put to the test, by investigation for classes of ulteriorly exactly-soluble models (i.e. ones which are soluble by techniques outside the Semiclassical Approach.) An extension of this would be to use e.g. the naïve Schrödinger interpretation to provide the relative probabilities of experiencing a WKB regime within a given range of model universes.

However, I find that even addressing the question of where it holds in simple toy model cases is not in practice clear-cut, due to there being of the order of additional, distinct quantum-cosmological approximations that one requires to make alongside it. In existing literature, these are mostly tacit or not touched upon by leaving vague the full extent of what the words ‘adiabatic’ and ‘WKB’ need to mean to fully cover all the simplifications required. This may not entirely be fair once order of magnitude estimates are brought in, though the literature is not clear on that either. Various of the below considerations also act to worsen this proliferation of approximations. Thus even explicit investigations in ulteriorly exactly soluble models only concern small pieces of configuration space characterized by all the other approximations being made. [Thus RPM’s are valuable conceptually and to test whether one should be *qualitatively* confident in the assumptions and approximations made in such schemes.]

13.7.4 The many approximations problem.

One contribution the present article makes to the above debate is that it is hard to meaningfully isolate the testing of whether the WKB condition applies due to the plethora of other approximations made. It would seem that at best one can make a number of such and then test whether the WKB approximation holds in the small region of the configuration space where all those approximations are applicable. Thus it would in general be very drawn-out to carry out the above-suggested programs.

Some papers [378, 104, 377] investigate Quantum Cosmology by expanding in 1 parameter. That however there are multiple parameters was pointed out by Padmanabhan [495], and is investigated explicitly in the present article. While [495] proceeded by considering which parameter to expand in, in the present article I point out rather that 1-parameter expansions in no matter what parameter will not in general suffice for beyond a corner of the Quantum Cosmology solution space. In general one would have to expand in many independent parameters. Careful theoretical arguments may however then match

certain frameworks with less parameters to certain relevant situations to various degrees of accuracy. For some consideration as to what regimes are required in GR cosmology, see [23].

13.8 Commentary: detailed meaning of approximations used

13.8.1 Adiabaticity (Analogy 89)

N.B. that there are a lot of adiabaticity conditions used.

The context for this is an adiabatic loop in phase space, whence this scheme is underlied by being in a *classically-adiabatic* regime (i.e. that classical h-processes are much slower than classical l-processes):

$$\epsilon_{\text{Ad}} = \Omega_{\text{h}}/\omega_{\text{l}} = t_{\text{l}}/t_{\text{a}} \quad (867)$$

for ‘characteristic frequencies’ Ω_{h} and ω_{l} . The suffix ‘a’ indeed stands here for ‘adiabatic’, and is much used below as there are numerous different adiabatic approximations at the quantum level. For us, non-adiabatic terms are h1-h3, l1-l3, l9 and l10.

At the quantum level, there are two different ‘pure forms’ that adiabaticity can take. Firstly, there are quantities that are small through $|\chi\rangle$ being far less sensitive to changes in h-subsystem physics than to changes in l-subsystem physics, which I label ‘p’. Secondly, there are quantities that are small through $|\chi\rangle$ being far less sensitive to changes in l-subsystem physics than ϕ is sensitive to changes in h-subsystem physics, which I label by ‘q’.

Note 1) None of the above in general follow from the smallness of the classical adiabatic parameter ϵ_{Ad} . For, some wave-functions can be very steep or wiggly even for slow processes - like the thousandth Hermite function for the slower oscillator, say. However, high wiggleness is related to high occupation number via quantum states increasing in number of nodes as one increases the corresponding quantum numbers, and high occupation number itself is a characterization of semiclassicality.

Note 2) Inspection of the h and l equations furthermore reveals that both p and q occur in terms also containing an $\epsilon_{\text{hl}} = |J_{\text{hl}}/V_{\text{h}}|$ type factor. Thus, overall, these terms in the equations are particularly small.

Note 3) Massar and Parentani’s work including non-adiabatic effects [504] in the minisuperspace arena is similar in spirit to the present paper in choosing to keep yet another sort of terms that are usually neglected in Quantum Cosmology. In this case, the effects found are expanding universe–contracting universe matter state couplings and the quantum-cosmological case of the Klein paradox (backward-travelling waves being generated from an initially forward-travelling wave).

13.8.2 Higher derivatives (Analogy 90)

One often neglects the extra t^{em} -derivative terms whether by discarding them prior to noticing they are also convertible into $t_{\{0\}}^{\text{em}}$ -derivatives or by arguing that \hbar^2 is small or ρ variation is slow. Moreover there is a potential danger in ignoring higher derivative terms even if they are small (c.f. Navier–Stokes equation versus Euler equation in fluid dynamics). For us, higher-derivative terms are h2, l2 and l9, with l9 especially significant through being the unaveraged one.

One would expect there to be some regions of configuration space where the emergent time dependent wave equation of behaving more like a Klein–Gordon equation than a time-dependent Schrödinger equation, and is in all fullness more general than either of these. Thus the guarantee of appropriate interpretability that accompanies time-dependent Schrödinger equations is replaced by a difficult study of a more general wave equation. It is worth noting that Klein–Gordon-like but more complicated equations are prone to substantial extra impasses (see, e.g. [397] or Sec 11.6.1).

13.8.3 Averaged terms (Analogy 91)

Expectation/averaged terms are often dropped in the Quantum Cosmology literature.⁸¹ For us, these terms are h1-h4, l1-l4 and l8. The usual line given for this in that literature is that these are argued to be negligible by the

Riemann–Lebesgue theorem, which is the mathematics corresponding to the physical idea of **destructive interference**.

I add that people in the field probably do not want such terms to be around due to non-amenability to exact treatment that they confer upon the equations if included. However, here are some reasons to keep it.

1) Here are RPM counterexamples to these terms being small. For 3-stop metroland, $O(\hbar^2)\langle\partial_{\varphi}^2\rangle$ and the $|\psi_{\text{d}}^{\text{t}\{0\}}\rangle$ are eigenfunctions, so this one goes as $O(\hbar^2)$ – an example of an averaged term not being smaller. As another example of this, for triangle land, $O(\hbar^2)\langle\Delta_{\mathbf{5}(3,2)}\rangle$, and the \mathcal{Y} ’s are eigenfunctions, so this one goes as $O(\hbar^2)$ to leading order.

2) **Geometric phase** being a significant consideration is a reason for considering some averaged terms that is more widespread within the Quantum Cosmology literature itself; this is in analogy with Berry Phase in ordinary QM.

⁸¹Incidentally, the idea of neglecting averaged terms is a brutal illustration of how dimensional analysis is not all, since averaged and unaveraged versions of a quantity clearly have the same ‘dimensionless groups’ (meant in the sense of fluid mechanics).

3) I have pointed out [40] an analogy with Atomic and Molecular Physics, where the counterparts of such terms require a **self-consistent** variational–numerical approach, an example being the iterative techniques of the **Hartree–Fock scheme**. In Atomic and Molecular Physics, it is then conceded that this ensuing non-exactly tractable mathematics is necessary so as to get passably correct answers (comparison with experiments confirms this).

While there are a number of differences between Molecular Physics and Quantum Cosmology, Hartree–Fock theory in fact is known to span those differences. E.g. it is available in t -dependent for sure, and involving a plain rather than antisymmetrized wavefunction, and for field theory (c.f. the Condensed Matter Physics literature [180, 281, 196, 320, 554, 197, 610]).

Question. Variationally justify this Hartree–Fock self-consistent procedure, and then study its outcome.

Question*. It is not as yet clear how to extend the self-consistent treatment to this non-negligible back-reaction case. It is something like

$$\left\{ \begin{array}{l} \text{(emergent-time-dependent Hartree–Fock scheme)} \\ \text{(chroniferous expectation-corrected variant of the Hamilton–Jacobi equation)} \end{array} \right\}, \quad (868)$$

which, by this stage, is *not* a mathematical format that I have ever seen elsewhere in the Physics literature. As such, this investigation is not just of qualitative confidence in the Halliwell–Hawking scheme but important also as regards the detailed robustness of the Semiclassical Approach’s time-emergence itself.

Note 1) Is there any promise to additionally incorporating the approximate atom-by-atom product form of Hartree–Fock wavefunctions to make a clump-by-clump analysis of inhomogeneous cosmology? The nonlinearity of GR is ultimately sure to cause problems here, though this clump-by-clump possibility might just be possible within the RPM toy models themselves.

Note 2) If a Hartree–Fock scheme is deemed to be appropriate for Quantum Cosmology, whether it iteratively converges will become an issue and the nonstandardness of the system (868) will likely make this hard to check for.

13.9 End-check against semiclassical time’s problems

Semiclassical Problem-1. Having invoked a Wheeler–DeWitt equation results in inheriting some of its problems [400, 335]. Of course, for RPM’s this is less severe: there is no inner product problem, no functional derivatives or need of regularization. Semiclassical Problem-2. Making the WKB approximation requires justification. I showed how in RPM’s (semi)classical conditions need not always occur.

Semiclassical Problem-4. It is furthermore unclear how to relate the probability interpretation of the approximation with that for the underlying Wheeler–DeWitt equation itself [335, 400]. Considering various less approximated equations will help with this. I have not looked into it, but RPM’s are a suitable arena in which to do so.

Semiclassical Problem-5. The status of the Spacetime Reconstruction Problem is unclear for the Semiclassical Approach. RPM’s cannot address this issue since these have no spacetime to reconstruct.

Semiclassical Problem-6. The Multiple Choice Problem remains present in such schemes in detail [400]. This is in principle investigable using RPM’s, but has not yet been done.

13.10 Appendix A: geometric phase issues

13.10.1 Berry phase

$$D_h^{\alpha i'} = \partial_h^{\alpha i'} + i A_h^{\alpha i'} , \quad D_h^{* \alpha i'} = \partial_h^{\alpha i'} - i A_h^{\alpha i'} \quad (869)$$

denote the Berry covariant derivative and its conjugate. Here, $A_h^{\alpha i'}$ is the Berry connection [123, 575], i.e. the vector gauge potential induced by l-physics on h-physics of a nondegenerate quantum state that corresponds to its $U(1)$ freedom in phase,

$$A_h^{\alpha i'} = -i \langle \chi_j | \partial_h^{\alpha i'} | \chi_j \rangle . \quad (870)$$

h-equations in place of the Born–Oppenheimer equation are e.g. Mead–Truhlar’s [454] or Berry–Simon’s [123, 575, 124] geometrical form[124]. One merit of Berry’s scheme as opposed to BO’s is that it is one means of including back-reaction. Moreover, Berry’s scheme includes this in a geometrically insightful way which is more precise and indeed more correct in laboratory situations. (Effects have been observed [429], the explanation of which has been found to rest on geometric phase [454, 123, 575]).

Question. It is not clear what happens phase-geometrically upon applying the WKB ansatz. I.e. Berry rendered BO very clear in this respect, but on applying the WKB ansatz too, I have never seen anybody use a clear geometrical interpretation of that. Why?

13.10.2 Absolute or relative geometrical phase?

Using l-equations with no averaging corrections amounts to removing phase, and this is in general inadmissible [190] due to this phase possessing a physical meaning.

This issue is moreover quite subtle, requiring quite some conceptualization and nomenclature to discuss. The point of including this Appendix is that this issue is tied both to time issues and to relational considerations, and moreover may override the preceding SSSec's no-go as concerns the present article's concrete modelling. Let γ_{TNS} be the *total phase* in the case in which the presence of an external time causes this to be naturally splittable into

$$\gamma_{\text{TNS}} = \gamma_{\text{geometric}} + \gamma_{\text{evolutionary}} \quad (871)$$

In the case in which there is no such external time, however, one has, rather, a total phase γ_{T} for which there is no natural such split; moreover this case can be thought of as entirely due to quantum phase geometry,

$$\gamma_{\text{T}} = \gamma'_{\text{geometric}}. \quad (872)$$

Datta's argument is that the primed and unprimed notions of quantum phase geometry above are geometrically distinct. In particular, γ_{T} is both gauge invariant (as befits its nature as a geometric quantum phase) and MRI (as befits a total dynamical phase of a theory with no external time, of which Quantum Cosmology is surely an example.) This means overall that the primed, *relative quantum phase geometry* has more unphysical quantities ('overall gauge freedom') than the unprimed *absolute quantum phase geometry*. This means that more terms 'are gauge rather than physical' if relative quantum phase is in use. Thus there are differences in each case as regards what is purportedly physical, so one should take due care to use whichever is appropriate for the situation in hand. Non-external time requires relative rather than absolute quantum phase geometry. Assuming this argument is correct, this Chapter's treatment would need to be upgraded.

One consequence of Datta's relative phase is that gauge choice in the original sense (no inverted commas) also causes the zero-point energy to be shifted so that J_{hl} is renormalized to $J_{\text{hl}} - \langle J_{\text{hl}} \rangle$. This then causes $\langle J_{\text{hl}} \rangle$ to drop out of the h-equation. The physics is then only in the fluctuating quantities, invalidating the unbarring of the l-equations also.

Question Use RPM's to further investigate how this alternative type of back-reaction works, at the level of a small perturbation.

A second consequence is that the enlargement of what is gauge circumvents the argument that a 1- d configuration space for the h-subsystem entails automatic neglect of connection terms. Thus the very simplest toy models can be investigated to see if relative phase effects cause major or minor alterations to 'conventional wisdom'. That includes RPM's due to their implementation of temporal relationalism, even in the easiest cases in which nontrivial spatial relationalism is absent.

Question. Carry out this investigation for simple RPM's. What is the relative phase geometry for these? How are this Sec's equations to be modified due to this, and how does this affect the nature of the emergent Semiclassical Approach's equations and significances?

13.11 Appendix B: decoherence in standard QM, decoherence in QC and any differences between the two

Why do we not observe superpositions of macroscopic objects, such as of dead and live cats? An explanation is that interaction with the environment swiftly measures such a system, reducing it to an entirely live or dead state. On the other hand, QM phenomena indicate that very small things are not swiftly reduced in such a way. For example, the ammonia molecule behaves as a superposition of umbrellas and wind-blown umbrellas. There is then the question of for what sort of size do such superpositions quickly become reduced. The outcome of this is that sugar molecules are observed to stay in one chirality, unlike ammonia. Therefore, the boundary for such behaviour lies somewhere between the characteristic size of ammonia molecules and that of sugar molecules (which are much smaller than cats). For standard QM decoherence, see e.e. [260] and references within.

Decoherence in Quantum Cosmology (see e.g. [375, 372]) has somewhat distinct features. As regards explaining the semiclassical approach to Quantum Cosmology, one idea is that $A \exp(iW_+/\hbar) + B \exp(iW_-/\hbar)$ is a superposition along the above lines. Though the situation for whole universes is not the same – there is no pre-existing time in which to decohere...

The basic scheme is that one can do further perturbation terms with standard mathematics to begin to investigate the effects of not neglecting each of the commonly neglected terms and Part II has exact comparers. Thus the present article contains a laboratory for qualitative investigation of how reliable our understanding of the quantum-cosmological origin of the structure in the universe is. Though, clearly, the current article has only just begun to use this machinery (I will priorly list what else to do as Questions).

Question and Analogy 92) Decoherence time is the (suitable notion of) time taken for off-diagonal components of the density matrix to effectively vanish. This is typically extremely short for everyday, macroscale process within the usual QM

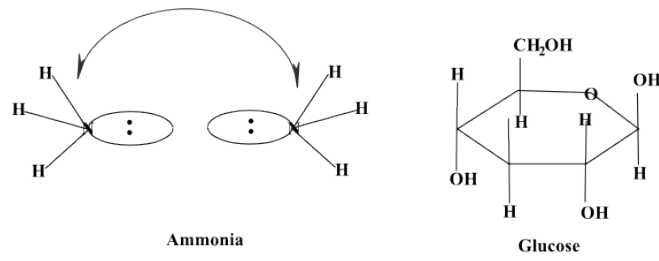


Figure 52: There's a QM behaviour size interface somewhere between the sizes of the depicted molecules.

framework. Some physical examples are useful in understanding the nature of this quantity. $t_{\text{Dec}} = \hbar/2\pi|V|^2$ for V an interaction function CaLo05. t_{Dec} for a macroscopic body is within a few orders of magnitude of the Planck time. In cases involving a thermal bath, $t_{\text{Dec}} = \hbar/4\pi K_{\text{B}}T$. Generally, ML^2/τ is classical action and \hbar is what it is compared to. Then t_{Dec} goes as $\tau\hbar^2/ML^2KT$, i.e. as $\tau(\lambda_{\text{DB}}/L_{\text{char}})^2$ where 'DB' stands for De Broglie. These examples are, moreover general enough to apply e.g. to the triangle with various potentials. This will go into questions concerning e.g. the base decohereing the position of the apex. These are, in turn, as good a detector model as is possible within the triangle. Investigate.

14 Tempus Nihil Est 1

14.1 Some levels of structure for timeless approaches

I assume below that QM on configuration space suffices for the study of timeless records (and is arguable to be privileged for such as per Appendix 4.B). Then Sec 3 on the structure of configuration space is both a useful start and, in fact, not detailed enough, so that I need to introduce a number of further layers of structure below.

14.1.1 Subconfiguration spaces

Sec 3's configuration spaces are taken to be for whole-universe models. However, in physics, in practise one usually deals with subconfigurations that correspond to subsystems (they are furthermore to be the primary objects in Records Theory). One can then define subconfiguration spaces for these in direct parallel to defining configuration spaces for whole systems.

Moreover, I argue that the general mathematical class of subconfiguration spaces is not particularly of physical interest. What is of interest is, rather, the cases of subconfiguration space that are maintained by the physics or which correspond to only partially observing a system. For triangleland, an example of the more interesting case is keeping track of the length of the base of a triangle $I_{\text{base}} = I_1$ (ignore $I_{\text{median}} = I_2$ and Φ). An example of the less interesting case is restricting attention to the isosceles triangles (by fixing Φ to be $\pm\pi/2$, which is prone to not being maintained by the actual dynamics of the system).

These examples illustrate that the subconfiguration spaces of interest are more likely to be families of subspaces, one for each value of irrelevant coordinates rather than the mathematically cleaner, but often less physical, picture of a single subspace of a given space.

The more interesting example above is, mathematically, a half-line which sits in the radial direction within $\mathfrak{R}(3, 2)$.

At least in general, not all values of the ignored quantity correspond to submanifolds of the same type for the observed quantities, and for some values, may not even be defined. I note that this example is fairly localized in space by involving the base's cluster, though subconfigurations need not be localized.

A useful example here is the scale of the system and the overall shape of the system are subconfigurations to which each particle in the model contributes, so that they are highly nonlocal. As well as for RPM's, this applies to inhomogeneous GR and to anisotropic minisuperspace.

I also note that some subconfigurations have particular meanings pinned on them: studied subsystem versus environment, heavy slow h subsystem versus light fast l subsystem. Nature can hardly be expected to always kindly align h-l, light-fast and studied-environment for us. Finally, a useful recollection from Appendix 5.C is that subconfiguration spaces are not necessarily themselves smaller RPM configuration spaces since they can each possess nontrivial properties that the whole RPM universe cannot possess, such as nonzero angular momentum (or, in the case of pure-shape RPM, a subconfiguration can possess an overall dilatation/expansion factor). Thus e.g. the subconfiguration spaces of a zero total angular momentum universe are certainly not themselves zero total angular momentum.

14.1.2 Some side-issues about subconfigurations and 'contextual' relationalism

The main role of subconfigurations in this Sec is that Records are localized subconfigurations of a single instant. Moreover, localized subconfigurations also play an underlying role in Crane's thinking [188]. This involves the postulate that I term

Contextualism 1) **QM ONLY makes sense for subsystems** (my caps).⁸² that In the quantum GR arena, he deals with this by considering splits of the universe with the observer residing on the surface of that split.⁸³ Each of these splits then has its own Hilbert space. I use the notation $\text{Context}(\mathbf{q}) = \langle \text{set of all physically meaningful Sub}\mathbf{q} \rangle$.

Note 1) This 'only' includes the claim that there is no whole-universe Hilbert space. I argue against this in Sec 16.3.

Note 2) Considering subsystems, whether by its particular practicality or as a matter of no choice, takes one away from where RPM is an important part of the modelling as opposed to the highly unrestricted emergent Newtonian Mechanics of localized subsystems as per Appendix 5.C.2. Using specifically subsystems of small RPM's allows one to investigate what happens as the subsystems one considers get to have enough contents to be the whole-universe model, whilst keeping the mathematics (if not necessarily the standardness of the QM interpretation) simple.

Note 3) Crane also postulates that whole-universe physics does make sense in a semiclassical limit. Whilst he is almost certainly referring to a different technical definition of semiclassical limit from that in Sec 13, this is at least conceptually in line with Sec 13's viewpoint.

⁸²Bear in mind that Crane himself calls these *relational* postulates, so my terminology here is just my way of trying to maintain some clarity in subsequent discussion that contains large numbers of different ideas that are usually referred to as 'relational'. Hopefully my intended meaning of **contextual** is clear: it is that what is observed depends on the context given to that observation by the nature of the observer involved, e.g. their position, motion, and sensor capacity and of the subsystem-environment split. I also credit Lee Smolin for having been involved in the conception of these contextual ideas, and for publicizing some of them in [586], which was very useful education to me.

⁸³In fact, Crane [187] defines the observer to be such a boundary, though I caution that not all boundaries will have actual observers upon them and that sizeable boundaries would need to be populated by many observers. Crane's view (and Rovelli's less specific one [540]) of observers both strip them of any connotations of animateness or manifestly capacitated for processing information.

Note 4) Whilst I have no evidence for such a connection, I comment that on the one hand one has multiplicity by inequivalent quantum theories and on the other hand multiplicity by multiple observer perspectives. Are there ever any connections between these two multiplicities?

Contextualism 2) Crane also allows for the QM of **observers observing other observers observing subsystems**, i.e. of larger subsystems containing a smaller subsystem under observation and the observer performing that observation (see also Rovelli [540]). This provides a further level of structure between the plethora of Hilbert spaces associated with all the contexts of Contextualism 1).

Note 4) In the Conclusion, I argue that Rovelli's Partial Observables and Bojowald et al's [131, 132] type of notion of localized clocks are also manifestations of contextualism.

Note 5) In RPM's, the contextual approach can be toy-modelled using the notion of a localized fictitious test-observers who can measure nearby inter-particle distances and perhaps relative angles about their particular position (c.f. Appendices 3.E and 4.B), archetypes being considering the base subsystem of a triangleland or the base alongside the relative angle (the 'fictitious test-observer more or less at the centre of mass of a localized binary'). This can accommodate Contextualism 1) to some extent, but cannot meaningfully incorporate Contextualism 2) which explicitly requires non-negligibility of observer material properties.

Note 6) The involvement of subsystems requires for, conventionally, SubQuant: (SubPhase, SubCan) \rightarrow (SubHilb, SubUni), or (D)A versions of this in the presence of a nontrivial \mathfrak{g} , noting that global constraints do not apply, at least to a very large extent. [If the universe has a fixed positive energy E_{Uni} and the complement to the subsystem has to have a positive energy, say, to admit solution then there is still an inequality on the energy of the subsystem itself, $E < E_{\text{Uni}}$. But the linear constraints are free. How that is so variationally if just the subsystem is of interest? It would appear that even if one neglects the complement subsystem in all other respects one still has to do the \mathfrak{g} -variation for the whole universe. E.g. for a 2-island universe model with each island conserving its own angular momentum, the study of one island needs to involve $(\rho \times \pi)_{\text{subsystem}} = -(\rho \times \pi)_{\text{environment}}$ which is then to take a free (or observed) value. This reflects that the best matching argument has an inherently whole-universe character: rotation has to be of the whole universe rather than just of the subsystem under study, since rotation that subsystem and not the complement involves an in-principle discernible.] Or possibly (Sub \mathfrak{q} , SubPoint) or (SubRigPhase, SubPoint) versions if one adheres to operational distinguishability implementing morphisms only – the further 'Contextual' expansion of Relationalism 3) It should be clear from the above that in the presence of a nontrivial \mathfrak{g} , Sub \mathfrak{q} is to include variable E and \mathcal{J} parameters as part of its specification (with SubPhase, RigSubPhase and SubHilb following suit).

Note 7) The Crane set-up then involves 'the entire set of SubQuant's', with Contextualism 2 necessitating yet further structure beyond that, as well as nontrivial modelling of observers as quantum systems (an idea nobody can really do, which also features in the thinking of Page and of Hartle; the state of the art here is Hartle's IGUS model for observers). My preceding SSsec makes the possible advance that 'the entire set' requires physical Sub \mathfrak{q} 's rather than the greater mathematical generality of Sub \mathfrak{q} 's, and that it is additionally contingent on a definition of localization in space to be obeyed as a further practical consideration in the selection of a set. I add that these desirable features would not appear to be conducive to the implementing sets being mathematically simple; the notion of localization in space in particular would introduce a source of ambiguity into the model: *how* local and by what criterion.

For now, I just return to my development of Records Theory, within the restriction of humanity not being ready to cope with the mathematical implementation of the previous sentence! [See Appendix 14.B for a simple RPM example of subsystem QM and 16.3 for a final comparison and synthesis of the various people's notions of relationalism.]

14.1.3 Imperfect knowledge of a (sub)system's configuration

A priori, there are two distinct notions of imperfect knowledge of a (sub)system's configuration.

- 1) Imperfect knowledge of the system's contents e.g. modelling a labelled triangle $\{1, 23\}$ by $\{1, \text{COM}(23)\}$.
- 2) Imperfect knowledge of the system's state.

However, $\{1, \text{COM}(23)\}$ is I_{median} with I_{base} and Φ arbitrary which is clearly a special case of imperfect knowledge of state. This argument extends to show that 1) is but a subcase of 2), which is in accord with the literature usually taking grainings to mean 2) (whether for configuration space, phase space or histories space).

Note 1) 1) above also requires as good a consideration of union of configuration spaces as is possible (hence Sec 14.1.6). Though one might restrict oneself to using a space they all sit in and then using a map $M : \mathfrak{q} \rightarrow \text{Sub}\mathfrak{q}$. For this SSsec's example, it sends $\mathfrak{A}(3, 2) = \mathbb{R}^3$ to \mathbb{R}_+ which is any radial half-line. But sometimes such a map will only act on some parts of the configuration space, and may map to multiple types of space (mathematical niceness of such a map is not in general guaranteed).

Note 2) One may well place particular emphasis on localized subsets of \mathfrak{q} as regards representing one's imperfect knowledge of a (sub)system's state; this requires having a suitable notion of distance on \mathfrak{q} , and so is not developed until Sec 14.8.

14.1.4 Examples of physically/geometrically significant small regions of RPM configuration spaces

Sec 3.11 detailed significant points, edges and wedges on the configuration spaces of small RPM's, corresponding to such as equilateral, isosceles and regular triangles, collinearities, and various sorts of collisions and mergers. However, these are zero-measure portions of \mathbf{q} . Thus one needs approximate notions of these significant properties which cover nonzero-measure portions of \mathbf{q} (see Fig 53). The work of Kendall [368] is a useful pointer toward how to conceive of such (as well as providing substantial statistical machinery for the classical analysis of questions concerning shape, which lie beyond the scope of the present article). As a final piece of motivation, I note that the study of regions of configuration space already featured in Sec 13, as well as playing a role in Sec 15.

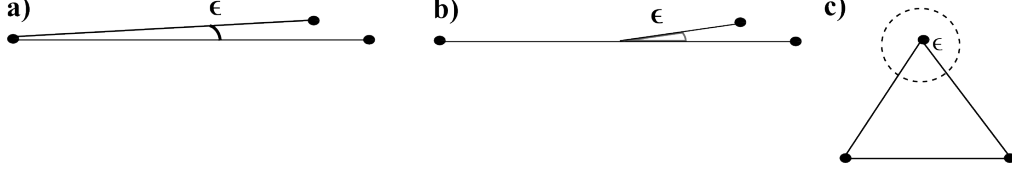


Figure 53: A meaning in space for a) Kendall's [368] ϵ -blunt notion of collinearity (in fact he considers the min for this over all choices of construction) and my notions [34] of b) ϵ -collinear (adapted to the dynamically useful Jacobi variables) [34] and c) ϵ -equilateral [34].

3-stop metroland has notions of near-collision and near-merger. For the pure-shape case, these are arcs centred about the D and M points. For the scaled case, these are wedges centred about the D and M half-lines, and there is also a disc of near-maximal collision. 4-stop metroland has notions of near-T and near-DD collisions. These are spherical caps in the pure-shape case [Fig 54a) has an example] or their extension in the hyperradial direction to form cones in the scaled case. There are also now near-D-arc belts [Fig 54a)] and (multi)lunes [Fig 54b)] in the pure-shape case, or their extension in the hyperradial direction to form coins and wedges of solid angle respectively in the scaled case. One can just as easily construct the same kinds of regions as in the preceding sentence around M-points and M-arcs, and there is a sphere of near-maximal collision. Triangleland has notions of near-equilaterality, near-D and near-M. These are spherical caps in the pure-shape case [Fig 54c) has an example], or their extension in the radial I-direction to form cones in the scaled case. There is also now a near-collinearity belt [Fig 54c)] and near-isoscelesness and near-regularity (multi)lunes [Fig 54d)] or their extension in the I-direction to form coins and wedges of solid angle respectively in the scaled case. There is also a sphere of near-maximal collision. Finally, some questions concern compositions of the above regions under union, intersection and negation. (e.g. spherical shells and segments or wedge portions of cones).

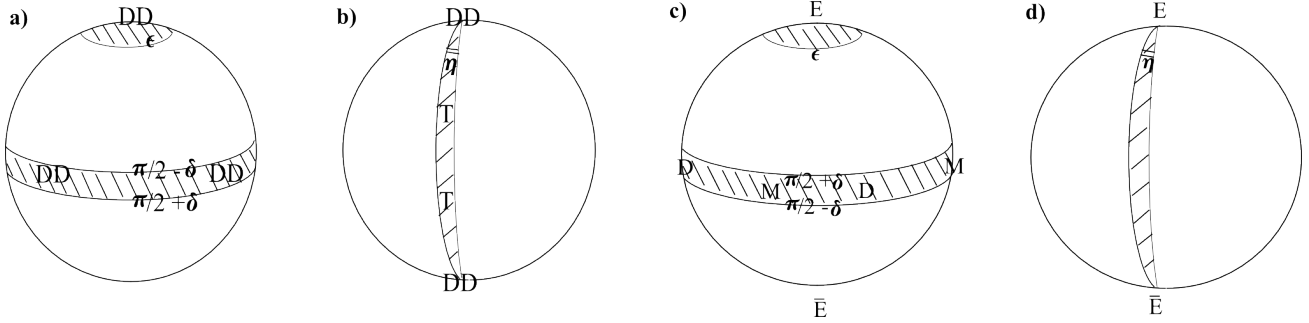


Figure 54: For 4-stop metroland, a) illustrates spherical caps and belts as constant azimuthal angle constructs. The spherical cap $\theta \leq \epsilon$ is a notion of closeness to a particular DD collision – that of the {12} and {34} clusters. The belt $\pi/2 - \delta \leq \theta \leq \pi/2 + \delta$ then corresponds to the opposite merger notion, i.e. to the centres of mass of {12} and {34} being near each other so that these clusters largely overlap (which is, in a certain sense, a more 'homogeneous' universe model).

b) illustrates a lune, i.e. a constant polar angle construct about a meridian, $\Phi_0 - \eta \leq \phi \leq \Phi_0 + \eta$. E.g. being in the lune around the Greenwich meridian means that the {34} cluster is localized (as does the lune about the meridian antipodal to that). Being in the bilune perpendicular to that means that the {12} cluster is localized. Finally, being in the tetralune at $\pi/4$ to all of these signifies that clusters {12} and {34} are of similar size, i.e. η -close to contents homogeneity.

For triangleland c) indicates the spherical cap of near-equilaterality $\Theta \leq \epsilon$ and the belt of near-collinear configurations correspond to the belt around the equator, $\pi/2 - \delta \leq \Theta \leq \pi/2 + \delta$. d) is an example of lune, $\Phi_0 - \eta \leq \Phi \leq \Phi_0 + \eta$; statements of near-isoscelesness and near-regularness correspond to (multi)lunes.

14.1.5 Grainings of universe contents

Example 1) In ordinary mechanics or RPM's one can define an operation by which a (localized) cluster is approximated by replacing it by the total mass of the cluster at the centre of mass of the cluster. This appeared in 3 as regards the three coarse-graining triangles constructible from a given quadrilateral in a given Jacobi coordinate system.

Example 2) Analogously, replacing GR's two anisotropy parameters by a single representative (e.g. the average one).

Example 3) Finally, in inhomogeneous GR one could (try to) define a local averaging operation that approximates complicated/multiple lumps by simple/single ones.

14.1.6 Unions of configuration spaces

While each \mathbf{q}_A corresponds to a given model with a fixed list of contents, one may not know which model a given (e.g. observed) (sub)configuration belongs to, or the theory may admit operations that alter the list of contents of the universe. Then one has a collection⁸⁴ of (sub)configuration spaces of instants, $\mathbf{q}_{A(\beta)}$, where β parametrizes the collection. The subconfiguration map idea will at least sometimes be useful here too: if there is a unique biggest true space or good enough model, the less detailed ones can all be thought of as subsets of its configuration space.

Example 1) use $\bigcup_{N \in \mathbb{N}_0} \mathbf{q}(N, d)$ for a mechanics theory that allows for particle coalescence/splitting or creation/annihilation. Stratified RPM configuration spaces can at least in some cases be regarded as already being such unions of configuration spaces, allowing e.g. for N m-masses and any coalescences among these. One would need to do more than that to allow for more general splittings (so that a 3m mass might forget it is made up of 3 1-m masses and resplit instead into 2 1.5m masses).

Example 2) use $\bigcup_{\Sigma \text{ compact without boundary}} \text{Riem}(\Sigma)$ for a formulation of GR that allows for spatial topology change.

One can see that unions of configuration spaces allow for more general physical processes, but can be considerably harder to do mathematics with. Analogy 93) is between the above 2 examples.

From now on, I use \mathbf{q} to denote a general (sub)configuration space or union of these.

Note 1) As per Secs 1 and 3, a configuration space is not just a set, it has many layers of mathematical structure that it is not at all clear if and how they might be extended to unions of configuration spaces.

Note 2) A main modelling point that this SSSec accommodates is that each \mathbf{q}_A corresponds to a given model with a fixed list of contents. However, one may not know which model a given (e.g. observed) (sub)configuration belongs to, or the theory may admit operations that alter the list of contents of the universe.

14.2 Naïve Schrödinger interpretation

I consider this as a simple example of Problem of Time approach whose computations are explicitly doable for concrete RPM toy models. N.B. I am only modestly explicitly exemplifying answerable questions of being rather than trying to supplant more general questions by questions of being (which in any are not possible within the Naïve Schrödinger Interpretation). Thus I am not addressing Problems 11.8.1-1 and -3. I do however make contact with issues concerning uniform states.

14.2.1 Pure-shape 4-stop metroland examples of Naïve Schrödinger Interpretation

Example 1) Consider quantifying Prob(universe is large), in the sense that the two clusters under study are but specks in the firmament, by

$$\text{Prob}(\text{model universe is } \epsilon\text{-close to the } \{12,34\} \text{ DD collision}) . \quad (873)$$

This means, at the level of the configurations themselves, that the magnitude of $\sqrt{n_1^2 + n_2^2}/n_3$ lies between 1 and $1 - \epsilon^2/2$. [I drop (H2) cluster labels in this SSec.] In configuration space terms, this means that one is in the ϵ -caps about each pole [Fig 54.a)]. Then from the latter and by the Naïve Schrödinger Interpretation, this quantity is

$$\propto \int_{\epsilon\text{-caps in } \mathbf{S}(4,1) = \mathbb{S}^2} |\Psi|^2 \mathbb{D}\mathbf{S} = \int_{\phi=0}^{2\pi} \left\{ \int_{\theta=0}^{\epsilon} + \int_{\theta=\pi-\epsilon}^{\pi} \right\} |\Psi(\theta, \phi)|^2 \sin \theta \, d\theta \, d\phi . \quad (874)$$

So, e.g. for the free/very special HO ground state and first excited state, this is $\propto \epsilon^2 + O(\epsilon^4)$, while for the $D = 1$, $|d| = 1$ states, this is $\propto \epsilon^4 + O(\epsilon^6)$.

Example 2) Consider quantifying Prob(the two clusters nominally under study are in fact merged), which is connected to localization and isolated systems issues, by

$$\text{Prob}(\text{model universe is } \delta\text{-close to } \{12,34\} \text{ merger}) . \quad (875)$$

This means, at the level of the configurations themselves, that the size of n_3 does not exceed the small number δ , and, in configuration space terms, that one is in the δ -belt around the equator [Fig 54.a)]. The Naïve Schrödinger Interpretation then gives this quantity to be

$$\propto \int_{\delta\text{-belt in } \mathbf{S}(4,1) = \mathbb{S}^2} |\Psi|^2 \mathbb{D}\mathbf{S} , \quad (876)$$

which, in the free/very special HO case, works out to be $\propto \delta^3 + O(\delta^5)$ for the $D = 1$ $d = 0$ state and to $\delta + O(\delta^3)$ for the other three lowest-lying states.

Example 3) Consider quantifying Prob(universe is contents-homogeneous) in the sense that the two clusters under study in Jacobi H-coordinates are similar to each other, by the magnitude of n_1/n_2 departing from 1 by no more than 2η . This is

⁸⁴Barbour uses ‘heap of instants’ for these [78, 80, 83], though he largely treats this as an unstructured set, whilst I view it as much as possible as a more structured space or as structured a union as possible of such [28].

qualitatively relevant to structure formation. The region of configuration space that this corresponds to consists of four lunes of width η [Fig 54b)], so the Naïve Schrödinger Interpretation gives

$$\text{Prob}(\text{universe is } \eta\text{-contents-homogeneous}) \propto \int_{4 \text{ lunes of width } \eta \text{ in } S(4,1) = S^2} |\Psi|^2 \mathbb{D}S, \quad (877)$$

which, in the very special case, comes out as proportional to η for all four of the lowest-lying states.

Example 4) Example 1's question also makes sense for the small-regime special HO solution. One now obtains proportionality to $\epsilon^2 \sqrt{\omega/\hbar}$ to leading order. I.e., one has the same '(small)²' factor as in the very special problem but now with an opposing ' $\sqrt{\text{large}}$ ' factor. This amounts to the small regime's potential well (Fig 36) concentrating the wavefunction near the poles, i.e. in the region of the configuration space that corresponds to large universes in the sense described above.

Example 5) repeating Example 1 for the wavefunctions with first order perturbative corrections in B included, there is now proportionality to $\epsilon^2 \{1 - 8BI^2/9\hbar^2\} + O(B^2) + O(\epsilon^4)$ for the ground state, to $\epsilon^2 \{1 - 8BI^2/25\hbar^2\} + O(B^2) + O(\epsilon^4)$ for D = 1, d = 0, and $\epsilon^4 \{1 - 36I^2B/25\hbar^2\} + O(B^2) + O(\epsilon^6)$ for the D = 1, d = 1 states. The signs of these corrections conform with intuition. For, (Fig 36) B > 0 corresponds to placing a potential barrier at the poles and a well around the equator. This should indeed decrease the amount of wavefunction there, i.e. making large universes less probable, and vice versa for B < 0.

14.2.2 Pure-shape triangleland examples of the Naïve Schrödinger Interpretation

Example 1) Consider quantifying Prob(model universe is uniform), for which triangleland has particularly clean equilateral and almost-equilateral criteria. A suitable notion of uniformity (Sec 10.7 is maximization of the democratic invariant $\text{dra}_2 = 4 \times (\text{area per unit moment of inertia})$, which peaks about the equilateral triangle. [I drop (1)-cluster and [1]-basis subscripts in this SSSec.] Fig 54c) then presents a notion of ϵ -equilaterality which quantifies approximate uniformity. This corresponds to the polar cap region indicated in Fig 54c). Thus, the Naïve Schrödinger Interpretation gives

$$\text{Prob}(\text{triangular model universe is } \epsilon\text{-equilateral}) \propto \int_{\epsilon\text{-caps in } S(3,2) = S^2} |\Psi|^2 \mathbb{D}S = \int_{\Phi=0}^{2\pi} \left\{ \int_{\Theta=0}^{\epsilon} + \int_{\pi-\epsilon}^{\pi} \right\} |\Psi(\Theta, \Phi)|^2 \sin \Theta d\Theta d\Phi, \quad (878)$$

which, for the free/very special HO case, is e.g. $\propto \epsilon^2 + O(\epsilon^4)$ for the ground state and the S = 1, s = 0 state, and $\propto \epsilon^4 + O(\epsilon^6)$ for the S = 1 = |s| state. This conforms with S = 1, |s| = 1 pointing along axes in the plane of collinearity.

Example 2) Collinear configurations have zero area per unit moment of inertia, and so represent a nonunique opposite of the preceding notion of uniformity. Fig 54d) then presents a notion of δ -collinearity. This corresponds to the equatorial belt region indicated in Fig 54d). Thus, the Naïve Schrödinger Interpretation gives

$$\text{Prob}(\text{triangular model universe is } \delta\text{-collinear}) \propto \int_{\delta\text{-belt in } S(3,2) = S^2} |\Psi|^2 \mathbb{D}S, \quad (879)$$

which, for the free/the very special HO case is e.g. $\propto \delta + O(\delta^3)$ for the ground state and the S = 1 = |s| state, and $\propto \delta^3$ for the S = 1, s = 0 state. This conforms with S = 1, s = 0 pointing along the EĒ axis with a node in the plane of collinearity. Example 2) continues to make sense for the small regime of the special problem. The result here for the lowest 4 states is, using the (1)-basis and a η -collinear bilune,

$$\text{Prob}(\text{triangular model is } \eta\text{-collinear}) \propto \eta \sqrt{\omega/\hbar}. \quad (880)$$

Thus this case has a sizeable concentrating factor $\sqrt{\omega/\hbar}$ as compared to the very special case, i.e. the potential is trapping more of the wavefunction near the collinearity plane. [The lack of a third-order correction is down to the small difference in area between the bilune used here and the belt used above.]

Example 3) Consider Prob(model universe is isosceles). It is useful to first consider Prob(model universe is sharp isosceles). A zone quantifying this with respect to the (1) clustering is the η -lune centred on the Greenwich meridian, so

$$\text{Prob}(\text{triangular model universe is } \eta\text{-(sharp isosceles) with regard to the (1)-clustering}) \propto \int_{\eta\text{-lune in } S(3,2) = S^2} |\Psi|^2 \mathbb{D}S. \quad (881)$$

For the free/very special HO solution, this gives $\propto \eta$ for the ground state and the S = 1, s = 0 state, $\propto \eta + O(\eta^3)$ for the S = 1, |s| = 1 cosine solution and $\propto \eta^3 + O(\eta^5)$ for the S = 1, |s| = 1 sine solution. (This last solution is particularly lacking in intersection with that lune). For the (2)- and (3)-clustering, the relevant lunes are centred around meridians at $\pm\pi/3$ to the principal one. Then all four of the above states give $\propto \eta + O(\eta^3)$. As regards Prob(triangular model universe is η -(sharp isosceles)) i.e. Prob(triangular model universe is η -(sharp isosceles) for any clustering), one should sum over the three clusterings (or equivalently integrate over the obvious trilune), which, for these examples, retains the form $\propto \eta + O(\eta^3)$. This exemplifies the less neat but more widely available notion of 'democracy by summing over all clusterings', which is another use of group invariance by summing over the actions of all of the relevant group's elements. Prob(universe is contents homogeneous) in the preceding SSSec also attains democracy in this way.

Next, for each of the above η -sharp results there is a corresponding η -flat result of the same form, obtained by using the antipodal counterparts of each meridian. Then sum over both of these to obtain Prob(triangular model universe is η -isosceles) can now be obtained by summing the η -sharp and η -flat results. [Here, the meaning in space of η -isosceles is that the magnitude of aniso weighted by $1/\sqrt{1 - 4 \times \text{area}^2}$ does not exceed the small number η .]

Finally, note that investigating Prob(model universe is regular) involves a hexalune at $\pi/6$ to the above. This is a more interesting case since it quantifies ‘size’ relative to contents and also whether there are isolated model island universes.

14.2.3 Scaled RPM examples of the Naïve Schrödinger Interpretation

The preceding calculations hold again in the scaled case by the scale–shape split. One can now also investigate questions about the scale of the model universe. E.g. what is Prob(moment of inertia of the model universe lies between 0 and I): a direct quantifier of size. I consider this question for the ground states of two scale-shape separable RPM models.

Example 1) for scaled 3-stop metroland model, this is

$$\propto \int_{\rho' \leq \rho = \sqrt{I}} |\Psi(\rho, \varphi)|^2 \rho d\rho d\psi \propto \int_{\rho'=0}^{\rho} \mathcal{I}^2(\rho') \rho' d\rho' \quad (882)$$

which is, for the special multi-HO, by (585), $\propto 1 - \exp(-2I/I_{\text{HO}})\{1 + 2I/I_{\text{HO}}\}$. Thus it tends to 0 for $I \ll I_{\text{HO}}$ (the characteristic value of the moment of inertia for this particular problem) and to 1 for $I \gg I_{\text{HO}}$.

Example 2) for scaled triangleland, this is

$$\propto \int_{I' \leq I} |\Psi(I', \Theta, \Phi)|^2 I' \sin \Theta dI' d\Theta d\Phi \propto \int_{I'=0}^I \mathcal{I}^2(I') I' dI' \quad (883)$$

which is, for the special multi-HO, by (667) $\propto 1 - \exp(-2I/I_0)\{1 - I/I_0\}$. The small and large I analysis for this parallels that of Example 1.

14.2.4 Extensions to bigger RPM models

While [41] covers quadrilateralland at the level of characterizing regions, it awaits [44] for wavefunction inputs so as to be able to finish these Naïve Schrödinger Interpretation calculations off.

14.3 A note on Conditional Probabilities Interpretation workings for RPM’s

Investigating the Conditional Probabilities Interpretation with RPM’s is also possible. Given explicit wavefunctions such as this article’s, one can build up projectors and mixed states as required for the Conditional Probabilities Interpretation (including with environment portions traced out). Then one can construct conditional probabilities [502] for pairs of universe properties. In the standard Conditional Probabilities Interpretation, one of the correlated subsystems is now a clock for the other. Thus one could here investigate the goodness of various possible clock variables for such as N -stop metroland or triangleland. Some suggestions for clocks and rulers are as follows. Scaled triangleland could be viewed as one ruler, one clock variable and one piece of physics. E.g. Φ could be a clockhand angle, the base could play the role of ruler, leaving the physical subsystem under study to be the ellipticity of the triangle. However, none of the relative angle, the base or the overall scale come with any widespread guarantees of monotonicity. See Sec 16.12.2 for more on the Conditional Probabilities Interpretation and for a number of further developments in timeless strategy classification and new conceptual combinations.

Question/Analogy 94) The Gambini–Porto–Pullin version of Conditional Probabilities Interpretation can also be investigated using RPM’s, with the usual value of being a closed-universe model with parallels to structure formation. As things stand, I view this as the more promising branch of the Conditional Probabilities Interpretation, making this one of the more likely questions I will work on in the future.

14.4 Notions of distance

In this SSec, I mean distance in the sense of metric spaces; thus the metric space parts of Sec 3, as well as the Riemannian metrics part of Sec 3 that lead to path metrics are useful again in this SSec, though I use a yet wider range of length structures. It is one of the inputs for Prerecords Theory. Some of these extend the scope of the configurational-relational construction of Secs 1 and 2. The variety of such present is part-driven by the following

14.4.1 Axioms of distance

For $\text{Dist}(x, y)$ to be a bona fide *distance function* between points x and y in a set X ,

$$\text{Dist}(x, y) \geq 0 \quad \forall x, y \in X \quad (\text{non-negativity}) \quad (884)$$

$$\text{If } \text{Dist}(x, y) = 0, \text{ then } x = y \quad (\text{separation}) \quad (885)$$

$$\text{Dist}(x, y) = \text{Dist}(y, x) \text{ (symmetry) .} \quad (886)$$

$$\text{Dist}(x, y) \leq \text{Dist}(x, z) + \text{Dist}(z, y) \text{ (triangle inequality) .} \quad (887)$$

Dist is well-known to furnish the following useful things: a notion of limit, closeness, openness, continuity and distance-decreasing (Lipschitz) map. Without separation, it is a so-called *pseudometric* [e.g. in eq (207)].

14.4.2 Some useful infrastructure for notions of distance between shapes

Structure 1) The Euclidean norm $|| \cdot ||$ and its generalization to $|| \cdot ||_{\mathbf{M}}$ norm and $(\cdot, \cdot)_{\mathbf{M}}$ inner product with respect to an array \mathbf{M} (in general nonconstant: a function of the configuration space objects, which allows for it to be a curved-geometry metric.)

Structure 2) Taking an inf or a sup.

Structure 3) Integrating over a path (finite models) or a tube (field-theoretic models), or the sums corresponding to each world- in discrete models.

Structure 4) Performing intrinsic computations from a single shape,

$$\iota : X \longrightarrow \mathbb{R}^P . \quad (888)$$

Structure 5) Perform comparison computations involving two shapes.

$$C : X \times X \longrightarrow \mathbb{R}^P . \quad (889)$$

Note: some notions of closeness on the collection depend on a fuller notion of comparison *between* instants, i.e. their joint consideration rather than a subsequent comparison of real numbers extracted from each individually. That may either be a means of judging which instants are similar or of judging which instants can evolve into each other along dynamical trajectories. Some criteria to determine which notion should be used are adherence to the axioms of distance, gauge or 3-diffeomorphism invariance as suitable, and, for some applications, whether they can be applied to unions of configuration spaces.

14.4.3 Examples of distances

Example 1) Using Structure 1), the *Euclidean distance* in Euclidean space is

$$(\text{Euclidean Dist})(\underline{x}, \underline{y}) = ||\underline{x} - \underline{y}||^2 , \quad (890)$$

based on the Euclidean norm $|| \cdot ||$, or the generalization of this to mass-weighted and curved-space version $|| \cdot ||_{\mathbf{M}}$. As per Secs 1–3, these are relevant in simple mechanics configuration spaces such as particle position space $\mathbf{q}(N, d) = \mathbb{R}^{Nd}$ or relative particle position space $\mathbf{r}(N, d) = \mathbb{R}^{nd}$.

Note: there are problems with generalizing objects based on Structure 1), to be used as notions of distance, to the case of semi-Riemannian metrics – one loses the non-negativity and separation properties.

Example 2) Using Structure 2), one has the supremum norm notion: for f a bounded function, $\sup_{x \in X} |f(x)|$.

Example 3) Using Structures 2 and 3 [with Structure 1) in the ds]: on a sufficiently smooth Riemannian manifold and for σ a continuous and everywhere positive-definite *weight function*, the corresponding *path metric* is

$$(\text{Path Dist})[\mathbf{g}, w](\text{point 1}, \text{point 2}) = \inf_{\Gamma \text{ joining point 1 and point 2}} \left(\int_{\Gamma} w \, ds \right) \quad (891)$$

where ds is the arclength $g_{\mu\nu} dx^\mu dx^\nu$.

Note 1) as a generalization, the action for field theories (including on curved spaces) involves integrating over the NOS as well. [Insert a $\int_{\Sigma_p} d\Sigma_p$ factor into (891).]

Note 2) Sec 14.5, 14.7.1, 14.7.2 and 14.7.3 contain numerous further generalizations of this kind of example.

Example 4) Plugging Structure 4) [or a finite vector's worth thereof] into Structure 1),

$$\iota\text{-Dist}(x, y) := (\text{Euclidean Dist})(\iota(x) - \iota(y)) = ||\iota(x) - \iota(y)||^2 , \quad (892)$$

though it should be noted that ι in general has a nontrivial kernel and hence Dist_ι misses out on the separation property. If separation fails, one can usually (see e.g. [277]) quotient to leave one with a notion of distance. But sometimes this leaves one with a single object, making it a trivial notion of distance, while it is sometimes limited or inappropriate to use such a distance as the originally intended space X and not the quotient has significance attached to it. Then

$$\iota\text{-Dist}(\tilde{x}, \tilde{y}) \quad (893)$$

is a bona fide distance on $\tilde{X} = X/\text{Ker}(\iota)$ which may or may not be of use for the originally intended problem on X .

Example 5) Some cases of Structure 5 and 1) hybrids include, firstly, a prototype of Kendall's two-input finite comparer [368],

$$(\text{Kendall Proto-Dist}) = (\mathbf{Q}, \mathbf{Q})_{\mathbf{M}} . \quad (894)$$

Secondly, a prototype of Barbour's one-input infinitesimal comparer (c.f. Sec 1.4) is

$$(\text{Barbour Proto-Dist}) = \|\mathbf{dQ}\|_{\mathbf{M}}^2 . \quad (895)$$

Thirdly, there is also a prototype of DeWitt's two-input infinitesimal comparer [203]

$$(\text{DeWitt Proto-Dist}) = (\dot{\mathbf{Q}}, \dot{\mathbf{Q}}')_{\mathbf{M}} . \quad (896)$$

Note 1) finite and infinitesimal here are in the same sense as 'thick sandwich' and 'thin sandwich'.

Note 2) for the Kendall and DeWitt cases, if $\mathbf{M} = \mathbf{M}(\mathbf{Q})$ then it is ambiguous which of the pair being compared the kinetic term is to pertain to. In DeWitt's case, he resolved this by 'priming half the factors' of each term in \mathcal{M}' [203].

Note 3) as presented, these are distance-squared candidates. Thus take their square roots to more strictly have distance candidates.

A particular construct is then [combining with Structure 3)]

$$(\text{Path Dist})[\mathbf{M}, \sqrt{2\mathbf{W}}](\text{Shape 1}, \text{Shape 2}) := \inf_{\text{shape 1 and shape 2}} \Gamma \text{ joining } 2 \circ \left\{ \int_{\Gamma} \circ \int_{\text{NOS}} \circ \times \sqrt{2\mathbf{W}} \circ \text{Sqrt} \circ (\text{Barbour Proto-Dist}) \right\} . \quad (897)$$

The above bracketed combination was termed \mathbf{S}_J in Sec 1, terming the map from \mathbf{q} to it 'JS' for Jacobi–Synge.

14.5 Relational notions of distance

So far, I have not made any configurationally-relational requirement. Now also demand one's notion of distance does no depend on physically irrelevant variables, requires refining/discarding various of the previously-discussed notions of distance.

14.5.1 \mathfrak{g} -dependent cores as \mathfrak{g} -on moves to be paired with \mathfrak{g} -off moves

In cases in which objects are represented redundantly, then the notion of distance used should have the corresponding \mathfrak{g} -invariance. Three families of such can be built up from the preceding SSsec's Proto-Distances,

$$(\text{Kendall } \mathfrak{g}\text{-Dist})(\mathbf{Q}, \mathbf{Q}') = (\mathbf{Q}, \overrightarrow{\mathfrak{g}}_{\mathbf{g}} \mathbf{Q}')_{\mathbf{M}} , \quad (898)$$

$$(\text{Barbour } \mathfrak{g}\text{-Dist}) = \mathbf{dg} \in \inf_{\text{infinitesimal } \mathfrak{g}\text{-transformations}} \|\overrightarrow{\mathfrak{g}}_{\mathbf{g}} \mathbf{Q}\|_{\mathbf{M}}^2 , \quad (899)$$

$$(\text{DeWitt } \mathfrak{g}\text{-Dist}) = (\overrightarrow{\mathfrak{g}}_{\mathbf{dg}} \mathbf{Q}, \overrightarrow{\mathfrak{g}}_{\mathbf{dg}} \mathbf{Q}')_{\mathbf{M}} . \quad (900)$$

One is to regard the above as ' \mathfrak{g} -on' first halves of moves, the second halves of which are ' \mathfrak{g} -off' so that the overall effect is to render the object in question \mathfrak{g} -invariant. The most obvious and useful ' \mathfrak{g} -off' move in the present context is

$$\inf_{\mathfrak{g} \in \mathfrak{g}\text{-transformations}} \quad (901)$$

over the finite ones when paired with the Kendall core here and over the infinitesimal ones when paired with the Barbour or DeWitt cores. Other possible \mathfrak{g} -off moves are sum/integrate/average over \mathfrak{g} (and the slight if undesirable generalization of inf to extremum obtained via the variational procedure with respect to \mathfrak{g}).⁸⁵ I use the collective notation $\mathbf{S}_{\mathfrak{g}}$ for \mathfrak{g} -off moves.

Note 1) A well-known situation that roughly parallels this is the adjoint action, except that our current situation is not a uniqueness-of-inverse pairing: any action for \mathfrak{g} -on and \mathfrak{g} -off will do.

Note 2) In cases that are variational principles, in general one extremizes rather than taking an inf, which either requires being more lax or checking variational solutions to be of this nature by such as higher order variational calculus and knowing all of the solutions.

Obvious but diversity-inducing lemma \mathfrak{g} -on, \mathfrak{g} -off freeing from \mathfrak{g} -dependence does not necessitate \mathfrak{g} -on and \mathfrak{g} -off to be consecutive maps.

Thus we are talking about the class $\mathbf{S}_{\mathfrak{g}} \circ \text{Maps} \circ \overrightarrow{\mathfrak{g}} O$ for some set of objects O that we wish to render \mathfrak{g} -invariant. In the present Sec, these are being used to construct candidate distances, but do note the wider examples list.

Note 1) I am not saying that such notions of e.g. distance will *coincide* for Maps_1 and Maps_2 , but rather that they are in both cases \mathfrak{g} -invariant.

⁸⁵Undesirable since an extremum can be a max or a min and carries less connotations of existence and, in particular, uniqueness.

Example 1) The best-matching implementation of configurational relationalism is of this form:

$$\text{BM}(\mathbf{q}) = \underset{\mathbf{g} \in \mathfrak{g}}{\text{extremum}} \circ \left\{ \int \circ \int_{\text{NOS}} \circ \times \sqrt{2\mathbf{W}} \circ \text{Sqrt} \right\} \circ (\mathfrak{g}\text{-Barbour Dist}) . \quad (902)$$

Note 1) Indeed the current SSSec vastly generalizes Relationalism 4) as regards indirect implementations of configurational relationalism from particular tacit choices in constructing actions to full revelation of the ambiguities in constructing any such kind of physical object. See Appendix 14.A for even greater generality.

Note 2) The bracketed Maps here is indeed what I previously referred to as JS; also

JBB = JS \circ \mathfrak{g} -Bundle(\mathbf{q}) = Maps \circ (\mathfrak{g} -Barbour Dist).

Example 2) The action itself furnishes a candidate path distance,

$$(\text{Path Dist})[\mathbf{M}, \sqrt{2\mathbf{W}}](\text{Shape 1}, \text{Shape 2}) := \underset{\text{shape 1 and shape 2}}{\inf \Gamma \text{ joining}} \circ \underset{\mathbf{g} \in \mathfrak{g}}{\text{extremum}} \circ \int_{\Gamma} \circ \int_{\text{NOS}} \circ \times \sqrt{2\mathbf{W}} \circ \text{Sqrt} \circ (\mathfrak{g}\text{-Barbour Dist}) . \quad (903)$$

Example 3) A simple and very well-known example from elementary group and representation theory is group summing/integrating/averaging

$$S_{\mathfrak{g}}\{\vec{\mathbf{g}} \circ\} , \quad S_{\mathfrak{g}} = \sum_{\mathbf{g} \in \mathfrak{g}} , \quad \int_{\mathfrak{g}} \mathbb{D}\mathbf{g} , \quad \frac{1}{|\mathfrak{g}|} \sum_{\mathbf{g} \in \mathfrak{g}} \quad \text{or} \quad \int_{\mathfrak{g}} \mathbb{D}\mathbf{g} \times \left/ \left\{ \int_{\mathfrak{g}} \mathbb{D}\mathbf{g} \right\} \right. \text{ as appropriate} , \quad (904)$$

of which 6.4 has an example with exponentiated adjoint action, and the ‘democracy by summing over clusters’ in Sec 14.2.2 is an example with the simplest action for the permutation group.

14.5.2 The rarely-available r-option

One could work directly with (or reduce down to) \mathbf{q}/\mathfrak{g} objects if one has the fortune of being able to explicitly find and use enough of them.

14.5.3 The secondary natural objects option

One could base notions of distance on secondary natural objects that do not exhibit the redundancy, such as constructions based on eigenvalues of a differential operator.

I consider that explicitly group-averaging with respect to Diff would be too tall an order. I more generally make a connection between this and the preceding SSSec’s material concerning group-averaging notions in QM as relational impositions useful in formulating QM for relational theories.

14.6 Further criteria for a suitable notion of distance

1) Allowing consideration of unions of configuration spaces could reduce the number of useable measures, e.g. some could not cope with comparing 3-metrics on 3-spaces with different topologies as per Appendix 14.A. [This may be too stringent a requirement, however, e.g. conventional geometrodynamics is of fixed topology and much particle mechanics study preclude collisions or even collinearities.]

2) **Common underlying structures conjecture** One should use notions that contain a natural object from the (classical fundamental) laws of physics. One should have preference for dynamical rather than merely kinematical notions, i.e. those that behave well under the action of the natural laws. One may consider extending this structural compatibility to involve structures common with such as Quantum Mechanics or Statistical Mechanics.

N.B. one can aim to use the freedom in the preceding SSSec’s Maps toward obtain compatibilities of this nature.

14.7 Specific examples of (candidate) notions of distance

14.7.1 Comparers in RPM’s

1) The *Kendall comparer* for scaled RPM is

$$\min_{R \in \text{Rot}(d)} (\rho^i, \underline{R}\rho^j) , \quad (905)$$

for $R_{\beta}{}^{\gamma}$ the d -dimensional rotation matrix (the 2- d form of this plays a part in Sec 3).

2) On the other hand, the infinitesimal core objects are

$$\|\circ \underline{B}\rho\|^2 \text{ (scaled RPM)} , \quad \|\circ \underline{B}_{,C}\rho\|^2 \text{ (pure-shape RPM)} , \quad (906)$$

Some particular objects built out of this are as follows.

Example 1) The path distance built from the Barbour Rot(d)-distance based on the scaled RPM action weighting $\sqrt{2W}$ is

$$(\text{JBB Path Dist})[|| ||, \sqrt{2W}](\rho_{\text{in}}, \rho_{\text{fin}}) = \inf_{\rho_{\text{in}} \text{ and } \rho_{\text{fin}}} \Gamma \text{ joining } \left(\underset{\underline{B} \text{ of Rot}(d)}{\text{extremum}} \left(\sqrt{2} \int_{\Gamma} ||d\underline{B}\rho|| \sqrt{W(\rho)} \right) \right). \quad (907)$$

In 1- d , this is just

$$(\text{Jacobi Path Dist})[|| ||, \sqrt{2W}](N\text{-stop } 1, N\text{-stop } 2) = \inf_{N\text{-stop } 1 \text{ and } N\text{-stop } 2} \Gamma \text{ joining } \left(\sqrt{2} \int_{\Gamma} ||d\rho|| \sqrt{W(\rho)} \right). \quad (908)$$

In indirect form in 2- d , it is

$$(\text{JBB Path Dist})[|| ||, \sqrt{2W}](N\text{-a-gon } 1, N\text{-a-gon } 2) = \inf_{N\text{-a-gon } 1 \text{ and } N\text{-a-gon } 2} \Gamma \text{ joining } \left(\underset{\underline{B} \in SO(2)}{\text{extremum}} \left(\sqrt{2} \int_{\Gamma} ||d\underline{B}\rho|| \sqrt{W(\rho)} \right) \right), \quad (909)$$

whilst in r-form it is

$$(\text{Jacobi Path Dist})[\mathbf{M}_{C(\text{FS})}, \sqrt{2W}](N\text{-a-gon } 1, N\text{-a-gon } 2) = \inf_{N\text{-a-gon } 1 \text{ and } N\text{-a-gon } 2} \Gamma \text{ joining } \left(\sqrt{2} \int_{\Gamma} \sqrt{d\rho^2 + \rho^2 ||d\mathbf{Z}||_{\mathbf{M}_{\text{FS}}}^2} \sqrt{W(\rho, \mathbf{Z})} \right). \quad (910)$$

For triangleland, after the extremization or straight off in the relationalspace approach, the notion of distance is then

$$(\text{Jacobi Path Dist})[|| ||, \sqrt{2W}](\triangle_1, \triangle_2) = \inf_{\triangle_1 \text{ and } \triangle_2} \Gamma \text{ joining } \left(\sqrt{2} \int_{\Gamma} \sqrt{\{d\mathbf{I}^2 + \mathbf{I}^2 \{d\Theta^2 + \sin^2 \Theta d\Phi^2\}\}} \check{W}(\mathbf{I}, \Theta, \Phi) \right). \quad (911)$$

Example 2) The path distance built from the Barbour Rot(d)-distance based on the emergent time weighting $1/\sqrt{2W}$ is

$$(\text{Time Path Dist})_t(\rho_1, \rho_2) = t^{\text{em}} - t^{\text{em}(\text{JBB})}(0) = \inf_{\rho_{\text{in}} \text{ and } \rho_{\text{fin}}} \Gamma \text{ joining } \left(\underset{\underline{B} \in \text{Rot}(d) \text{ of } \mathbf{s}^{\text{ERPM}}}{\text{extremum}} \left(\int_{\Gamma} ||d\underline{B}\mathbf{R}||_{\rho} / \sqrt{2W(\rho)} \right) \right). \quad (912)$$

14.7.2 The analogous comparers in geometrodynamics

These are based on the GR configuration space metric (inverse of DeWitt supermetric) which is natural as regards the physical laws.

1) The Kendall-type comparer for geometrodynamics is some kind of thick-sandwich comparer. It cannot be written more than formally, through not having an explicit formula for the finite action of $\text{Diff}(\Sigma)$.

2) Barbour-type comparers for geometrodynamics are built from the infinitesimal group action of $\text{Diff}(\Sigma)$ generated by F_{μ} . The infinitesimal core object is then

$$||d_F \mathbf{h}||_{\mathcal{M}}^2. \quad (913)$$

Example 1) the F^{μ} -extremization of the BFO-A action (149)

$$(\text{JBB Path 'Dist'})[\mathcal{M}, \sqrt{2W}](\text{geom } 1, \text{geom } 2) = \inf_{\text{geom } 1 \text{ and } \text{geom } 2} \Gamma \text{ joining } \left(\underset{F^{\mu} \in \text{Diff}(\Sigma)}{\text{extremum}} \left(\int_{\Gamma} \int_{\Sigma} \sqrt{2\{\text{Ric}(h) - 2\Lambda\}} ||d_F \mathbf{h}||_{\mathcal{M}} \right) \right). \quad (914)$$

Note: the non-availability of GR emergent time here due to its not containing \int_{NOS} .

Example 2) The

$$\int_{\text{NOS}} \circ \text{Sqrt} \quad \text{versus} \quad \text{Sqrt} \circ \int_{\text{NOS}} \circ \int_{\text{NOS}} \quad (915)$$

ambiguity is a local versus global square root ambiguity in non-prejudgingly attempting to construct relational actions/distances. A global square root ordering object gives the DeWitt metric functional which (in velocity of frame presentation) is

$$(\text{DeWitt Path Dist})[\mathcal{M}](\text{geom } 1, \text{geom } 2) = \inf_{\text{geom } 1 \text{ and } \text{geom } 2} \Gamma \text{ joining } \left(\underset{F^{\mu} \in \text{Diff}(\Sigma)}{\text{extremum}} \left(\sqrt{\int_{\Sigma} \int_{\Sigma} (\circ_F \mathbf{h}, \circ_F \mathbf{h}')_{\mathcal{M}} \sqrt{h} d^3 x \sqrt{h'} d^3 x'} \right) \right). \quad (916)$$

Note 1) In these examples, it is the local choice that gives good actions (via the Dirac procedure) and the global choice that gives good semi-Riemannian geometry (local is not even Finsler by reason of degeneracy).

Note 2) in finite theory cases this ambiguity vanishes, e.g. for minisuperspace

$$(\text{MSS Path Dist})[M, \sqrt{2\{\text{Ric}(h) - 2\Lambda\}}](\text{MSS geom } 1, \text{MSS geom } 2) = \inf_{\text{MSS geom } 1 \text{ and } \text{MSS geom } 2} \Gamma \text{ joining } \left(\int_{\Gamma} \sqrt{\{\text{Ric}(h) - 2\Lambda\}} d\mathbf{s}_{\text{MSS}}^{\text{GR}} \right) \quad (917)$$

for M the minisupermetric though it does retain a local versus global sum over particles counterpart in RPM's.

Note 3) The emergent time does not furnish a path distance for non-minisuperspace GR since it does not contain a spatial integral. This non-generalization is a further reason beyond the extremization leaning on a distinct object why to favour action-type path distances over emergent time-type ones.

There is a serious problem with all of the above, however, namely that the configuration space metric involved is indefinite, so these do not furnish bona fide distances.

Difference 36) GR does not have a bona fide notion of distance based on the configuration space metric, as another consequence of this being indefinite for GR.

14.7.3 Way out 1: pure-shape and conformogeometrodynamics comparers

I use

$$\mathbf{S}_{\text{ABFO}} = \mathbf{S}_{\text{BFO-A}}[\phi^4 h_{\mu\nu}]/\text{Vol}^{2/3} = \int \int_{\Sigma} d^3x \sqrt{h} \phi^4 \sqrt{\{\text{Ric}(\mathbf{h}) - 8\{\Delta_{\mathbf{h}}\phi\}/\phi\}} \|\mathbf{dh} + 4d\phi\mathbf{h}/\phi\|_{\mathcal{M}}/\text{Vol}^{2/3}. \quad (918)$$

One can view this as made from $\text{Diff}(\Sigma)$ and $\text{Conf}(\Sigma)$ correcting $\|\mathbf{dh}\|_{\mathcal{M}}$. However, it also encodes $\pi = 0$, by which we are now free to replace $\mathcal{M}^{\mu\nu\rho\sigma}$ by

$$\mathcal{U}^{\mu\nu\rho\gamma} = h^{\mu\rho} h^{\nu\sigma}, \quad (919)$$

which is indeed a positive-definite supermetric (and on CRiem), so it furnishes bona fide distances between the preshapes of GR. Thus

$$\mathbf{S}_{\text{ABFO}} = \mathbf{S}_{\text{BFO-A}}[\phi^4 h_{\mu\nu}]/\text{Vol}^{2/3} = \int \int_{\Sigma} d^3x \sqrt{h} \phi^4 \sqrt{\{\text{Ric}(\mathbf{h}) - 8\{\Delta_{\mathbf{h}}\phi\}/\phi\}} \|\mathbf{dh} + 4d\phi\mathbf{h}/\phi\|_{\mathcal{U}}/\text{Vol}^{2/3} \quad (920)$$

and the ensuing notion of distance is

$$\begin{aligned} & (\text{CS Barbour Path Dist})[\mathcal{U}, \sqrt{\text{Ric}(\mathbf{h}) - 3\{\Delta_{\mathbf{h}}\phi\}/\phi}](\text{conf geom 1}, \text{conf geom 2}) = \\ & \inf_{\text{conf geom 1 and conf geom 2}} \Gamma \text{ joining} \left(\text{extremum}_{\text{F} \in \text{Diff}(\Sigma), \phi \in \text{Conf}(\Sigma)} \left(\int_{\Gamma} \int_{\Sigma} d^3x \sqrt{h} \phi^4 \sqrt{\{\text{Ric}(\mathbf{h}) - 8\{\Delta_{\mathbf{h}}\phi\}/\phi\}} \|\mathbf{dh} + 4d\phi\mathbf{h}/\phi\|_{\mathcal{U}}/\text{Vol}^{2/3} \right) \right). \end{aligned} \quad (921)$$

I also note the 'DeWitt' counterpart

$$\begin{aligned} \mathbf{S}_{\text{ABFO-D}} &= \mathbf{S}_{\text{BFO-A}}[\phi^4 h_{\mu\nu}]/V^{2/3} = \\ & \int \sqrt{\int_{\Sigma} \int_{\Sigma} d^3x d^3x' \sqrt{h} \phi^4 \sqrt{\text{Ric}(\mathbf{h}) - 8\{\Delta_{\mathbf{h}}\phi\}/\phi} \sqrt{h'} \phi'^4 \sqrt{\text{Ric}(\mathbf{h}') - 8\{\Delta_{\mathbf{h}'}\phi'\}/\phi'} (\mathbf{dh} + 4d\phi\mathbf{h}/\phi, \mathbf{dh}' + 4d\phi'\mathbf{h}'/\phi')}_{\mathcal{U}}/\text{Vol}^{2/3} \end{aligned} \quad (922)$$

for \mathcal{U} the mixed-index version of the above configuration space metric.

As regards mathematical compatibility, I note that \mathcal{U} is not in the Wheeler-DeWitt equation but it *does* in the emergent-time-dependent Tomonaga-Schwinger Einstein-Schrödinger equation for the $l = \text{shape}$ degrees of freedom,

$$i\hbar \left\{ \frac{\partial}{\partial \mathcal{T}} - \frac{\partial \underline{B}}{\partial \mathcal{T}} \cdot \hat{\underline{L}} \right\} |\chi\rangle = -\frac{\hbar^2}{2} \Delta_{\langle \mathbf{P}(N,d), \mathbf{M}_{\text{sphc}} \rangle}^c |\chi\rangle + \dots \quad (923)$$

versus

$$i\hbar \left\{ \frac{\delta}{\delta t^{\text{em}}} - \frac{\delta \underline{\mathbf{F}}^{\mu}}{\delta t^{\text{em}}} \widehat{\mathcal{M}}_{\mu} \right\} |\chi\rangle = -\hbar^2 \Delta_{\langle \text{CRiem}(\Sigma), \mathcal{U} \rangle}^c |\chi\rangle + \dots \quad (924)$$

There is no difficulty accommodating minimally-coupled matter here, via $\widehat{\mathcal{M}} \longrightarrow \widehat{\mathcal{M}} + \widehat{\mathcal{M}}^{\text{matter}}$ and $\text{CRiem}(\Sigma) \longrightarrow \text{CRiem}(\Sigma) \oplus \text{Min}$, for Min the minimally coupled matter configuration space. The general case of the above equation is

$$i\hbar \left\{ \frac{\nabla}{\nabla t} - \frac{\nabla \epsilon^Z}{\nabla t} \hat{\underline{L}}_{\text{in}\mu} \right\} |\chi\rangle \propto -\hbar^2 \Delta_{\text{Preshape}}^c |\chi\rangle + \dots \quad (925)$$

for t an emergent, or possibly rectified, time. Overall,

Analogy 95) Preshape space and CRiem 's simple and positive-definite configuration space metrics furnish good notions of distance in each case (for GR, being a field theory, these come in distinct Barbour and DeWitt forms), and are compatible with physical laws.

The SRPM counterpart is the SRPM action; in indirectly-formulated form in $1-d$,

$$(\text{JBB Path Dist})[||| ||, \sqrt{W}](N\text{-stop 1}, N\text{-stop 2}) =_{N\text{-stop 1 and } N\text{-stop 2}} \inf_{\Gamma \text{ joining}} \left(\text{extremum}_{C \in \text{Dil}} \left(\sqrt{2} \int_{\Gamma} \|d_C \rho\| \sqrt{W(\rho)} \right) \right) \quad (926)$$

with corresponding r-form

$$(\text{Jacobi Path Dist})[\mathbf{M}_{\text{sph}}, \sqrt{2W}](N\text{-stop } 1, N\text{-stop } 2) = \inf_{N\text{-stop } 1 \text{ and } N\text{-stop } 2} \Gamma \text{ joining} \left(\sqrt{2} \int \|\mathrm{d}\boldsymbol{\Theta}\|_{\mathbf{M}_{\text{sph}}} \sqrt{W(\boldsymbol{\rho})} \right) . \quad (927)$$

In 2- d , the indirectly formulated form is

$$(\text{JBB Path Dist})[|||, \sqrt{2W}](N\text{-a-gon } 1, N\text{-a-gon } 2) = \inf_{N\text{-a-gon } 1 \text{ and } N\text{-a-gon } 2} \Gamma \text{ joining} \left(\underset{\substack{\text{extremum} \\ \underline{B} \in SO(2), C \in \text{Dil}}}{\left(\sqrt{2} \int_{\Gamma} \|\mathrm{d}_{\underline{B}, C} \boldsymbol{\rho}\|_{\mathbf{M}} \sqrt{W(\boldsymbol{\rho})} \right)} \right) \quad (928)$$

with corresponding r-form

$$(\text{Jacobi Path Dist})[\mathbf{M}_{\text{FS}}, \sqrt{2W}](N\text{-a-gon } 1, N\text{-a-gon } 2) = \left(\inf_{N\text{-a-gon } 1 \text{ and } N\text{-a-gon } 2} \Gamma \text{ joining} \left(\sqrt{2} \int_{\Gamma} \|\mathrm{d}\mathbf{Z}\|_{\mathbf{M}_{\text{FS}}} \sqrt{W(\mathbf{Z})} \right) \right) . \quad (929)$$

For triangle land, after the extremization or straight off in the relational space approach, the notion of distance is then

$$(\text{Jacobi Path Dist})[\mathbf{M}_{\text{sph}}, \sqrt{2W}](\triangle_1, \triangle_2) = \inf_{\triangle_1 \text{ and } \triangle_2} \Gamma \text{ joining} \left(\sqrt{2} \int_{\Gamma} \sqrt{\{\mathrm{d}\Theta^2 + \sin^2 \Theta \mathrm{d}\Phi^2\}} W(\Theta, \Phi) \right) . \quad (930)$$

Question. Consider using anisotropy as used as a notion of distance on mini-CS.

14.7.4 Way out 2: The Gromov–Hausdorff notion of distance

This approach [277] shuffles both the metrics being compared and is built via an inf rather than an integral. This is a *metric space* construction

$$(\text{Gromov–Hausdorff Dist}) := \inf_Z \left((\text{Hausdorff Dist})^Z(f(X), g(Y)) \right) \quad (931)$$

over all metric spaces Z , with f, g being isometric embeddings of X and Y into Z , and where

$$(\text{Hausdorff Dist})(A, B) = \inf_{\epsilon > 0} \{ B \subset U_{\epsilon}(A), A \subset U_{\epsilon}(B) \} \quad (932)$$

between subsets of a given metric space.

Note: (Gromov–Hausdorff Dist) is well-defined for compact metric spaces.

Question. Does this work for Riemannian metrics? Certainly a Riemannian metric furnishes a natural metric space notion of metric, via the path metric construction (891) It might however be very impractical to calculate with, needing to consider the set of all possible isometric embeddings... One physical price to pay is that it does not relate to the ‘natural DeWitt structure’.⁸⁶ Are these embeddings here isometric in the Riemannian geometry sense then?

Question. Have GR configuration spaces been studied from this perspective?

14.7.5 Further constructs for geometrodynamics

The attitude of this SSSec and the next one is to consider a wide range of structures. This is partly to look for analogues in cases which make sense (to see if there are some suitable notions that apply to a wide range of (toy) models). Partly due to limitations in some of the structures for some of the theories. And partly due to the possibility/need for compatibility between notions of distance and notions of information and features of the physical laws of each of the models. Thus expect reasons of nongenerality or incompatibility will dismiss a number of the structures considered, and thus I provide a substantial list of possibilities for the moment. I see future work following from the present SSSec as involving bigger specific RPM examples and midisuperspace examples, as more strenuous tests both of the capacity to compute specific objects and of which objects still make clean conceptual sense in these bigger settings. And also looking at notions of information (both classical and quantum-mechanical) as the next stage in building an explicit records-theoretic framework.

A) **Poisson sprinkling** [588] is a procedure by which n points can be embedded at random onto a manifold. If done for two manifolds, one can then build a (comparer) norm based on just these points, such as a supremum norm or a Euclidean norm. An obstacle to this, however, is that different people doing it would get different answers, though one might get round this by applying the procedure many times and invoking statistical theory. Another problem is that these norms do not have anything to do with the ‘natural objects’ already present in the physical laws. Sprinkling can be on whichever interpretation for a mathematical space: physical space or configuration space.⁸⁷ But there may be little motivation to discretely approximate when the manifold is to be interpreted as a *mechanical configuration space*.

⁸⁶In the study of spacetimes, there is a counterpart of this that serves to compare causal structures [490].

⁸⁷By its generality, I prefer this scheme to Bookstein’s using of special ‘landmark’ points on somewhat symmetric manifold models of biological objects [133].

B) **Curvature invariants.** For 3-spaces there are just three built with up to second derivatives of the metric. One is the Ricci scalar. One could then compute what value this takes on average over the manifold and use that as an i-map, but this will have a big kernel, there being e.g. many vacuum spaces ($\text{Ric} = 0$). One could also compare curvature at random points, or consider $\text{Ric} - \langle \text{Ric} \rangle$ or $\langle \text{Ric}^2 \rangle - \langle \text{Ric} \rangle^2$ rather than Ric ; these are simple measures of bumpiness/inhomogeneity.

C) **Spectral measures** extracted from certain operators. Examples (see refs 14-17 of [570]) demonstrate that there can be isospectrality (nonuniqueness of geometries giving the same spectrum, which Kac publicized as ‘hearing the shape of a drum’). So that spectral measures are to be suspected of not obeying the separable axiom. But is that a problem with all operators or is there some operator (or set of operators) that are entirely spectrally-discerning? One benefit in generality of Strategy 3 over Strategy 1 is that comparison between topologies is possible.

C.I) **Matzner’s spectral measure** [449]. Using $\iota = \iota_{\text{Matzner}}(\Sigma, h_{\mu\nu}) :=$ (smallest eigenvalue of the Yano–Bochner operator $D_\mu\{D^\mu\delta_\rho^\nu + D^\nu\delta_\rho^\mu\}$ on $(\Sigma, h_{\mu\nu})$) gives

$$(\text{Matzner Dist})(\langle \Sigma, h_{\mu\nu} \rangle, \langle \Sigma', h'_{\mu\nu} \rangle) = |\iota_{\text{Matzner}}(\Sigma, h_{\mu\nu}) - \iota_{\text{Matzner}}(\Sigma', h'_{\mu\nu})|^2 \quad (933)$$

which obeys all the other distance axioms [449].

C.II) **Seriu’s spectral measure** is a comparer,

$$(\text{Seriu Dist})(\langle \Sigma, h_{\mu\nu} \rangle, \langle \Sigma', h'_{\mu\nu} \rangle) = \sum_{k=1}^{\infty} \log\left(\sqrt{\lambda_k/\lambda'_k} + \sqrt{\lambda'_k/\lambda_k}\right) / 2. \quad (934)$$

where $\{\lambda_k, k = 0 \text{ to } \infty\}$ and $\{\lambda'_k, k = 0 \text{ to } \infty\}$ are the whole spectrum of the Laplace operator on the primed and unprimed manifolds respectively. This form is motivated by $\exp(-(\text{Seriu Dist}))$ being a factor in off-diagonal elements of reduced density matrix of the universe, so the larger this comparer’s value is, the more decoherence there is between corresponding spatial cuts of Einstein–scalar spaces. It can be viewed as built out of secondary quantities that are guaranteed to have the suitable invariances. This (and spectral measures in general) allow for comparison between geometries with different topology, and is in principle of interest in Quantum Cosmology. However, as a notion of distance, it is not only robbed of separation by the isospectral problem but also fails to obey the triangle inequality.

D) **Inhomogeneity measures.** A simple notion of inhomogeneity comes from partitioning space up and computing the energy density ε (GR) or normalized number density $\varepsilon/\int d\Omega \varepsilon$ (RPM) for each. One has the problem however of showing that this is insensitive to how the partitions are carried out. That in GR is a major part of the averaging problem, which is particularly acute due to the nonlinearity of the Einstein equations.

For the moment, consider the phenomenological matter in terms of which such things have already been started to be investigated in the literature. Firstly, consider

$$\iota = \iota_{\text{density contrast}} = \varepsilon/\langle \varepsilon \rangle \quad (935)$$

or its integrated counterpart, possibly with volume of integration and volume in the average being distinct, followed by construction 1 gives⁸⁸

$$(\text{Density Contrast Dist})(\epsilon_1, \epsilon_2) = |\iota_{\text{density contrast}}(\epsilon_1) - \iota_{\text{density contrast}}(\epsilon_2)|^2. \quad (936)$$

A similar notion which has yet further connotations as a relative information is the Hosoya–Buchert–Morita (HBM) quantity [324]

$$\iota = I_{\text{HBM}}(\varepsilon) = \int d\Omega \varepsilon \log(\varepsilon/\langle \varepsilon \rangle) \quad (937)$$

The asymmetric input of ε and $\langle \varepsilon \rangle$ exploits the asymmetric nature of the Kullback-Leibler quantity [186] itself and is a special pairing making the triangle fault nonsensical. Then use construction 1 to extract

$$(\text{HBM Dist}) = |I_{\text{HBM}}(\varepsilon) - I_{\text{HBM}}(\varepsilon')|^2 \quad (938)$$

which is guaranteed to obey everything except separation. Separation is clearly a problem for inhomogeneities: these take a common value when the space is homogeneous (1 for the contrast and 0 for the HBM quantity). Using a single spatial topology may help, as may quotienting out the homogeneous spacetimes (perturbations about a particular homogeneous universe whose overall shape is an approximate fit for cosmological data). This may not remove all of the inhomogeneity kernel: two spaces can be distinct and yet give the same value for each of these inhomogeneity indices, via only differing in their ‘higher moments’. Indeed the above notion of distance is poor, but that come from the inhomogeneity quantifier in question

⁸⁸ ε is the energy density. 2B’s notions depend also on Σ and $h_{\mu\nu}$ through the integrals involved.

being very limited in information content; one could well construct distances from sufficiently discerning quantifications of inhomogeneity.

The corresponding contrast object is

$$\varepsilon_{\Pi}/\langle\varepsilon\rangle \quad (939)$$

for $\langle A \rangle = \int A \mathbb{D}\Omega / \int \mathbb{D}\Omega$. One can then furthermore build the Shannon entropy of $f_{\Pi} \equiv \varepsilon_{\Pi} / \int \varepsilon \mathbb{D}\Omega$,

$$I_{\text{Shannon}} = \int \mathbb{D}\Omega \frac{\varepsilon_{\Pi}}{\int \varepsilon \mathbb{D}\Omega} \log \left(\frac{\varepsilon_{\Pi}}{\int \varepsilon \mathbb{D}\Omega} \right) , \quad (940)$$

and the relative information of ε versus $\langle\varepsilon\rangle$

$$I(f_{\Pi}||\langle f \rangle) = \int f_{\Pi} \log \left(\frac{f_{\Pi}}{\langle f \rangle} \right) = \left\langle \frac{\varepsilon}{\langle\varepsilon\rangle} \log \left(\frac{\varepsilon}{\langle\varepsilon\rangle} \right) \right\rangle \quad (941)$$

which by its second form is also an object of contrast form.

Then HBM-type objects are

$$I_{\text{HBM}}(\varepsilon_{\Pi}) = \int \mathbb{D}\Omega \varepsilon_{\Pi} \log \left(\frac{\varepsilon_{\Pi}}{\langle\varepsilon\rangle} \right) \quad (\text{relative information form}) \quad (942)$$

$$I'_{\text{HBM}}(\varepsilon_{\Pi}) = \left\langle \varepsilon_{\Pi} \log \left(\frac{\varepsilon_{\Pi}}{\langle\varepsilon\rangle} \right) \right\rangle \quad (\text{expectation form}) \quad (943)$$

which bear the relations

$$I_A = I_{\text{HBM}} / \int \mathbb{D}\Omega \varepsilon_{\Pi} = I'_{\text{HBM}} / \langle\varepsilon\rangle \quad (944)$$

(i.e. differ in normalization convention alone).

The normalized variance object is

$$\{\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2\} / \langle\varepsilon\rangle^2 \quad (945)$$

One can build ι -maps based on each of these or on vectors made out of multiple such.

Note 1) Relative information and I_{HBM} are not themselves itself distances – symmetry and the triangle inequality are irrelevant, as it is for comparing a configuration with its average distribution (which is a piece of info about shape rather than a distance between two shapes).

Note 2) One can build ι -maps from the base object, the three simple objects or the interesting 1, 2, 3, 4 objects.

Note 3) Also note that I_{HBM} is not very discerning: it provides only 1 piece of information, but the variety of different possible configurations cannot in general be captured with so little information: $\text{Ker}(I_{\text{HBM}})$ is generally nontrivial, continues to be so for the iota construction.

Note 4) The Gromov–Hausdorff, embedding point, curvature invariant and spectral measure approaches perhaps overly favour geometry over matter. The Kendall and Barbour comparers treat the two on an equal footing. Finally the HBM object and related such is based on the matter content.

14.7.6 Which of the preceding SSec's notions of distance have RPM counterparts?

The configuration spaces of RPM's are too simple to furnish much of a classification by curvature or by the spectrum of some associated operator. There are however inhomogeneity (clumping) notions for RPM's partition into pieces of space define number density N_{Π} for each partition and then build up various significant objects.

Three subsequent simple compound objects are the *normalized number density* $n_{\Pi} = N_{\Pi} / \sum_{\Pi} N_{\Pi}$, the associated '*contrast object*' $N_{\Pi} / \langle N \rangle$ where $\langle A \rangle = \sum_{\Pi=1}^n A_{\Pi} / n$ and the *normalized variance comparer*

$$\frac{\langle N_{\Pi}^2 \rangle - \langle N_{\Pi} \rangle^2}{\langle N_{\Pi} \rangle^2} . \quad (946)$$

One can then build further significant composite objects such as 1) the Shannon information of $n_{\Pi} \equiv N_{\Pi} / \sum_{\Lambda} N_{\Lambda}$,

$$I_{\text{Shannon}} = \sum_{\Pi} n_{\Pi} \log(N_{\Pi}) . \quad (947)$$

2) The relative information of N_{Π} versus $\langle N_{\Pi} \rangle$,

$$I_{\text{rel}} = I(n|\langle n \rangle) = \sum n \log \left(\frac{n}{\langle n \rangle} \right) = \left\langle \frac{N}{\langle N \rangle} \log \left(\frac{N}{\langle N \rangle} \right) \right\rangle \quad (948)$$

by which last form it is thus also an object of contrast form.

3) HBM-type objects

$$I_{\text{HBM}}^{\text{mech}}(N_{\Pi}(Q_A)) = \sum_{\Pi} N_{\Pi} \log(N_{\Pi}/\langle N \rangle) \quad (\text{relative information form}) \quad (949)$$

$$\text{and } I_{\text{HBM}}^{\text{mech}'}(N_{\Pi}(Q_A)) = \langle N_{\Pi} \log(N_{\Pi}/\langle N \rangle) \rangle \quad (\text{expectation form}) . \quad (950)$$

Note the inter-relations

$$I_A^{\text{mech}} = I_{\text{HBM}}^{\text{mech}} / \sum_{\Pi} N_{\Pi} = I_{\text{HBM}}^{\text{mech}'} / \langle N \rangle . \quad (951)$$

14.8 Yet further configuration space structure for Records Theory

14.8.1 Notions of localization in space

Example 1) In triangleland, $I_{\text{base}} \ll I_{\text{median}}$ means that the base can be treated as localized.

Example 2) $\langle \mathbb{R}^d, || || \rangle$ and $\langle \Sigma, \mathbf{h} \rangle$ metrics furnish good distances.

14.8.2 Localized subconfiguration spaces and localized configurational records

Now the other hitherto ignored coordinates may in parts become relevant. That is through these other coordinates entering the notion of localization itself. Following from the above Example 1, consider the base of the triangle as a localized subsystem. One notion of localized here is for the apex to have to be outside of the circle of regularity (i.e. $I_{\text{median}} = I_2 > I_1 = I_{\text{base}}$). In considering this, one passes from being on any open half-line emanating from the triple collision to being on any such that lies within the D-hemisphere. I.e., in this example, one passes from the mathematics of the Dirac string to that of the Iwai string (c.f. Sec 4.6.1). Note: I do not mean to imply that these need to be subspaces of configuration space in any particular sense. They are physically desirable things to study but not necessarily mathematically nice.

This localized subconfiguration space notion alongside some notion of localized in space furnish a notion of records. RPM's with their local particle clusters, and inhomogeneous perturbations about minisuperspace with their localized bumps, are two modellable cases of this.

N.B. that RPM strict subsystems are just like Newtonian mechanics ones and records are held to be a type of localized subsystems. Thus RPM records and Newtonian ones will very often coincide (relational ones may need to have more particles each to have nontriviality, so it is a subset, but that is not a big portion of the combinations.

Localization in space also ties in well with basic aspects of Crane's program.

There is also the issue of notions of distance between QM states on such as Hilb, \mathfrak{g} -MidHilb or SubHilb. On the one hand, new space, new study of suitable notions of distance! On the other hand, at least crude possibilities may be inherited from the classical \mathfrak{q} (particularly if primality is giveable to this) as in e.g. the Naïve Schrödinger Interpretation. It is a fair point that this is beyond where one should really have to face fermions, as such I place an extra problem on the Naïve Schrödinger Interpretation: how does its configuration space region specification face up to generalizing to models including fermions? In particular, whether and how does its usual canonical GR specification carry over to Einstein–Dirac theory. All in all, I leave this as

Question* Provide the quantum state space equivalent of this Section's treatment up to this point for RPM's, and for GR (at least formally in the latter case). Can this be done so as to comply with the common structures conjecture as regards the next SSec's quantum notions of information?

14.9 Are records typically useful?

14.9.1 Notions of information

Prerecords Theory requires subconfigurations to be capable of holding enough information to address at least some nontrivial propositions. If Records Theory is by itself to stand as a complete formulation of all physics, replace “at least some nontrivial propositions” with “all propositions to which those subconfigurations are relevant”, noting that some propositions involve multiple subconfigurations, the addressing of which may then require up to all of the subconfigurations involved.

In discussing information with a number of other physicists, I have noted that it is important to distinguish between observable information as opposed to information content, e.g. the spectrum of the Sun as opposed to the Sun itself. This heterogeneity gives a choice: stars as records versus photos of stars as records. Records are in practise often of the former sort, since one can have under one's nose far more pictures of spectra than stars, and likewise many more accounts purporting to be from different times rather than the one instantaneous state of the object itself in that present. The price to pay is that photos/spectra of stars may well not allow for one to answer all questions about stars themselves.

Information Theory [519] is then an appropriate discipline to be furthermore specific about the information content of records. Here it is (more or less: arguments for include those in [195, 114]) professed that information is negentropy, so that an incipient classical notion is the Boltzmann-like expression⁸⁹

$$I_{\text{Boltzmann}} = -\log W . \quad (952)$$

Here, W is the number of microstates, which is evaluated combinatorially in the discrete case or taken to be proportional to the phase space volume in the continuous case. The Shannon entropy (already encountered in Sec 14.7.5) is then

$$I_{\text{Shannon}}(p_x) = \sum_x p_x \log p_x \quad (953)$$

for p_x a discrete probability distribution for the records, or

$$I_{\text{Shannon}}[\sigma] = \int d\Omega \sigma \log \sigma \quad (954)$$

for σ a continuous probability distribution. If one operates rather on the quantum level, there is the von Neumann information,

$$I_{\text{von Neumann}}[\rho] = \text{Tr}(\rho \log \rho) \quad (955)$$

for ρ the density matrix of the quantum system.

Note 1) The Shannon and von Neumann notions are both based on the $x \log x$ function. (This is *the* positive continuous function that is consistent with regraining; it then has many further useful properties [626]). These notions furthermore tie in well with each other as regards classical–QM correspondence (see e.g. [626]).

Note 2) One can generally obtain an SM if one knows the underlying QM. Furthermore, perturbative solution of the quantum theory suffices [417] in order to build an approximate statistical mechanics. This provides further uses for the perturbative treatments in Part II.

Note 3) Furthermore the von Neumann notion survives the transition to relativistic QM, and that to QFT modulo a short-distance cutoff [155, 622]. As regards GR, it has been used in the context of black holes (see e.g. [622]). One contention in interpreting (735) at the general level required for developing a Problem of Time strategy is that information is minus the entropy. However, classical (never mind quantum) gravitational entropy is a not well-understood concept [508, 135, 261, 278, 456, 582, 138, 534, 324, 622]. Quantum Gravity may well lead to some notion of information

$$I[\rho_{\text{QGrav}}] = \text{Tr} \rho_{\text{QGrav}} \log \rho_{\text{QGrav}} . \quad (956)$$

However, either the quantum-gravitational density matrix ρ_{QGrav} is an unknown object since the underlying microstates are unknown. Or, alternatively, one would need to import into Records Theory an extra procedure for obtaining this, such as how to solve and interpret the Wheeler–DeWitt equation, which would be fraught with numerous further technical and conceptual problems. Rather than a notion of gravitational information that is completely general, a notion of entropy suitable for approximate classical and quantum cosmologies may suffice for the present study. Quite a lot of candidate objects of this kind have been proposed. However, it is unclear how some of these would arise from the above fundamental picture, while for others it is not clear that the candidate does in fact possess properties that make it a bona fide entropy.

Note 4) Monotonicity is an often-mentioned property for a bona fide entropy to have. Gravitational information candidates based on the Weyl tensor [508, 135, 261, 278, 456] have run into problems with this. [This monotonicity may be connectable to that of time itself via some ‘Arrow of Time’ demonstration.] In the context of Records Theory, this monotonicity would need to be with respect to a semblance of time. Moreover, information/entropy is characterized by a number of further properties [626], and it is not clear to me whether the gravitational candidates have been screened for these.

Note 5) Examples of cosmologically relevant notions of information proposed to date that are manifestly related to conventional notions of information are Rothman–Anninos’ [534] use of the continuous form of (952) and Brandenberger et al.’s [138] continuous version of (954).

14.9.2 Notions of mutual and relative information

Mutual information is one concept of possible use for Records Theory; it is given by

$$M(A, B) = I(A) + I(B) - I(AB) \quad (957)$$

for AB the joint distribution of A and B for each of classical Shannon or QM von Neumann information [519].

Relative information is a conceptually-similar quantity given by

$$I_{\text{relative}}[p, q] = \sum_x p_x \log(p_x/q_x) \quad (\text{discrete case}) , \quad I_{\text{relative}}[\sigma, \tau] = \int d\Omega \sigma \log(\sigma/\tau) \quad (\text{continuous case}) , \quad (958)$$

⁸⁹I choose units such that Boltzmann’s constant is 1.

(of which we saw a particular example in Sec 14.7 by Hosoya–Buchert–Morita). *Tsallis relative information* [243] is a more general such notion which includes both (958 i) and Shannon information as special cases. This goes further than HBM in incorporating more discerning ‘higher moment’ information.

The QM counterpart of relative information is [489]

$$I_{\text{relative}}[\rho_1, \rho_2] = \text{Tr}(\rho_1 \{\log \rho_1 - \log \rho_2\}) \quad (959)$$

There is also a quantum-mechanical analogue of mutual information (and a further closely-related notion, *Holevo information* [519], and a quantum analogue of Tsallis information [243]). These may well be of use in setting up the quantum cosmological counterpart of Hosoya–Buchert–Morita type inhomogeneity measures.

Question. Does it really make sense for matter to provide a notion of information in generally-relativistic physics? Is that an approximation and how crude is it?

14.9.3 Notions of correlation

Example 1) The family of notions of correlator/spatial n-point function that occur in cosmology, QFT, and indeed QFT on a cosmological background is one well-known example. I consider the 2-point function for simplicity (but the analysis readily generalizes to the n-point function). In classical cosmology, one works with the 2-point function for such as mass density or galaxy number density [178],

$$T(\underline{r}) = \{\langle \sigma(\underline{r}') \sigma(\underline{r}' - \underline{r}) \rangle - \langle \sigma(\underline{r}) \rangle \langle \sigma(\underline{r}' - \underline{r}) \rangle\} / \langle \sigma(\underline{r}) \rangle \langle \sigma(\underline{r}' - \underline{r}) \rangle \quad (960)$$

(or some integrated, angular or Fourier-transformed version of this, the Fourier transform being the well-known *power spectrum* quantity). QM has an extra kind of correlations that classical theory does not have, due to entanglement. Ordinary (Minkowski spacetime) QFT has n-point functions of the same kind [512], where $\langle \rangle$ now includes inserting the ground-state wavefunction at each end. This notion carries over to n-point functions for a simple ‘Mukhanov variables QFT’ picture of Quantum Cosmology [483].

Example 2) In Cosmology the search for circles in the microwave background data as possible signatures of the large-scale shape of the universe [184].

Example 3) In the general GR setting, the Zalaletdinov correlation tensor (a comparer) [657] applies to the more general inhomogeneous setting. On the one hand, these objects are higher moments, which indeed in general would be expected to carry more detail of structure than the above relative information quantities. However, on the other hand, it is not clear whether they can catch particular subtle details such as quasi-alignments between all triplets of included objects.

Example 4) Giddings–Marolf–Hartle presents a useful treatment of correlators for Quantum Cosmology in [256]. LQG has difficulties through standard machinery for building n-point functions being unavailable on account of background-independence.

14.9.4 Further notions of possible informational relevance to Records Theory

Deteriorated/doctored records Information can be lost from a record ‘after its formative event’ – the word “stored” in (0) can also be problematic. Photos yellow with age and can be defaced or doctored. And some characteristics of the microwave ‘background’ radiation that we observe have in part been formed since last scattering by such as the integrated Sachs–Wolfe effect or foreground effects [178].

Question (patterns versus information content). Is there any notion of pattern in a record or collection of records that involves more than just how much information is contained within.

Two similar-size samples of the same kind of sand could be [526] a hoof-print and a random pattern due to the wind blowing. What one requires is a general quantification of there being a pattern. There should be at least a partial link between this and information content, in that at least some complicated patterns require a minimum amount of information in order to be realized. Records theory is, intuitively, about drawing conclusions from similar patterns in different records. Can one tell a game’s rules from a series of snapshots of the game’s positions? Does complexity help with this? Can total randomness and strategic input be recognized? (E.g. in inferring the rules of Snakes and Ladders versus those of Chess, in each case from large amount of data on game positions).

14.9.5 Notions of information for RPM’s

Possible **problem of ensembles** E fixed but N varies if fusion/fission is allowed, and this is a very uncommonly used ensemble, rather than microcanonical (both fixed), canonical (N fixed, E free) or grand (both free). I get round this for now by not allowing fusion/fission, so that the ensemble is microcanonical in the usual E fixed interpretation, and canonical in the varying- E multiverse interpretation. I note that the perturbative model context from which one would build a perturbative SM from Sec 8 and 9’s perturbative QM requires the canonical ensemble.

Problem of small particle number effects. One would expect these for such as triangleland. In fact these effects rapidly diminish in importance even for a moderate increase in particle number, but at least nonrelational models of such effects suggest that 3 or 4 particles are *somewhat* too few (despite not being vastly too few) for such effects not to be very significant. By this, it may be a rare case of where using a general N (assumed fairly large) is more straightforward than using some concrete small N (and N -stop metroland is particularly amenable to such a study, so that beginning to understand how to set up a relational SM is a major application of those models).

Possible problem of boxes In setting up a classical SM, one usually proceeds by covering a region in imaginary boxes and counting how many particles are in each. It may then be problematic to do this in a relational theory due to various combinations of absolutely distinct boxes now being indiscernible, so that the construction and the overall counting process may need unusual new relational input. Moreover, the primary notion "there is one particle in this imaginary box" is not relational since the box is imaginary and the one particle contains no relational information. It might then be necessary to have some other kind of primary notion based on multiple particles to avoid the latter, and possibly even not on boxes (or only using boxes that are crafted in a relationally invariant manner) to avoid the former. This would involve some notion of 'these particles are all near neighbours as compared with almost any of the other particles'. This feeds into how to define suitably relational N on which to subsequently base the classical information formulae. An indirect alternative would be to consider applying \mathfrak{g} -on, \mathfrak{g} -off moves to a non-relational notion of N for particles such as one would use in ordinary mechanics. The specifics of this, by either approach, are not yet entirely clear to me.

Quantum non-impasse Given a QM, one can in principle construct the corresponding SM. Furthermore, by Part II, a wide range of quantum RPM models have been solved, and are thus available to construct ensuing SM's which should inherit their relationalism from coming from a relationalist QM. The levels of structure beyond the wavefunctions and inner products solved for are firstly density matrices and then combinations of these such as the von Neumann entropy and the relative information. I do note that for now not much of the solving was localized subsystem-wise (e.g. finding separate wavefunctions for the base pair and the apex of the triangle) as required as inputs for relative-type notions of information (see however Appendix B). (But, at least for metrolands and triangleland, the coordinates in question for doing so are clear enough – relative Jacobi coordinates for the former and parabolic-type coordinates for the latter – and are at least sometimes analytically tractable. Thus, while Part II's study is not specifically geared toward SM/quantum information, it is reasonably straightforward to proceed in that way for *somesuch* models). Moreover, by Note 2) of Sec 14.9.1, the perturbative results of Secs 8.1.5 and 8.3.11 continue to be relevant here as regards furnishing nontrivial but at least to start off with computable examples of approximate SM construction; due to the approximate nature of E in these perturbative examples, this should be treated within a canonical ensemble.

How relational are the mutual and relative notions of information? They do carry connotations of the subsystems/contextual notion of relational. But how do they fare as regards the Leibniz–Mach–Barbour type of relationalism underlying the current article? Mutual information is a linear combination of already-computed objects and so inherits relational characteristics in cases in which these simpler information objects are already-relational. Relative information is a linear combination too, though one of the 2 objects is a 2-subsystem index generalization of the preceding, necessitating a small amount of extra work. The outcome is that, if built out of r -formulation objects, relationalism is automatic (they are also timeless objects). But, as ever, that is of course a luxury that one cannot afford in the case of GR itself. The more general alternative is to consider \mathfrak{g} -on, \mathfrak{g} -off versions of these objects.

Question. What is the entropy of a classical triangle? And of a quantum one? What is the mutual information of the two 'base' particles and of the third 'apex' particle? The relative information for this? The quantum relative information? What are the notions of homogeneity and of correlation for each of a classical and a quantum triangle?

Triangleland has some limitations in constructing working models of Records Theories as compared to 4-stop metroland, due to the latter possessing splits into two nontrivial (2-particle) subsystems: the H-clusterings.

14.9.6 Notions of correlation for RPM's

One can re-ask the preceding paragraph's questions about the standard type of n -point function correlators. Here, I note that translationally and rotationally invariant forms of correlators are known within standard contexts such as QM. These depend only on $\underline{x} - \underline{x}'$; they sometimes involve taking the dot product of this with an arbitrary-direction vector and then integrating over all directions, which amounts to a \mathfrak{g} -on, \mathfrak{g} -off manoeuvre.

Moreover, RPM's provide examples of entirely different-looking relational notions of correlation. These concern the particles themselves rather than sampling over all pairs of points within the space, which, for particle theories, is the more relational notion. These examples should be of qualitative theoretical value as regards being sufficiently open-minded in seeking notions of correlation in other background-independent contexts such as LQG. At the very least they provide counter-examples to standard n -point function technology being the only way to characterize correlation in a physical theory.

Barbour Records Doubt-1 Is Joos–Zeh a more generic paradigm? If so, can the purported selection principle sufficiently compensate for this?

Consider also the situation in which information in a curve or in a wave pulse that is detectable by/storeable in a detector in terms of approximands or modes. Detectors could happen to be tuned to pick up the harmonics that are principal contributors in the signal. Much like a bubble chamber is attuned to seeing tracks, a detector will often only detect certain (expected) frequencies. In this way one can obtain a good approximation to a curve from relatively little information. E.g. compare the square wave with the almost-square wave that is comprised of the first 10 harmonics of the square wave. That is clearly specific information as opposed to information storage capacity in general. This overall middles between the Joos–Zeh and Mott–Bell–Barbour paradigms, suggesting that Joos–Zeh is not too dominant for some natural physics to fall under the Mott–Bell–Barbour paradigm.

Note: Barbour 1) and 2) place emphasis on \mathfrak{q} as per Relationalism 3).

Barbour-Records Doubt-2. Barbour 2) needs qualifying, in the sense that obviously the form of the potential and total energy also contribute (and are in fact far more well-known for having this effect at the elementary level). Barbour agrees with this [87]. The global problem with h–l 12.1.6 in Sec 12.1 sharpens one part of Barbour’s assertion of the importance of the configuration space geometry [83].

Barbour Records Doubt-3. At least in Barbour’s earlier arguments [79, 83], he suggests the wedge-shape of his representation of triangleland could play a role, but (Sec 3.8.5) this is a representation-dependent rather than irreducible feature of triangleland. I.e. Barbour 1) was originally not well-identified, suggesting there was a lack of correct concrete mechanism in mind when Barbour 2) was proposed.

Perhaps it can then be argued instead to have to do with representation-independent features like the maximal collision itself or, more generally, the presence of strata or of a particularly uniform state.

Barbour Records Doubt-4. In modelling small atoms and cosmic strings [360], the effect on the dynamics of representation-invariant features such as stratifications, while non-negligible, only impart a small distortion on the wavefunction. [In the case of small atoms, these approximations are of reasonable success in comparison with observation.]

Barbour Records Doubt-5. Nobody has been able to supply any evidence for Barbour 3). Butterfield already commented on the lack of evidence for time capsules being probable in [152]; Healey has commented more widely on the lack of clarity and consistency of Barbour’s writings [317].

Some results, and possibilities, at the level of studying the semblance of dynamics studied within RPM models are as follows.

Modelling 1) N -body study in 2- d is likely to benefit from some sophisticated tests for the significance of patterns at the classical level.⁹²

Modelling 2) Does relational space geometry drastically affect the distribution of Ψ ? Preliminarily, my first few simple RPM models [25, 30, 50, 34, 36, 37] would *not* seem to exhibit any substantial peaking about configurations of the required sort.

Modelling 3) However, cleanly getting round Doubt 14.10-3 requires the configuration space to be conformally non-flat. [In conformally-flat spaces, configuration space geometry can be entirely sent to a redefined potential factor by a PPST. The potential factor, at least locally away from its zeros, can be sent to a PPST-redefined configuration space geometry. But in non-conformally-flat spaces, the configuration space geometry has an irreducible part that is *not* re-encodeable as potential. Moreover, the present article makes it clear that the simplest such relational dynamics is quadrilateralland, which itself lies beyond the scope of the specific examples presented here.] As regards RPM’s producing elliptic time-independent Schrödinger equations, these are a class of problem known to be *capable* of producing some sort of pattern which reflects the underlying shape [423, 17].

Modelling 4) Specifically investigating whether maximal collisions play a role requires scaled models. The hydrogen-like model of triangleland has this effectively excised by the potential being singular there. The wavefunction is zero there, though the wavefunction is centred about there.

Modelling 5) Specifically investigating whether strata play a role requires potentials that call for excision or 3- d models, which lies beyond the scope of the present article.

Modelling 6) Specifically investigating whether notions of uniform state play a role is looked into in this article For the moment, I note that my simple pure-shape RPM model would not seem to exhibit very heavy peaking around its equilateral configuration.

Modelling 7) **Question(Effect of distinguishability): what is the spectral centre of a triangle?** Ground state eigenfunctions of the Laplacian (and thus of the free Schrödinger equation) peak about ‘the middle’ of the region concerned. However, there are multiple notions of centre of a triangle. Which, if any, out of these does the free QM Schrödinger equation favour? One particular case of interest is the $\pi/2, \pi/3, \pi/3$ isosceles spherical triangle that corresponds to indistinguishable

⁹²In [368] and references therein, whether collinearities are statistically significant is studied in the context of 2- d shape spaces.

triangleland. An even simpler (but non-RPM) toy model case that serves to illustrate this question (without being an RPM model) is the $\pi/2, \pi/4, \pi/4$ isosceles triangle in flat space. Here, I find the QM/its Laplacian operator favours a centre of its own that moreover lies rather close to the centroid–orthocentre coincidence that is the most geometrically obvious centre for an isosceles triangle, thus producing a ‘small number’ which might have some significance in toy models of small departures from uniformity. This is also linked to the use of spectral measures in Records Theory, insofar as differential operators encode geometrical information...

If one is willing to assert that records are subconfigurations and then tries out Barbour’s conjecture in this setting, the following further objection arises.

Barbour-Records Doubt-6 In the study of **branching processes**, one learns that Barbour’s ‘how probable a subconfiguration is’ can depend strongly on the precise extent of its contents. As an example (closely paralleling Reichenbach [526]), suppose we see two patches of sand exhibiting hoof-shaped cavities. Here there are past interactions of these two patches of sand with a third presently unseen subsystem – a horse that has subsequently become quasi-isolated from the two patches. By these there is a clear capacity of rendering the individually improbable (low entropy and hence high information) configurations of each patch of sand collectively probable (high entropy, low information) for the many sand patches–horse subsystem. This still does not explain why useful records appear to be common in nature: a separate argument would be needed to account for why branching processes are common. The problem Whitrow (p 338) [635] has with this is that it depends on the entropy of a main system. Moreover, then one runs out of being able to resort to such an explanation as one’s increases in subsystem size tend to occupying the whole universe.

14.10.2 Semblance in Page’s extended Conditional Probabilities Interpretation scheme

This is as already presented in 11.8.7, since RPM’s do not tangibly contribute to its discussion.

14.10.3 Becoming in the Gambini–Porto–Pullin Conditional Probabilities Interpretation scheme

Here in contrast with Page’s extended scheme, there is a specific mathematically-implemented mechanism of emergence of becoming. As already outlined in Sec 11, decoherence from a less standard source imprecision of the clocks (and the rods) leads to a modified version of the Heisenberg equations of motion of the Lindblad type, so it gets to have emergent becoming by a specific mechanism. Moreover, this approach does have some composite elements in it or attachable to it (histories, observables), which discussion is best left to the overall Conclusion. Indeed the title of this SSSec is a give-away that, conceptually, this cannot be a purely timeless approach.

14.10.4 One is just sitting inside Histories Theory?

Is the semblance of dynamics/history in this scheme to be recovered by the same decoherence mechanism as in Halliwell’s scheme in the last 2 SSSecs of Sec 15?

14.10.5 Semiclassicality might either explain or supplant Barbour’s selection principle

Is it in regions in which the WKB ansatz happens to apply that have a semblance of time? Barbour agrees to some extent with this. Additionally, there are two a priori unrelated selection principles in the literature, which could be viewed either as competitors or as features that Barbour’s scheme should be checked to be able to account for.

14.10.6 End-Notes

End-Note 1) Barbour and Page are more general and minimalistic, perhaps Gambini–Porto–Pullin too, though it does begin to lean on histories ideas. The semiclassical way has less generality, and Gell-Mann–Hartle–Halliwell entirely rests on Histories theory (and the Halliwell part of it on semiclassicality as well).

End-Note 2) the current article’s RPM’s do not suffice to investigate Page’s approach, or a number of features of Barbour’s conjectures. Mirror image-identified RPM’s and 3-*d* RPM’s suffice to cover some of the latter.

End-Note 3) Barbour favours his ‘being’ perspective so as to be open to the possibility of explaining the Arrow of Time as emergent from that. On the other hand, Castagnino’s scheme [164] (which builds in a time asymmetry in the choice of admitted solutions), and Page’s scheme are not open to such a possibility. Nor is Gell-Mann–Hartle–Halliwell’s scheme, by its presupposition of histories, which come with a given time-ordering.

14.11 Appendix A. Even more general configurational relationalism

This expands on ways and places to apply relationalism 8): Configurational Relationalism. Best matching assumed continuous isometries, but the map-weighted \mathfrak{g} -on \mathfrak{g} -off procedure is in no way restricted to continuous isometries, or indeed to isometries. For continuous or discrete isometries (or a mixture), this is a metric background invariantizing (MBI) map

$$\text{MBI} = S_{\mathfrak{g}} \circ \text{Maps} \circ \vec{\mathfrak{g}} \circ O \quad (963)$$

for whichever object O upon which \mathfrak{g} is able to act and whichever Maps render this into a physically-interesting combination such as an action, a QM operator or one of the notions of distance/information/correlation of Sec 14.

14.11.1 Topological relationalism?

In canonical GR, there is an obvious extra level of background structure: the fixed spatial 3-topology. The general MBI blueprint extends to this, albeit not the details of many of its implementations:

$$\text{TBI} = \mathbf{S}_\tau \circ \text{Maps} \circ \vec{\tau} O . \quad (964)$$

This corresponds to $\text{Riem} = \bigcup_\Sigma \text{Riem}(\Sigma)$, which is possibly still a restrictive sum e.g. restricted to the 1) orientable Σ , 2) the closed without boundary Σ or 3) the Σ of a given dimension.⁹³

Note 1) The τ group is *not* a group of the conceptually simpler kinds that occur in topology (such as topological group - both a topological space and a group -, or one of homotopy, homology or cohomology - *characterizers* of topology. It is, rather, a ‘ripping group’ of topology-*altering* operations. E.g. for 2- d orientable closed without boundary spaces, it is generated by the handle-adding operation that changes the genus. In slightly more detail, rippings are the operations of *surgery theory*; *cobordisms* are a type of ripping: via there being a 1- d higher manifold linking the two manifolds of distinct topology. I do not then know the extent to which cobordisms suffice as regards spanning all types of rippings.

Note 2) Indirectness means one does not have to face the consequences of this to start off with, e.g. considering each genus in turn. One would then add, average or extremize over the genus to render the object topologically background independent. A common problem already present in this simple example is that adding and averaging will be infinite contributions (and ratios/other compositions of such infinities).

Note 3) Inf-taking is definitely accommodating of being over a discrete parameter, since it is free of the specific technical limitations of the variational calculus and indeed held to be very widely applicable in mathematics. But extremization/inf taking looks strange in that it could well pick but one contributor. It is far from clear whether it would make any sense to assume the universe has three holes just because the action for a three-holed universe is smaller than that for any other number of holes! This is a discrete parameter extremization of actions selecting dynamical paths involves extremizing continuous parameters. At the very least one would have to use an extended version of the calculus of variations to justify this, and how to do so is not clear.

Note 4) MBI and TBI can be composed via τ -on, \mathfrak{g} -on, \mathfrak{g} -off, τ -off making sense, in that given a particular topology, one can establish a \mathfrak{g} -corrected version of \mathfrak{q} on it.

Question Some theories are only topological. To what extent has relationalism been considered for these theories?

Question Discrete approaches may more readily link to topological considerations, thus furnishing another arena of study.

Note 5) All of Matzner, Seriu and some inhomogeneity comparers are useable between manifolds of different spatial topology.

Basing physical modelling on fixed topology, or even on topology at all is not relational, in the sense that topologically distinct spaces with metric structure thereupon and physical entities built on top of that can be physically indiscernible and thus identical. One does not know how to calculate without starting by stating the topology, but some results will not be affected. The truly physical notion is the large-scale structure of space. This is some new kind of mathematics: topology itself coarsened by length concepts, so that, somehow, large handles/tubes count, and small ones do not. (‘Large’ here is with respect to the probing capacity of the observers.) Sometimes global effects will serve to discern, but I do not expect these to always be the case. I view Seriu’s work as an at least qualitatively useful prototype, since observations are tied to observed wavelengths and thus to spectral information about the operators on the space that also occur in the physical laws. Though

⁹³The motivations for these restrictions are 1) an inbuilt time-orientability both for its own sake and for understood and well-defined mathematics of evolution. 2) The putative connection between relationalism/Machianism and closedness, alongside it remaining plausible our universe is closed, and alongside the minimality of not having to consider the forms of boundaries and subsequently boundary conditions on them (which, in the lack of evidence for boundaries, could be seen as extraneous and un-Machian). 3) 3- d is the smallest dimension that is sufficient to describe all experimentally/observationally confirmed physics to date so it is a good minimalistic and non-hypothesis-making choice.

However, it could well be that in a deeper theory dimension, orientability and whichever of openness, closedness and boundaries are emergent phenomena. These could indeed still be viewed as possibly background structure if treated as pre-determined inputs.

I note that observationally the universe could indeed as well be open, and have previously argued that (building on some arguments of Hawking, in particular his connecting tubes of negligible action), it is always possible to provide a closed model indiscernible from a given open one using current observational sensitivity, and vice versa. I now underpin this argument with some relational thinking. Sometimes then Occam’s razor may favour one of the pair, whilst some relational/Machian thinking might favour the closed one. Though the construction used to have that could be as bad as postulating a boundary for no reason, e.g. that an apparently open universe solution closes up by curvature or by topological identification on a scale sufficiently in excess of the Hubble radius to not be detectable, or that an apparently closed universe has an undetectably thin aperture leading to an open region. But to the Leibnizian, these constructs demonstrate that openness or closedness is not true property of the world, since the two are indiscernible. It is then just a case that often one of the two will be a simpler description than the other. (Both geometrically, which is what the arguments establish, and as regards stability conditions, which notably remains unestablished - do undetectably small apertures pinch off, particularly in worlds not constrained to be of fixed spatial topology, and might there be restrictions on whole-universe size or on the plausibility of such as topologically-identified FLRW.)

it is still indirect: given a metrized topology, compute from the Laplacian, and it still does not maximize the connection to observability (or yet establish insensitivity to which of the physical differential operators are used). If spectra are everything, the isospectral problem is relevant. I posit that a relational view of the isospectral problem is that if spectra are all that is relevant to physics and there is isospectrality for all operators occurring in physics, then the identity of indiscernibles would have it that we live in an equivalence class of geometries with no physical meaning being ascribable to the mathematically different representatives of this. This is much like Sec 2's 3-particle universes having no sense of dimension beyond 1 and 'more than 1'.

An example to bear in mind in the opposite direction is that it is held that 1 monopole (a defect type) being present in the universe would suffice to explain charge quantization (Dirac). Here, the conventional argument allows for this monopole to be too weak and far away for direct detection. Does this argument hold in the whole-universe GR setting, and could variants of such render the actually-used mathematically standard notion of topology indirectly detectable. E.g. the Dirac argument in curved spacetime, whether the monopole being outside the Hubble radius would affect the result, and, if not, is Dirac's argument Machian/Leibnizian (such a monopole acts but isn't at all meaningfully acted upon?)

As regards RPM's as toy models of this, the most obvious is to use all N 's and/or all d 's indiscernible to each N for all N 's. Particle fusion/fission is the RPM analogy of GR's topology change [c.f. Analogy 93].

Analogy 97) Then one expects topological interaction terms much as one has particle-nonconserving interaction terms. These are unclear at the classical level (except possibly in a hydrodynamical limit with blobs), but at the quantum level, QFT is obviously of that nature. So, can we have a particle creation-annihilation incorporating theory of finite number of particles via some partly-QFT-like version of RPM's? Is there any way of classically modelling variable-number RPM? This gives the idea that sum over topologies is a free theory, and the interacting theory would have topology-changing 'ripping' operations. Sum over topologies is common in path integral approaches, as is transition amplitude for topology change. Thus these are a starting point. Can lessons be learnt from those which are applicable to configuration space? The path integral form is however already ready-tailored for Histories Theory. This is one advantage of such which I point out as notable from perspective that fixed topology is itself a deeper absolutist intrusion into physics. But there is a problem insofar as the interpretation of second quantizing is model universe creation rather than particle creation? Or are both somehow possible interpretations of second-quantizing the RPM? Second-quantizing nonrelativistic mechanics is well known. It is a QFT in standard sense: infinite, variable-particle-number. One would postulate that the second ψ has the wavefunction of the universe interpretation now. This is the opposite of second-quantizing as a toy model of third-quantizing as a Problem of Time scheme. It is nonstandard in being constrained (Dirac), or in being a curved-space theory (r-formulation).

14.12 Appendix B. Simple RPM example of parts of the contextual approach

For the scaled triangleland model with Φ -independent, and concentrating on observation around the base, the energy equation and \mathcal{J} conservation equation are all, giving a quadrature for the orbit shape and another for the approximate emergent timefunction in parallel to whichever 2 degree of freedom working in Sec 5. The free quantum problem is then solved by

$$\Psi_m^{\text{base}} \propto \sin(m\Phi) J(\rho_{\text{base}}/\rho_{\text{base}}^0) \quad (965)$$

for m the usual kind of angular momentum quantum number. Thus these form a SubHilb, and the ones concentrating on observation about the median do also. The HO problem with an HO along the base is a more convenient example by boundedness; its wavefunctions are given by

$$\Psi_{m N_{\text{base}}}^{\text{base}} \propto \sin(m\Phi) L_{N_{\text{base}}}^{|m|} (I_{\text{base}}/N_{\text{base}} I_{\text{base}}^0) \exp(-I_{\text{base}}/2N_{\text{base}} I_{\text{base}}^0) I_{\text{base}}^{|m|/2}. \quad (966)$$

In each of these cases, these lie within a total subsystem Hilbert space as per Sec 9 (which does exhibit less of a spectrum by closed-universe effects, and does necessitate some nonstandard interpretations, but would still nevertheless appear to be characterizable as a Hilbert space).

15 Histories Theory and Halliwell's approach

I give here a brief account of how RPM's can be set up as Histories Theories. I presently consider the RPM program to benefit more from this than the Histories Theory program does (due to mathematically similar examples already being in the Histories Theory literature: ordinary mechanics [306, 345, 346], relativistic mechanics [306, 559], minisuperspace [12, 371] and quantum field theories [10].) However that only applies to the first half of this chapter, as the work started in Sec 15.4 putting together of Histories, Records and Semiclassical approaches benefitted from being done in a whole-universe context (and also the quadrilateral example is not mathematically similar to pre-existing Histories Theory literature).

15.1 Hartle-type formulation Histories Theory for RPM's

15.1.1 Computation of decoherence functionals in the \mathfrak{g} -free case

$$\langle \mathbf{Q}^{\text{fin}} || C_\alpha || \mathbf{Q}^{\text{in}} \rangle = \sum_{\text{path} \in \mathbf{Q}^{\text{in}}_{C_\alpha} \mathbf{Q}^{\text{fin}}} \exp(i\mathbf{S}[\text{path}]) \quad (967)$$

for a given coarse-graining C_α made up of classes $\{c_\alpha\}$, and summing over all paths beginning at configuration \mathbf{Q}^{in} and ending at configuration \mathbf{Q}^{fin} . Then

$$\text{Dec}(\alpha, \alpha') = \mathcal{N} \sum_{ij} \text{Prob}_i^{\text{fin}} \langle \psi_i^{\text{fin}} | C_{\alpha'} | \psi_j^{\text{in}} \rangle \langle \psi_i^{\text{fin}} | C_\alpha | \psi_j^{\text{in}} \rangle^* \text{Prob}_j^{\text{in}} \quad (968)$$

where the probabilities are initial and final inputs alongside the initial and final states.

$$\mathcal{N} = 1 / \sum_{ij} \text{Prob}_i^{\text{fin}} |\langle \psi_i^{\text{fin}} | C_S | \psi_j^{\text{in}} \rangle|^2 \text{Prob}_j^{\text{in}} \quad (969)$$

where C_S is the sum of all the paths in (967). This is built in terms of a 'kernel' path integral,

$$\langle \psi_i^{\text{fin}} | C_\alpha | \psi_j^{\text{in}} \rangle := \psi_i^{\text{fin}}(\mathbf{Q}^{\text{fin}}) \circ \langle \mathbf{Q}^{\text{fin}} || C_\alpha || \mathbf{Q}^{\text{in}} \rangle \circ \psi_j^{\text{in}}(\mathbf{Q}^{\text{in}}), \quad (970)$$

where \circ is some Hermitian but not necessarily positive inner product, $\psi^{\text{in}} = \psi^{\text{in}}(\mathbf{Q}^{\text{in}})$ and $\psi^{\text{fin}} = \psi^{\text{fin}}(\mathbf{Q}^{\text{fin}})$ are initial- and final-condition wavefunctions. Finally, the above 'kernel' path integral takes e.g. the MRI form

$$\langle \mathbf{Q}^{\text{fin}} || C_\alpha || \mathbf{Q}^{\text{in}} \rangle = \int_\alpha \mathbb{D}\mathbf{P} \mathbb{D}\mathbf{Q} \mathbb{D}\dot{\mathbf{I}} \delta[\mathcal{F}[\mathbf{Q}^{\text{B}}, \dot{\mathbf{I}}]] \exp(i\mathbf{S}[\mathbf{Q}^{\text{B}}, \mathbf{P}_{\text{B}}, \dot{\mathbf{I}}]). \quad (971)$$

In using $\dot{\mathbf{I}}$ (or $d\mathbf{I}$) in place of the Lagrange multiplier counterpart \mathbf{N} , I differ from Hartle's review [306]. This distinction renders Histories Theory compatible with this article's general ethos of temporal relationalism implemented by temporal relationalism via MRI/MPI; it also means that the precise form that the action is taken to have here is (as per Appendix 2.A.5) $(\int_\Sigma d\Sigma) \int \{\mathbf{P}_{\text{B}} d\mathbf{Q}^{\text{B}} - d\mathbf{A}\}$.

15.1.2 Histories theory on a temporally-relational footing

As the input action to be used in Histories theory is that related to the Hamiltonian by a Legendre transformation, for theories spanning both RPM's and GR, by the last form in eq (187), this can be built from the \mathbf{t}^{em} (in its JBB interpretation rather than carrying any semiclassical connotations). However, the temporally relational actions in question are linear in $d/d\mathbf{t}^{\text{em}}(\text{JBB})$ and in $d\mathbf{t}^{\text{em}}(\text{JBB})$ and so in fact they are MRI/MPI, and so actually one is free to use any other label time λ if one so wishes. So there is in fact no hard content to using the emergent time in the construction of the decoherence functional. This is clear e.g. from the second form of eq (187). This is also the nature of the time used in the next SSec.

15.1.3 Computation of decoherence functionals in the case with \mathfrak{g}

For a \mathfrak{g} -constrained formulation of some physical theory, the above 'kernel' path integral takes the e.g. the MRI form

$$\langle \mathbf{Q}^{\text{fin}} || C_\alpha || \mathbf{Q}^{\text{in}} \rangle = \int_\alpha \mathbb{D}\mathbf{P} \mathbb{D}\mathbf{Q} \mathbb{D}\dot{\mathbf{I}} \mathbb{D}\dot{\mathbf{c}} \mathcal{D}_{\mathcal{F}}[\mathbf{Q}^{\text{B}}, \dot{\mathbf{I}}, \dot{\mathbf{c}}^{\text{Z}}] \delta[\mathcal{F}[\mathbf{Q}^{\text{B}}, \dot{\mathbf{I}}, \dot{\mathbf{c}}^{\text{Z}}]] \exp(i\mathbf{S}[\mathbf{Q}^{\text{B}}, \mathbf{P}_{\text{B}}, \dot{\mathbf{I}}, \dot{\mathbf{c}}^{\text{Z}}]). \quad (972)$$

Note 1) I am also now departing from Hartle's review in using $\dot{\mathbf{c}}$ (or $d\mathbf{c}$) in place of multiplier coordinates. As usual, this conceptual/aesthetic upgrades alongside the accompanying FENOS variation (Sec 2.11) does not significantly alter the physics.

Note 2) The δ is a delta function imposing the basis set of gauge-fixing conditions \mathcal{F}_{B} and that the \mathbb{D} 's collectively form the Liouville measure on the almost-phase space.

Note 3) In (972) $\mathcal{D}_{\mathcal{F}}$ is the *Faddeev–Popov determinant* [226, 450]. This determinant can be seen to arise as the Jacobian in the phase space transformation by which the positions and momenta are formally split into the gauge-fixing-conditions-as-momenta, their conjugate coordinates and a further independent set of coordinates and momenta. From this mathematical

origin it is clear that in constraint-less cases (whether never-constrained or successfully reduced), the Faddeev–Popov factor is multiplicatively trivial. For use in the surviving cases, this determinant is, computationally $\det(\mathbf{M}_{\text{AB}})$ for

$$\mathbf{M}_{\text{AB}}\delta(t_1 - t_2) = \frac{\delta}{\delta u(t_1)} \frac{\delta}{\delta v(t_2)} \int \{\mathcal{C}_{\text{A}}[v], \mathcal{F}_{\text{B}}[u]\} d\mathbf{t} , \quad (973)$$

for whichever notion of \mathbf{t} the histories come with. Also here, \mathcal{C}_{A} are secondary first-class constraints and \mathcal{F}_{B} are gauge-fixing conditions for which it is assumed that $\{\mathcal{F}_{\text{B}}, \mathcal{F}_{\text{C}}\} = 0$. For RPM’s the finiteness (i.e. lack of spatial extent) simplifies this somewhat to

$$\det(\mathbf{M}_{\text{AB}}) = \int \{\mathcal{C}_{\text{A}}, \mathcal{F}_{\text{B}}\} d\mathbf{t}^{\text{em}} . \quad (974)$$

Analogy 98) Decoherence functional computation.

15.1.4 Scaled N -stop metroland example

I work in the scaled case, to aid the unification with semiclassical ideas and to better parallel GR Quantum Cosmology. Again, this is a useful first case through having no Dirac–reduced distinction. Classically, the configurations are N particles on a line, as described by n relative separations ρ^i and the histories are then sequences of these at e.g. emergent JBB times t_1, t_2, \dots, t_p . Or sequences of approximate such, the approximateness notion providing the coarse-graining. Hartle’s work covers contents-coarse-graining and coarse-graining by sampling at a subset of times Quantum-mechanically, configurations become probability densities on relational space $\mathfrak{R}(N, 1) = \mathbb{R}^n$ and the histories are then a sequence of projectors at given emergent JBB times onto (approximate) such states.

The decoherence functional (968, 971) is then further computed as follows. $\psi^{\text{in}} = \psi^{\text{in}}(\boldsymbol{\rho}^{\text{in}})$ and $\psi^{\text{fin}} = \psi^{\text{fin}}(\boldsymbol{\rho}^{\text{fin}})$. The $\mathbb{D}\boldsymbol{\rho}$ that plays the role of $\mathbb{D}\mathbf{Q}$ is trivial as the corresponding configuration space is flat with the ρ^i playing the role of Cartesian coordinates (Sec 3.10.3). The associated $\mathbb{D}\mathbf{p}$ that plays the role of $\mathbb{D}\mathbf{P}$ is likewise trivial. Use action (189), in particular in its second, MRI form. The Faddeev–Popov and gauge-condition-imposing factors are trivial for never-constrained theories. Thus,

$$\langle \rho^{\text{fin}} | C_{\alpha} | \rho^{\text{in}} \rangle = \int_{\alpha} \mathbb{D}\boldsymbol{\rho} \mathbb{D}\mathbf{p} \mathbb{D}dI \exp(i\mathbf{S}[\boldsymbol{\rho}, \mathbf{p}, dI]) , \quad (975)$$

and then $\text{Dec}(\alpha, \alpha')$ is built out of this under (968, 969, 970) using $\mathbf{Q}^{\text{A}} \longrightarrow \boldsymbol{\rho}$.

15.1.5 Triangleland example in the reduced formulation

Classically, the configurations are triangles, as described by e.g. three Dragt-type coordinates Dra^{Γ} and the histories are then sequences of these at e.g. emergent JBB times t_1, t_2, \dots, t_p . Or sequences of approximate such, the approximateness notion providing the coarse-graining. Quantum-mechanically, configurations become probability densities on $\mathfrak{R}(3, 2) = \mathbb{R}^3$ and the histories are then a sequence of projectors at given emergent JBB times onto (approximate) such states.

The decoherence functional (968, 971) is then further computed as follows. $\psi^{\text{in}} = \psi^{\text{in}}(\text{Dra}_{\text{in}}^{\Gamma})$ and $\psi^{\text{fin}} = \psi^{\text{fin}}(\text{Dra}_{\text{fin}}^{\Gamma})$. The $\mathbb{D}\text{Dra}$ that plays the role of $\mathbb{D}\mathbf{Q}$ in these equations is trivial as the corresponding configuration space is flat and the Dra^{Γ} play the role of Cartesian coordinates for it (Sec 3.10.4). The associated $\mathbb{D}\Pi^{\text{Dra}}$ that plays the role of $\mathbb{D}\mathbf{P}$ is likewise trivial. Use action (310), in particular in its second, MRI form. The Faddeev–Popov and gauge-condition-imposing factors are trivial for direct relationalspace/reduced formulations. Thus,

$$\langle \text{Dra}_{\text{fin}} | C_{\alpha} | \text{Dra}_{\text{in}} \rangle = \int_{\alpha} \mathbb{D}\text{Dra} \mathbb{D}\Pi^{\text{Dra}} \mathbb{D}dI \exp(i\mathbf{S}[\text{Dra}, \Pi^{\text{Dra}}, dI]) , \quad (976)$$

and then $\text{Dec}(\alpha, \alpha')$ is built out of this via (968, 969, 970) using $\mathbf{Q}^{\text{A}} \longrightarrow \text{Dra}^{\Gamma}$.

15.1.6 Triangleland case in the Dirac formulation

This one more clearly parallels full GR/midisuperspace (in involving rotations and associated linear constraints in parallel to GR involving 3-diffeomorphisms and associated linear constraints), for all that then making progress with it is more formal.

The configurations are now triangles redundantly described by the four components of the pair of relative Jacobi vectors, and then the classical histories are built up from sequences of these. Quantum-mechanically, configurations become probability densities on $\mathfrak{R}(3, 2) = \mathbb{R}^4$, and the histories are then a sequence of projectors at given e.g. emergent JBB times onto (approximate) such states. Throughout the above description, the $\mathcal{L} = 0$ constraint remains to be imposed.

The decoherence functional (968, 971) is then further computed as follows. $\psi^{\text{in}} = \psi^{\text{in}}(\boldsymbol{\rho}_{\text{in}})$ and $\psi^{\text{fin}} = \psi^{\text{fin}}(\boldsymbol{\rho}_{\text{fin}})$. The $\mathbb{D}\boldsymbol{\rho}$ that plays the role of $\mathbb{D}\mathbf{Q}$ in equations (968) and (971) is trivial as the corresponding configuration space is flat and the $\boldsymbol{\rho}$ play the role of Cartesian coordinates for it (Sec 2.2.4). The associated $\mathbb{D}\mathbf{p}$ that plays the role of $\mathbb{D}\mathbf{P}$ is likewise trivial. Use action (191), in particular in its second, MRI form. The Faddeev–Popov and gauge-condition-imposing factors are then in general nontrivial. Explicit evaluation of these requires making a gauge-fixing choice for scaled triangleland. I consider the base = x gauge’s $\mathcal{F}_{\text{M}} := \theta_1 = 0$ (see Sec 4.6.1). Then $\{\mathcal{F}_{\text{M}}, \mathcal{L}\} = 1$, so the Faddeev–Popov determinant does not contribute any nontriviality to the integration in this gauge. Sec 4.6.1 also makes it clear that this is not a globally valid gauge. Thus

one should also consider the median = x gauge's $\mathcal{F}_D := \theta_2 = 0$, which gives the same Poisson bracket as above, and split the region of integration into two charts within each of which one of these two gauges is entirely valid. As regards the delta function, \mathcal{F} can be written in Cartesian coordinates as $\arctan(\rho_y^1/\rho_x^1) = 0$, i.e. $\rho_y^1 = 0$ provided that $\rho_x^1 \neq 0$, which is guaranteed by the choice of region of integration in which this gauge-fixing is applied. Thus, using $\theta_M := \theta_1$ and $\theta_D := \theta_2$ and similarly for the other 1- and 2-coordinates, and denoting M or D collectively by the index F, for α lying entirely within one F,

$$\langle \rho_1 || C_\alpha || \rho_2 \rangle = \int_{\alpha \text{ represented in F-chart}} \mathbb{D}\rho \mathbb{D}\mathbf{p} \mathbb{D}dI \mathbb{D}dB \delta[\theta_F(\rho^F)] \exp(i\mathbf{S}[\rho, \mathbf{p}, dI, dB]) . \quad (977)$$

Then $\text{Dec}(\alpha, \alpha')$ is built out of this via (968, 969, 970) using $Q^A \longrightarrow \rho^i$ and $c^Z \longrightarrow B$. I certainly in general see exclusion of histories that require multiple charts for their description as a problem here, unless one can somehow patch multiple charts ('Gribov regions') together. I leave that as an open question for now.

In parallel, for full GR, the configurations are now 3-geometries redundantly described by the six components of the 3-metric $h_{\mu\nu}$, and then the classical histories are built up from sequences of these. Quantum-mechanically, configurations become probability densities on superspace(Σ), and the histories are then a sequence of projectors at given e.g. emergent JBB times onto (approximate) such states. Throughout the above description, the $\mathcal{M}_\mu = 0$ constraint remains to be imposed. The decoherence functional (968, 971) is then further computed as follows. $\psi^{\text{in}} = \psi^{\text{in}}(h_{\mu\nu}^{\text{in}})$ and $\psi^{\text{fin}} = \psi^{\text{fin}}(h_{\mu\nu}^{\text{fin}})$. Use action (188), in particular in its second form. The $\mathbb{D}h$ that plays the role of $\mathbb{D}Q$ in equations (968) and (971) and the associated $\mathbb{D}\pi$ that plays the role of $\mathbb{D}P$ need to be left formal here, and I leave the Faddeev–Popov and gauge-condition-imposing factors general too. Thus,

$$\langle h_{\mu\nu}^{\text{fin}}, \varsigma^{\text{fin}} || C_\alpha || h_{\mu\nu}^{\text{in}}, \varsigma^{\text{in}} \rangle = \int_{\alpha} \mathbb{D}h \mathbb{D}\varsigma \mathbb{D}\pi \mathbb{D}\pi_\varsigma \mathbb{D}dI \mathbb{D}dF \mathbb{D}\mathcal{F} [h_{\mu\nu}, \pi^{\mu\nu}, \varsigma, \pi_\varsigma, dI, dF^\Gamma] \delta[\mathcal{F}^\gamma[h_{\mu\nu}, \varsigma, dF^\Gamma]] \exp\{i\mathbf{S}[\pi^{\mu\nu}, \pi_\varsigma, h_{\mu\nu}, \varsigma, dF^\Gamma, dI]\} . \quad (978)$$

Then $\text{Dec}(\alpha, \alpha')$ is built out of this via (968, 969, 970) using $Q^A \longrightarrow h_{\mu\nu}$. One would in general expect a Gribov-type problem here too.

15.2 Isham–Linden-type formulation of Histories Theory for RPM's

15.2.1 N-stop metroland

The classical histories are now $\rho^i(\lambda)$ for λ a continuous label time. We can here take $\lambda = t^{\text{em}}$ as choice of label time. For 1- d scaled RPM's, the histories brackets algebra is then

$$\{\rho^i(t_1^{\text{em}}), \rho^j(t_2^{\text{em}})\} = 0 , \quad (979)$$

$$\{p_i(t_1^{\text{em}}), p_j(t_2^{\text{em}})\} = 0 , \quad (980)$$

$$\{\rho^i(t_1^{\text{em}}), p_j(t_2^{\text{em}})\} = \delta_j^i \delta(t_1^{\text{em}} - t_2^{\text{em}}) . \quad (981)$$

This is a known structure, matching the mathematics of the 1- d absolutist model with one particle more [345]. The histories energy constraint is $\mathcal{E}_{t^{\text{em}}} := \int dt^{\text{em}} \mathcal{E}(t^{\text{em}})$ for \mathcal{E} given by the N -stop metroland case of (74). The above brackets are straightforwardly promoteable to a quantum commutator algebra.

15.2.2 Scaled triangleland, reduced style

For scaled triangleland, I use that it has the same mathematics in conformally-transformed configuration space as for 3- d mechanics in space, which is covered in [346, 557]. [No such supporting work is known to be available for the case of the quadrilateral.] The classical histories can now likewise be taken to be $\text{Dra}^\Gamma(t^{\text{em}})$. The histories brackets algebra is

$$\{\text{Dra}^\Gamma(t_1^{\text{em}}), \text{Dra}^\Lambda(t_2^{\text{em}})\} = 0 , \quad (982)$$

$$\{\Pi_\Gamma^{\text{Dra}}(t_1^{\text{em}}), \Pi_\Lambda^{\text{Dra}}(t_2^{\text{em}})\} = 0 , \quad (983)$$

$$\{\text{Dra}^\Gamma(t_1^{\text{em}}), \Pi_\Lambda^{\text{Dra}}(t_2^{\text{em}})\} = \delta^\Gamma_\Lambda \delta(t_1^{\text{em}} - t_2^{\text{em}}) , \quad (984)$$

$$\{\mathcal{S}_\Gamma(t_1^{\text{em}}), \mathcal{S}_\Lambda(t_2^{\text{em}})\} = i\epsilon_{\Gamma\Lambda}^\Sigma \mathcal{S}_\Sigma(t_1^{\text{em}}) \delta(t_1^{\text{em}} - t_2^{\text{em}}) , \quad (985)$$

$$\{\text{Dra}^\Gamma(t_1^{\text{em}}), \mathcal{S}_\Lambda(t_2^{\text{em}})\} = i\epsilon^\Gamma_\Lambda{}^\Sigma \mathcal{S}_\Sigma(t_1^{\text{em}}) \delta(t_1^{\text{em}} - t_2^{\text{em}}) , \quad (986)$$

$$\{\Pi_\Gamma^{\text{Dra}}(t_1^{\text{em}}), \mathcal{S}_\Lambda(t_2^{\text{em}})\} = i\epsilon_{\Gamma\Lambda}^\Sigma \mathcal{S}_\Sigma(t_1^{\text{em}}) \delta(t_1^{\text{em}} - t_2^{\text{em}}) , \quad (987)$$

The histories energy constraint is $\mathcal{E}_{t^{\text{em}}} := \int dt^{\text{em}} \mathcal{E}(t^{\text{em}})$ with \mathcal{E} given by (340). The above brackets is also promoteable as it stands to a quantum commutator algebra (which is less straightforward to demonstrate due to needing to discount the possibility of a central term [346]).

Question. The quadrilateralland counterpart of this central check will be a new calculation in the quadrilateralland counterpart; even at the classical level, quadrilateralland should pose a mathematically new example of histories algebra).

15.2.3 Scaled triangleland, Dirac style

The classical histories can likewise be taken as $\rho(t^{\text{em}})$. The unreduced histories brackets algebra is

$$\{\rho^i(t_1^{\text{em}}), \rho^j(t_2^{\text{em}})\} = 0, \quad (988)$$

$$\{\pi_i(t_1^{\text{em}}), \pi_j(t_2^{\text{em}})\} = 0, \quad (989)$$

$$\{\rho^i(t_1^{\text{em}}), \pi_j(t_2^{\text{em}})\} = \delta_j^i \delta(t_1^{\text{em}} - t_2^{\text{em}}). \quad (990)$$

This now comes alongside the histories total zero angular momentum constraint $\mathcal{L}_{t^{\text{em}}} := \int dt^{\text{em}} \mathcal{L}(t^{\text{em}}) = 0$ and the histories energy constraint $\mathcal{E}_{t^{\text{em}}} := \int dt^{\text{em}} \mathcal{E}(t^{\text{em}})$ with \mathcal{E} given by the triangleland case of (74). Savvidou and Anastopoulos considered the relativistic particle in this context, however they proceeded by reduction from this point on, which, for us, would just send us back one subsection.

15.3 Some comments and open questions on Histories Theory

Difference 37) Let us next parallel Kouletsis and Kuchař [389], or Kouletsis [388]. RPM's, like Newtonian Mechanics, have a privileged time (now emergent, but still privileged). Thus their equivalent of how to split \mathcal{M}^4 is trivial, due to that privileged slicing. One has $\mathbb{R} \times \mathbb{R}^6$ in Dirac presentation and $\mathbb{R} \times \mathbb{R}^3$ with nonflat metric in reduced presentation. Due to that split, the notion of time map becomes trivial, $\tau : T \times \mathbb{R}^6 \longrightarrow T$, as does that of space map $\chi : T \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ (space map is even more trivial in the reduced presentation). There is, moreover no issue of spacetime symmetries and how these are split up. Foliation issues and constraint algebra nontrivialities are absent from RPM's.

Question. Consider RPM's as a theoretical probe of Savvidou's 2 times and canonical-and-covariant-at-once scheme: how much of that exists in a background-independent but non-spacetime scheme?

Note 1) This article's simple RPM models are free of Problems Histories-3 (measure problem), -5 (Functional Evolution), -6 (Sandwich Problem), -7 (specific to the diffeomorphisms) and -9 (foliation dependence). Moreover the reduced approach to the scaled triangle is also free of global issues (unlike the Dirac approach to the scaled triangle). Thus, at least for this model, one looks to have a rather good resolution of the Problem of Time, upon which the next Section builds an even better resolution.

Note 2) like 'q is primary', 'Hist is primary' can also be expanded categorically, propositionally and contextually. Prop(Hist) was an important motivation for Isham–Linden.

Question Are the momenta conjugate to histories themselves implemented by projectors? (One may well want to make propositions concerning these too).

The contextual version of histories, $\text{Context}(\text{Hist}) = \langle \text{set of physical SubHist's} \rangle$ is somewhat different from that of configurations. In the causal sets setting, this was covered by Markopoulou [443]. In jest, and yet with some lucidity, this 'concerns histories that depend on who is the historian (rather than history just being written from the victor's perspective) and historians studying histories that are partly protagonized by other historians (perhaps via some of the mighty choosing to heed history's lessons rather than being doomed to repeat them)'. What is less clear is whether historians are to be held to be some kind of privileged entities like observers are held to be in conventional QM. If historians are those who observe correlations in records (and thus deduce likely histories), then the two notions conflate.

Note 3) Considering $\text{Prop}(\text{Context}(\text{Hist}))$ combines Isham–Linden's program with Crane's. More generally, $\text{Prop}(\text{Context}(\))$ combines Mackey's Principle and Crane's program.

Question Does the Partial Observables Approach offer simpler ways of doing this than Isham-type approaches?

Question What is a suitable notion of distance between histories, i.e. on the space Hist?

Note 4) The issue of notions of information for histories has been considered by Hartle [307].

Note 5) As regards a notion of correlation between histories, the decoherence functional is already a key structural element.

15.4 Extending Halliwell 03 and its 09, 11 upgrades to RPM's.

Semiclassical provision of a timefunction frees one from Histories Problem-4 (Multiple choice) or any further mishaps arising from having to import internal time concepts (Histories Problem-8), with decoherence covering why semiclassical and records covering the issue of what decoheres what (Histories Problem-10).

The question addressed now is: for an eigenstate of the Hamiltonian, what is the probability of finding the system in a region (or series of regions) of configuration space without reference to time [294]?

15.4.1 Outline of the classical part of this program

Halliwell illustrated this with a free particle model; this has a counterpart for the reduced formulation of the scaled triangle free classical solution via the Dragt correspondence of Sec 3.10.4, which amounts to transcribing (much of?) Halliwell's mathematics to an arena in which it has whole-universe significance. Prob(Σ is intersected), closely related to region $\Sigma \times [0, t]$ having the dynamical path pass through it. For Σ a hypersurface with normal $n^{\Gamma'}$ in phase space, a *classical* phase space distribution w ,

$$\text{Prob}(\Sigma \text{ is intersected}) = \int dt \int_{\Sigma} \mathbb{D}\Pi^{\text{Dra}} \mathbb{D}_{\Sigma} \text{Dra}' n^{\Gamma'} \Pi_{\Gamma'}^{\text{Dra}'} w(\Pi^{\text{Dra}'}, \text{Dra}') , \quad (991)$$

where, the \mathbb{D}_{Σ} corresponds to the hypersurface. [The N -stop metroland counterpart is also straightforward, and the quadrilateralland counterpart would follow from having the geodesics on $C(\mathbb{CP}^2)$.]

15.4.2 Outline of the quantum part of this program

The Halliwell–Hawking semiclassical quantum cosmology of inhomogeneous perturbations [296] and the Hawking–Page Naïve Schrödinger Interpretation [315] are precursors of the approach used in quantum part of this paper, the first of these being specifically WKB-semiclassical and the second of these being specifically timeless. He works within Histories Theory, doing so in the Gell-Mann–Hartle formalism. This picks up the Problem of Time criticisms from the end of the last section albeit these are very largely absent from the present article's RPM's.

Halliwell alludes to the Mott–Bell–Barbour bubble chamber paradigm, and to Dirac observables (though he does not for now consider the effect of linear constraints). He implements propositions via regions of configuration space. In fact, Hartle and Halliwell both separately consider such schemes, via various versions of **class operators** that relate to Prob(enters a region R of configuration space); C_R refers to an α comprising of those histories that involves crossing over into region R . which for RPM's, includes cases of particularly lucid physical significance via Sec 3's tessellation interpretation.⁹⁴ The regions criticism from Naïve Schrödinger Interpretation does not seriously dint this: it is not claimed to be the way all propositions are made sense of; that would be understood to be associable within an underlying Isham–Linden scheme. On the other hand, it does mean that the technique in question is only being levelled at a subset of the physically relevant questions. A good further move to counter that is to point out that regions in phase space can be treated similarly (these obviously cover a wider range of physical questions).

Analogy 99) Halliwell-type class operators can be constructed for RPM's and for, at least, minisuperspace GR examples.

Halliwell then considers semiclassical WKB wavefunctions $\exp(iW)|\chi\rangle$, about which he notes that they have a particular form of Wigner function Wig, which is, approximately, and transcribing over to the case of scaled triangleand,

$$\text{Wig}(\Pi^{\text{Dra}}, \text{Dra}) = |\chi(\text{Dra})|^2 \delta(\Pi^{\text{Dra}} - \nabla W) \quad (992)$$

($\Pi^{\text{Dra}} = \nabla_{\text{Dra}} W$ for classical trajectories). I note that in my setting of interest, $|\chi\rangle = |\chi(\Theta, \Phi, I)\rangle$ and $W = W(I)$. The Wigner function is a Fourier transform of the density matrix, whose equation of motion is close to classical motion (up to a term that is third-order in \hbar). Halliwell's heuristic move is then to replace w by Wig in (991).

Question Whether this set-up works in cases with linear constraints is a natural next question to ask (see Sec 16.11 for extra reasons), and RPM's are a good model in which to investigate that qualitatively.

The RPM case of most interest to me is that with the radial 'scale of the universe' direction having particular \hbar -significance, by which the configurational 2-surface element is a sphere element, and the 3-momentum 3-surface element is the spherical polars one. Though I remind the reader that scaled triangleand is actually only conformal to flat space, so one needs to be careful to get the conformal factor contributions right in that explicit computation. Though, more generally, I really want to carry out this study for any region of configuration spaces (which will require more work). And yet, the W makes the evaluation of this in spherical polars natural, even if the sigma itself is totally unaligned with those. For sure, considering a Σ that is cleanly expressible as a region in spherical polars will be an example that can be taken particularly far computationally. So we want to rewrite (991) in conformal–spherical polar coordinates

$$\text{Prob}(\Sigma \text{ is intersected}) = \int dt \int \mathbb{D}\Pi_{\Theta} \mathbb{D}\Pi_{\Phi} \mathbb{D}\Pi_I \mathbb{D}\Theta \mathbb{D}\Phi \mathbb{D}I n^I \Pi_I \text{Wig}(\Theta, \Phi, I, \Pi_{\Theta}, \Pi_{\Phi}, \Pi_I) \quad (993)$$

⁹⁴In fact, Prob(does not enter a region R) plays the main role in Halliwell's program.

as well as substitute (992) into (991) to get

$$p_{\Sigma}^{\text{semicl}} = \int_{\Sigma} d\text{Dra}^{2'} n^{\Gamma'} \frac{\nabla W}{\nabla \text{Dra}^{\Gamma'}} |\chi(\text{Dra})|^2. \quad (994)$$

Now, $W = W(I)$ alone, so this becomes $n^I \partial W / \partial I$ factor plus two zeros (even if Σ is not part of a sphere). If Σ is part of a sphere centred on the origin, then this factor can be pulled out, leaving an integral over the Θ, Φ region. I note that a whole sphere has significance of “universe attains a given size”, which is an interesting if not generic question. In this case the ranges of integration are very straightforward, making further explicit computability highly likely.

The decoherence functional is of the form

$$\text{Dec}(\alpha, \alpha') = \int_{\alpha} \mathbb{D}\text{Dra} \int_{\alpha'} \mathbb{D}\text{Dra}' \exp(i\{\mathbf{S}[x(t)] - \mathbf{S}[y(t)]\}) \rho(\text{Dra}_{\text{in}}, \text{Dra}'_{\text{in}}). \quad (995)$$

The formulaic relation between class function and decoherence functional is

$$\text{Dec}(\alpha, \alpha') = \int d^3\text{Dra}_{\text{fin}} d^3\text{Dra}_{\text{in}} d^3\text{Dra}'_{\text{fin}} C_{\alpha}(\text{Dra}_{\text{fin}}, \text{Dra}_{\text{in}}) C_{\alpha'}(\text{Dra}'_{\text{fin}}, \text{Dra}'_{\text{in}}) \Psi(\text{Dra}_{\text{in}}) \Psi(\text{Dra}'_{\text{in}}) \quad (996)$$

For RPM's, there is an issue as regards assuming an environment in taking this calculation further. This final step is the one in which Halliwell's setting gives a good classical recovery with a smeared Wigner function in place of a classical probability distribution. The detail of influence functional computation involving environment-system interactions is in [298]. What happens if there is no environment or a very small environment (one more particle, say)? Can the above already be done for a subset of triangleland itself, e.g. using I_{base}, Φ alone rather than the Dragt coordinates? Then I_{median} and Φ would serve as a small environment. Could one use shape as the system and scale for the environment (or vice versa)? For I as system, the environment is the sphere variables which are in many ways analogous to cosmological inhomogeneities, so that may be a nice case to look at.

Question. Then at a technical level, half-line Wigner functions, class functions and decoherence functionals are needed. To which extent have these been covered in other contexts?

Next, we concentrate on details of defining class operators. Some differences between Hartle's and Halliwell's are that Hartle constructs his using sums over paths and these have problems with both the quantum Zeno effect (enough measurements of a subsystem prevent it from changing) and compatibility with the Hamiltonian constraint. Halliwell's [294] uses integrals over time, which resolves the latter but not the former. In detail, [294] used a top-hat-shaped window function, whilst his subsequent [295] uses a smoothed-out bump-shaped window function. This amounts to the region in question being taken to contain a potential, with the class operator being the corresponding S-matrix and the smoothed-out case representing a softening in the usual sense of scattering theory (albeit in configuration space rather than in space) [The smoothed-out case manages to avoid the quantum Zeno effect in addition to managing to still be compatible with the (quadratic, not for now linear) constraint.] This is part of why I select Halliwell's version over Hartle's for my own study; the rest of why concerns various features of the [292] and [294] schemes that I consider to be conceptually promising. His class operators are continuous products in label-time of positive operator-valued measures. Also note that $\{C_R, \mathcal{H}\} = 0$ via the window functions being 0 or 1, or, more generally, via group averaging (here integrating over t).

Semiclassicality helps at this particular point with explicit construction of class function in what seems to be an unambiguous manner.

Question The characteristic size of the region R , $l_{\text{Char}} \gg \lambda_{\text{Compton}}$ is required so as to get sharp tracks. What does this mean for a bubble chamber? What does it mean in the cosmological setting? For the triangleland toy model universe? Also note that for the universe models, the sharp track is in configuration space, but the bubble chamber as a record is itself in space. Does this not place Barbour and Bell–Halliwell at odds as regards their uses of bubble chamber paradigms?

Knowledge of the scaled triangle free quantum solution of Sec 9.3.1 is no doubt also useful in further developing Halliwell's scheme. Note: at this level not all the work will be mere transcription since triangleland's inner product differs from flat space's. Also, this model does lie within the good guarantees of calculability at the quantum level by involving just spherical harmonics and Bessel function mathematics.

Question. Extend these examples to contain harmonic oscillator potentials.

15.5 Extending Halliwell 99 to RPM's

While Halliwell's particular HO example's mathematics can be directly adapted to e.g. a $N = 3$, $d = 2$ scaled RPM setting, in doing so the h-l split is not scale-shape aligned, so I prefer to start with a new example. To date only a handful of models [292, 297] have been cast in records formulation, and all have external time. With GR in mind, this is an important deficiency to resolve, and the RPM's look to be a useful setting for doing this. How time enters this parallels how time enters Histories Theory.

15.5.1 A sketch of the steps involved

Given exact wavefunctions (e.g. free and HO ones from Sec 9.3) one can construct decoherence functionals from them. It is also not clear to me what kind of notion of temperature might exist within the context of whole-universe RPM's. Halliwell makes use of the Von Neumann information of the environment (c.f. Sec 14.9).

Analogy 100) Imperfect records are realized within both RPM's and GR.

This approach considers Records within Histories Theory. In particular, imperfect records are still possible in settings with very small environments (even 1 degree of freedom). This makes this work compatible with small RPM's in their aspect as small models. Conditional probability of records $\underline{\beta}$ [I use the notation $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ given the past alternatives $\underline{\alpha}$ is

$$\text{Prob}(\underline{\beta}|\underline{\alpha}) = \text{Prob}(\underline{\alpha}, \underline{\beta}) / \text{Prob}(\underline{\alpha}) = \text{Tr}(R_{\underline{\beta}} \rho_{\text{eff}}(\underline{\alpha})) , \quad (997)$$

for

$$\rho_{\text{eff}}(\underline{\alpha}) = \frac{C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger}{\text{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger)} . \quad (998)$$

Here, $R_{\underline{\beta}}$ denotes a **records projector**, which can be envisaged as the obvious subcase of the general histories projector. [Note also the relation

$$\text{Prob}(\underline{\alpha}) = \text{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger) = \text{Tr}(R_{\underline{\alpha}} \rho(t_n))] . \quad (999)$$

Perfect correlation between records and past alternatives is only guaranteed if $\text{Prob}(\underline{\beta}|\underline{\alpha}) = 1$, which only occurs for $\rho_{\text{eff}}(\underline{\alpha})$ pure. For the more general case of $\rho_{\text{eff}}(\underline{\alpha})$ mixed, $\text{Prob}(\underline{\beta}|\underline{\alpha}) < 1$ and this correlation is imperfect. *Thus in general one can expect the presence of imperfect records.* This approach would amount to considering detectors within the RPM setting: a base pair serving as detector for a passing apex.

Analogy 101) Records within histories theory occur in both RPM's and in GR.

15.5.2 Open questions

Modelling Question. This approach considers the information storage conjecture: information stored is related to extent of coarse graining. Given the multiplicity of such (Sec 14.9, what notion of information is most appropriate for this?

Question which Problem of Time facets remain in this approach? [Partial answer: most of those which Histories Theory fails to deal with; additionally, these are very largely absent from RPM models.] Do this for a shape-scale split RPM; by the shape-scale split, this no longer parallels the technical calculation in the Halliwell 99 paper itself. Start with the triangleland model of this.

Question Consider RPM's in the large- N limit so as to consider models with sizeable environments and sizeable amounts of records/records information within them.

16 Conclusion: Relationalism and the Problem of Time

‘Time’ in General Relativity (GR) and ‘time’ in ordinary Quantum Theory are mutually incompatible notions. This is problematic in trying to put these two theories together to form a theory of Quantum Gravity. Attempting to write down a canonical formulation for Quantum Gravity leads to an apparently frozen formalism, for which various strategies being proposed.

I) Tempus Ante Quantum: finding time before quantization, covering emergent Jacobi–Barbour–Bertotti (JBB) time, super-space time, hidden time, matter time, unimodular time.

II) Tempus Post Quantum: Finding time emerge after quantization under certain circumstances, covering another view on superspace time, third quantization and the Semiclassical Approach.

III) Tempus Nihil Est: dealing with as much physics as is possible from a timeless perspective, covering the Naïve Schrödinger Interpretation, the Conditional Probabilities Interpretation, Records Theory (which I collectively refer to as Type 1 approaches) and distinct observables approaches (Type 2 ‘Rovelli’ approaches).

IV) Non Tempus Sed Historia: ditching the notion of time for the notion of history instead, and re-starting the canonical quantization process using these instead of configurations.

Moreover, the above ‘Frozen Formalism Problem’ is but one of a number of facets of the Problem of Time. I have argued in this article that these facets form a coherent whole due to being based on a common cause (for which I used an ‘Ice Dragon’ mnemonic): GR and ordinary Quantum Theory relying upon mutually incompatible notions of ‘time’. Namely, the Inner Product, Functional Evolution, Foliation Dependence, Multiple Choice, Sandwich, Global, Spacetime Reconstruction and Observables Problems.

The above problems, and strategies were critically reviewed in detail in the early 90’s by Kuchař and Isham [398, 400, 335, 401]. The present article covers a number of the subsequent advances in the strategies and their toy modelling with RPM’s.

16.1 Summary of Problem of Time work over the past two decades

The principal advances have been as follows.

1) a wider range of Histories Theories have appeared [345, 558, 559, 560, 561, 12, 371].

2) The division between Rovelli (Type 2), and other previously proposed (Type 1), timeless strategies has continued to grow, with each side of this divide producing fruitful new work. On the one hand, Page, Barbour and I have made a number of proposals involving Timeless Records (e.g. [499, 500, 80, 83, 28] and the present article). Also representing part of 2), firstly, Records have also being found and studied within Histories Theory by Gell-Mann and Hartle, and Halliwell. Secondly, Gambini–Porto–Pullin [244, 522] have also recently proposed an extension of the Conditional Probabilities Interpretation. On the other side of the divide, the developments are of Rovelli’s Partial Observables Approach [539], Dittrich’s work [211] and the application of some such ideas to Loop Quantum Gravity (LQG), in particular by Thiemann [607, 606] (see also below, and [39] for further comments. The present article mostly studies Type-1, though comments on Type-2 are provided below for comparison.

3) [79, 22, 88] and this article clarify the status of JBB time and its alignment with WKB time.

4) Among the Tempus Ante Quantum strategies, more matter times [126, 325, 390], more dilational hidden times (this article), and a global patching argument have appeared [109, 131, 132].

5) There has been some other work on Problem of Time in context of Ashtekar variables and a small amount of work in context of alternative theories of gravity (see [39] for further comments on these, and the more recent [131, 132]).

6) A number of composite strategies have appeared (see Sec 16.13).

I consider 2) and 6) above to be the most significant developments over the past two decades.

16.2 Some means of judging formulations

I judge by the following criteria (I do not claim this list to furnish a complete judgement).

1) Value of whichever reformulation is involved.

1.0) Whether it is convincing in all aspects at the conceptual level [which connects it with 3) below].

1.i) Whether this affords insights the original formalism did not afford.

1.ii) Whether it affords smoother interpolation between theories usually formulated in ways that look more different.

1.iii) Whether it suggests any extension/alternative theories that are now natural but which were not apparent from the original formulation. (Universality is an anathema to this).

2) Candidate timefunction criteria in cases that have a timefunction, as per Sec 11.2.

3) By topicality in this article, I also judge by relational criteria, which, after various developments in this article, I synthesize as follows. [I use 1.0) to 1.iii) as text-labels, but not 2), 3).]

16.3 Relationalism

I first consider Leibniz–Mach–Barbour relationalism (with some sharpening and moderation of my own). Some intuitive if vague underlying relational principles (LMB) are

Relationalism 0) **Relational physics is to be solely about the relations between tangible entities.**

Relationalism 1) [Mach] These **testably act and are actable upon.**

Relationalism 2) [Leibniz] Any such which are **indiscernible are held to be identical.**

Relationalism 3) [Barbour] This concerns the possibility that the **configurations are the primary entities**. I.e. a central role for the configuration space \mathbf{q} . I expand on this in 3 ways:

categorizing (at least formally, following from the heuristic idea that quantization is a functor).

Propositioning: Mackey’s contention that science is entirely about propositions [437], and that these have a simpler structure in timeless theories to the extent that, subject to that atemporality, the propositions *constitute* what remains of the physics.

Contextualizing: $\text{Context}(\mathbf{q}) = \{\text{all Sub}\mathbf{q} \subseteq \mathbf{q}\}$ (see the next page for more about this).

One might also expand to phase space, though there is then the issue of whether (Phase, Can) is as/more fundamental than (\mathbf{q} , Point). All programs involve momenta, though perchance not the entirety of the canonical transformations, Can. This is on two different grounds: assuming stronger \mathbf{q} primality or the a wish to not induce nonunitarily-related QM. Some QG and Problem of Time strategy approaches use canonical transformations (e.g. Ashtekar Variables, York time). I have also argued in this article how primality of \mathbf{q} can be generalized to operational distinguishability between configuration and momentum measurements, which formulation can be expanded to a tripartite distinguishability that has fermionic variables as its additional part.

Histories Theory holds that, instead, **histories are the primary entities**; this amounts to supplanting Relationalism 3) and the ‘Tempus nihil est’ postulate that most naturally follows from it by the space of histories Hist and the ‘Non tempus sed historia’ postulate and admits all of the above expansions.

Note 1) [Isham–Linden] Propositioning first appeared in the context of Histories. The proposition-projector association is then the preferred and only provenly fully effective mathematization of this; one can also make use of this in Records Theory. Note 2) Propositioning and contextualizing combine in the form $\text{PhysProp}(\text{Context}(\))$, yielding overall the various possible groupings of structures as per Fig 55.

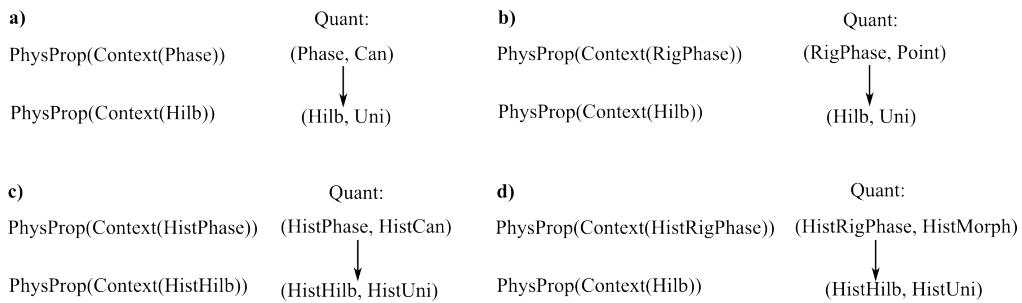


Figure 55: Enlargements of ‘ \mathbf{q} is primary’ or its substitute ‘Hist is primary’.

Relationalism 4) [Barbour’s ‘Cartesian’ element] At the classical level one is to implement relationalism at the level of the action.

One **builds up** compound objects as bona fide combinations of one’s primary objects (**tensor/bundle** structure). This applies to the relational action, but also more generally as a builder move, producing all of actions, notions of distance, notions of information and correlation, and quantum operators. The actions in question are Jacobi or Jacobi–Synge type actions.

Relationalism 5) [Leibniz] A physical theory is **temporally relational** if there is **no meaningful primary notion of time for the system as a whole** (e.g. the universe).

This is implemented by **manifest reparametrization invariance**, or, equivalently, **manifest parametrization irrelevance**. in each case with **freedom from extraneous time-related variables**, such as external absolute Newtonian time, or GR's 'lapse' variable, at the level of the action principle.

Relationalism 6) [Mach] **Time is to be abstracted from change.**

Relationalism 7) [Leibniz–Barbour] **The MOST PERFECT time is to be abstracted from THE TOTALITY OF change.** I refer to 6) and 7) together as the LMB view of time.

Relationalism 8) [my extension and part-reformulation of Barbour] There is a group of transformations \mathfrak{g} that are held to be physically irrelevant (**configurational relationalism**. This group is subject to [what were introduced as Relationalism 9) and 11)'s] $\mathfrak{q}, \mathfrak{g}$ pair compatibility issues:

- i) $\dim(\mathfrak{q}) > \dim(\mathfrak{g}) + 1$,
- ii) \mathfrak{g} has natural action on \mathfrak{q} , and
- iii) the constraint algebra a posteriori accepts \mathfrak{g} , both at the level of the classical Dirac procedure and at the level of the quantum commutator algebra.

\mathfrak{g} 's physical irrelevance can occasionally be directly implemented by \mathfrak{g} -independent objects, though far more usually one has to implement it indirectly via incorporating \mathfrak{g} -on, \mathfrak{g} -off pair of moves into the sequence of maps in one's build-up . This builds \mathfrak{g} -bundle objects/bibundle objects with one bundling involving \mathfrak{g} .

Relationalism 10) Quotienting out the \mathfrak{g} is to **remove all the structure in the \mathfrak{q} that is actually extraneous**. This is a good idea, but precisely what structures/features are extraneous in the case of the universe we live in is not so clear. See e.g. Sec 16.6.1 as regards whether to ditch scale. One passes to

$$\mathfrak{q}(\mathfrak{a})/\text{Aut}(\langle \mathfrak{a}, \mathcal{P} \rangle) \quad (1000)$$

for \mathcal{P} the structures up to a certain mathematical level (e.g. metric geometry or topology). Ideally, one might wish for relationalism to remove *all* levels of structure of \mathfrak{a} .

Note 3) Exact JBB time implements LMB time of Relationalism 7). My study of Jacobi–Barbour–Bertotti (JBB) time in Secs 1, 2 and 12 is thus an extension of the understanding of temporal relationalism.

Note 4) My study of notions of distance between shapes in Sec 14 is an extension of how one might go about implementing configurational relationalism. In particular, I considered Kendall, Barbour and DeWitt type comparers of configurations/shapes, and actions and other alternatives built out of these, parallels with group averaging (with QM applications) and with inf taking: the Gromov–Hausdorff approach (applications of this kind of which are, at least to me, unknown). This makes clear a number of steps in Barbour's approach that are actually choices, by providing alternatives to those choices.

Note 5) The above is at the level of the configuration space metric. What about topological relationalism? The arguments for relational and not absolute do not break down at this less structurally-endowed level. Some of the above indirect methodology may carry over to this case (summing over, averaging over, sup/inf/extremum notions over classes of topologies). One possibility here is the Gromov–Hausdorff notion of distance, assuming that it can be uplifted to Riemannian metrics serves to compare 3-metrics on different topologies. There is a potential problem as regards whether this distance is diffeomorphism-invariant and whether the notion of isometric embedding used uplifts to a good such at Riemannian level. Seriu's approach can be viewed as at least a prototype of comparer between metric geometries on different spatial topologies. These remain very early days for non-fixed topology/topologically relational physics. At a conceptual level, a canonical quantization of the trousers universe or of a universe developing a handle should be possible; our lack of machinery for doing so may indicate that it is premature to seek final quantum theories, or possibly indicate the demise of canonical quantization.

Note 6) (Hamiltonian Collapse problem) In passing to the Hamiltonian formalism prior to quantization, a lot of Barbour-relational versus nonrelational formalism differences get wiped out. However, the relational *ideas*, if not implementations, can be recycled at the quantum level.

Note 7) The Dirac and reduced formulations at the quantum level both embody configurational relationalism (though both of these techniques can be, and were, arrived at without any reference to relationalism). MRI label time remains a useful concept in path integral/histories formulations. Relationalism in expectations etc, \mathfrak{g} -averaged versions of operators: another application of \mathfrak{g} -on, \mathfrak{g} -off moves.

What of other relational programs? Rovelli has a sense of relationalism in the sense of objects not being located in spacetime (or space) but being located with respect to each other (though he does not develop this to the extent of the above LMB-A relational postulates). Rovelli and Crane have between them put forward 4 contextual postulates

Contextualism 1) **QM only makes sense for subsystems**. In the quantum GR arena, Crane implements this by considering splits of the universe with the observer residing on the surface of that split. Each of these splits then has its own Hilbert

space. Crane then maintains that the whole universe has no Hilbert space, though one is to be recovered in a semiclassical limit.

I dissect this as follows since I don't agree with one part of it.

- 1) QM makes sense for subsystems (true, and obvious); each has its own Hilbert space and admits the standard interpretation of QM.
- 2) Almost all QM concerns subsystems (true at an elementary level, but sometimes forgotten in Quantum Cosmology).
- 3) What would be QM for the whole universe does not possess a Hilbert space or the standard interpretation of QM. Here I disagree about the lack of Hilbert space (shrunk as it may be due to closed-universe effects as per Secs 7 and 10), but agree about the subsequent non-applicability of the standard interpretation (though this is well known and plenty of different groups of authors have been willing to work with that). Thus my notion of $\text{Context}(\mathbf{q}) = \{\text{Sub}\mathbf{q} \subseteq \mathbf{q}\}$, whereas Crane's is $\text{Context}(\mathbf{q}) = \{\text{Sub}\mathbf{q} \subset \mathbf{q}\}$. This difference also accounts for my replacing 'all' by 'almost'. I envisage the scalefactor of the universe as a possible whole-universe property that can plausibly enter one's physical propositions; at the very least, scale is not a *localized* subsystem. It is also not clear whether Crane's $\text{Sub}\mathbf{q}$ carry the same local physical connotations as mine (as per Sec 14.1.1).
- 4) QM makes sense for the universe as a whole in some kind of semiclassical limit. Here I agree with Crane, though in detail we may well not both mean exactly the same thing by a 'semiclassical limit'. Moreover 4) is less necessary from my perspective (it applies just to interpretation, rather than to the possession of a Hilbert space as well).

Contextualism 1W) (W for 'weakened') is the name by which I shall refer to my above version [1), 2) and the standard interpretation parts of 3) and 4)].

Note 8) Contextualism 1W) is what I posit as the contextual expansion of Relationalism 3) [though one *could* instead elect to use Contextualism 1) there].

Contextualism 2) allows for the QM of **observers observing other observers observing subsystems**, i.e. of larger subsystems containing a smaller subsystem under observation and the observer performing that observation. This provides a further level of structure between the plethora of Hilbert spaces associated with all the contexts of Contextualism 1). This approach contends that [540] **QM propositions only makes sense in the context of particular observers** (my rewording).

Contextualism 3) **Partial observables** are a good idea as regards physics being about correlations and as regards freeing one from the harder-to-find complete observables. These are contextual through involving pairs of subsystems. It is not however clear exactly what objects to compute in this approach in order to investigate whether subsystems exhibit correlation.

Contextualism 4) **Use anything as a time for anything else's motion**. This can be seen as a 'Relationalism 7W)' alternative to Relationalism 7) as regards how to abstract time from change; the W is for 'weakened'. I point out its similarity to '*tot tempora quot motus*' (so many times as there are motions, which is the Scholastics' view of Aristotle, see e.g. p 54 of [354]. Thus I also term this view of time *AMR time* after Aristotle, Mach and Rovelli.

Note 9) Thus we have two extreme interpretations of Mach's 'time is to be abstracted from change': Rovelli's 'a time can be abstracted from any change' to Leibniz and Barbour's 'time is best abstracted from the totality of change'. Each has a supporting argument: Rovelli's interpretation applies generically (regardless of physical regime or material composition of the system in question). On the other hand, LMB's interpretation, by using the totality of the change, produces an *incontestable* time, in that there is no more change elsewhere that can run in suspiciously irregular ways that cause of one to doubt one's timestandard. However, neither argument is all-embracing, and there is plenty of middle ground. Leibniz-Barbour's statement of classical bestness implies that there is a more general notion of **some times being better than others**, without having to go to the whole-universe extreme (which is impractical, and operationally undesirable - we know little of the motions and constitutions of very distant bodies). Such implies that some time standards are better than others, which as a partial argument against the Rovelli scheme. On the one hand, Rovelli does not state he is looking for 'good clocks' (regardless of whether just by criteria 1) to 3) of Sec 12.18 or additionally by criterion 4) as well: marching in step with the astronomical ephemeris). On the other hand, that is a good question to ask, and his approach to physics does not naturally envisage this question or any practical or conceptual means of determining which clocks are better than others. Moreover, Barbour's approach is very conducive toward asking and conceptually answering this question. In counterpoint, it might be argued that this is a question which in general has no answer, due to Rovelli's genericity. However, I propose a distinct LMB-A' resolution to this issue below. That is built on the wider view of Barbour and myself being that **MORE change is MORE reliable as a timestandard provider than less change**. This is tied to how timekeeping is not just about a stable system but also additionally reflects the usefulness of the read-off time for the wider physics: stable *and* marches in step with the system (and thus such clocks march in step with each other). I then implement this as follows.

Relationalism 7M) **Time is best in practise abstracted from a sufficient totality of local change**.

This is ‘middling’ (M) between LMB Relationalism 7) and AMR Relationalism 7W); it conforms to what consists sufficiently accurate timekeeping in practise. In comparison to AMR, it has the extra element of acknowledging some times are locally more useful to consider than others. In comparison with LMB time, the practically-used ephemeris-type method for the solar system is entirely adequate without having to consider the net physical effect (tidal effect) of distant massive bodies such as Andromeda (thus also rendering irrelevant our increasing lack of precise knowledge of the masses and positions as one considers more and more distant objects). I refer to 6) and 7M) together as the LMB-A view of time.

I began to develop Relationalism 7M) in Sec 12.1 with examples. I continue here in more general terms.

Relationalism 7M) is to be implemented by a **Generalized Local Ephemeris Procedure**. I.e., do not just use a change to abstract a time, but also check whether using this time in the equations of motion for other changes suffices to predict these to one’s desired accuracy. If the answer is yes, then we are done. If not, include further locally-significant changes in the definition of time, judging locally-significant at the relational level of tidal forces/geodesic deviation, and repeat the procedure. Then if this scheme converges without having to include the entirety of the universe, one has found a LMB-A time that is locally more robust than just using *any* change in order to abstract a time, and it is particularly useful to consider equations of motion in this time and propositions conditional on this time. [If the scheme requires the entirety of the universe to converge, then LMB time itself is the most robust time, at the above prices of impracticality; the LMB argument itself guarantees that ‘more robust’ is an attribute that is realized in all subsystems. The point of LMB-A is that many systems will have localized cut-offs by which an accurate enough time for one’s purposes can be abstracted from a practically manageable local portion of that system.]

Note 10) The generalized local ephemeris procedure gives a local timestandard (around the position of the observer).

Note 11) In the late universe in low-curvature regimes this is very much a local recovery of Newtonian time.

Note 12) To greater accuracy, SR and e.g. parametrized post-Newtonian PPN [639] GR effects can be included into the time-computing and time-calibrating procedure. Two separated observers using the generalized local ephemeris procedure in the solar system will get ‘marching in step’ from each applying this procedure, at least approximately. Once relativity is non-negligible, proper time is distinct for each observer: $t = t(x^\mu)$. This is for a given surface, corresponding to observers moving in a particular way. This is not necessarily complicit with, say, Earth and Jupiter being among the included observers moving in a particular way, so we can expect (GR as well as SR) deviations in proper time between Earth and Jupiter to build up. However, if each of Earth and Jupiter have chosen a time that marches in step with the totality of relevant local solar system dynamics, each would notice deviations from that and recalibrate one’s timestandard to compensate, and so, by keeping in step with the solar system to good accuracy, each would happen to be keeping in step with the other to comparable accuracy. This should however break down when the two observers are observing disjoint local phenomena (and the accuracy of marching in step should go down with decrease in overlap between observed phenomena). Thus one should not expect any kind of marching in step on greater than interstellar separation distances, unless both observers consider phenomena visible and accurately studyable by both; pulsars are an obvious suggestion. If each understands the natural variations in the pulses and factors them out, two distant civilizations’ clocks could roughly march in step by both of them marching in step with each’s set of observed pulsars, which two sets have a nontrivial intersection. A further issue then is whether such as Kepler and pulsar physics sufficiently universal to allow for marching in step in situations without overlap? A specific trouble for such a non-overlapping scheme lies in the various systems drifting apart within a GR universe. Perchance the accelerations experienced on each worldline would be measurable and incorporable, though the details of this remain to be worked out. Thus, all in all, the generalized localized ephemeris principle has a *tendency* to produce approximate marching in step.

Note 13) Cosmology can be done from each of multiple local observation points. Only very inaccurate timestandards ensue from such (at least at our own kind of technological level) but the ensuing timestandard could possibly be in accord as developed from each local observation points. Cosmological timestandards have a further LMB element in being global, but not however the stronger LMB element of using the totality of change in the universe.

Note 14) In extension of the incontestibility of LMB, the LMB-A practical timestandard involves considering a sufficiency of locally relevant change for one’s desired accuracy will be uncontestible for internal physics. It may still be contestible by external physics, e.g. if the Solar System does not keep time with the pulsars, which do we consider to be at fault? (This is not an issue in the LMB case itself since everything lies within the universe-system). But a ‘Machian synchronization’ of this type is not for now demonstrated to be highly accurate or entirely universal. It should be a useful conceptual awareness to have in the theory of larger-scale synchronization conventions that is largely complementary to the much more well-known SR conceptual awareness.

End-Note 15) Crane completes the suggestion of his program and particular model [188] by providing a fairly tentative technical suggestion to furnish it with a time (candidate). It might be plausible however to couple Rovelli’s or Barbour’s notion of time to this scheme.

End-Note 16) The building up implementation includes how to **g**, and Prop, as useful structures to include in the Rovelli approach too (these may be new structural insights).

End-Note 17) Prop(Context()) on **q**, Phase, Hist or PhaseHist strengthens not just LMB-A relationalism, but Rovelli–Crane relationalism also.

16.4 Relational content of the Problem of Time facets

The Frozen Formalism Problem ties directly to the quadratic constraint which arises from the MRI/MPI implementation of temporal relationalism. The Inner Product Problem at a very simple level is relationally linked (only inner products of wavefunctions are measurable and thus tangible); this problem is absent in RPM models. Accompanying linear constraints are a direct consequence of Configurational relationalism; in the GR case, these and the preceding constraint form the split-diffeomorphism (alias Dirac alias deformation) algebra. Foliation-dependence is then tied to this algebra (and moreover does not occur for RPM's). The thin sandwich problem is tied to the 3-diffeomorphisms; in this article I argued that this is a subcase of the best matching problem (and showed that for the RPM's studied in detail here, it is a resolved problem). Both of the preceding are also connected to notions of observer via Sec 11's account. Moreover, proper time pertains to general objects/blobs as well as to observers, thus it is not clear if foliation really has to be interpreted in terms of observers. Functional Evolution: view as part of a posteriori compatibility. absent in this article's RPM's The Multiple Choice problem: less emph on phase space helps, but does not cure this problem.

It might be hoped that in the contextual approach, different unitarily-inequivalent QT's correspond to different observers' perspectives. However, I know of no concrete evidence for this. This is likely much easier to refute by counterexample than it is to establish for a class of theories. Observers being local may go against some of the relevance of Global Problems of Time. As regards the global problem, contextualism places less importance on this, e.g. via Bojowald's fashionables: observables contextually tied to a clock being useable in that region and epoch (see Secs 16.13, 16.17 for more).

As regards the Spacetime Reconstruction Problem, space/configurations/dynamics are primary, and spacetime may not exist as a meaningful concept at the level of Quantum Gravity. Whether reconstruction procedures themselves are particularly relational needs somewhat more thought. This problem has some ties to the split diffeomorphism algebra. One can see on the basis of this that most of the facets are well-connected to relational ideas, so that relationalism may give a deeper account of whence some of these facets.

16.5 RPM's as a Problem of Time arena

As argued in the preface, GR can be viewed as a gestalt of reconciliation of relativity and gravitation and a freeing of physics from background structure. Then "Quantum Gravity" should be viewed as the quantum version of this gestalt. The present article then concerns a model of quantum background independence, which is of useful complementary value to a number of other studies and models. RPM's have many further analogies with geometrodynamics rather than in modelling the real world as an RPM (see Appendix 5.C, Sec 7 and Sec 10 for some comments on the latter), and 1- and 2- d models suffice to meet this interest whilst having all the benefits of being particularly tractable mathematically.

The O-case of my 3 particles in 2- d model in is also interpretable as a type of 3- d universe but this does not at all readily generalize to more particles. In 1- and 2- d , Barbour's 03 theory is equivalent to the Jacobi-Synge construction of the natural mechanics associated with a geometry to Kendall's shape geometry (the space of pure shapes). Likewise, BB82 is equivalent to the Jacobi-Synge construction applied to the cone over Kendall's shape geometry (the space of scaled shapes). This manifests a scale-shape split, by which the pure-shape study has the additional value of being a subproblem within scaled RPM's. The simplest configuration space geometries for 1- and 2- d RPM's are \mathbb{S}^{n-1} and \mathbb{CP}^{n-1} (pure shapes), and $C(\mathbb{S}^{n-1}) = \mathbb{R}^n$ and $C(\mathbb{CP}^{n-1})$, the first three of which are very well known as geometries and as regards subsequent classical and quantum mechanics thereupon and supporting linear methods of Mathematical Physics. These render many quantum and Problem of Time strategy calculations doable and available for comparison with each other, which is a rarity in the latter field. Triangleland is further aided in this way by $\mathbb{CP}^1 = \mathbb{S}^2$ and $C(\mathbb{CP}^1) = \mathbb{R}^3$ albeit the latter is not flat; it is, however, conformally flat. The simpleness of the ensuing mathematics, even well into the usually complex rearrangements necessary for the investigation of Problem of Time strategies, is a major asset in this RPM toy model arena, securing many computational successes beyond the usual points at which these break down for full GR/many other toy models.

The examples studied in detail in this article are:

- 1) 4-stop metroland (which has particles enough to be split up into two nontrivial subconfigurations: Jacobi H-coordinates adapted to this, or to possess a hierarchy of nontrivial subconfigurations: Jacobi K-coordinates are adapted to this, each of which are useful as regards Records Theory and structure formation).
- 2) Triangleland (the simplest model with nontrivial linear constraints in its quantum-cosmologically interesting scaled case so as to be able to parallel effects of the GR momentum constraint, which it attains via being the simplest RPM to possess physical angular momentum).
- 3) This article also made a start on the next most complicated model, quadrilateralland, which unifies the above two sets of desirable features and is the first nontrivially complex-projective-space case. (This is the RPM problem's 'Bianchi IX' as regards technical complexity and usefulness as a toy model of GR).

In RPM's, linear constraints and inhomogeneities are logically-independent features, in contrast with how in minisuperspace both are simultaneously trivialized by the derivative operator becoming meaningless. Furthermore, in GR nontriviality of these features requires an infinite number of degrees of freedom. Thus, overall, RPM's permit study of each of the midisuperspace-like features of linear constraints and inhomogeneities without these complicating each other or being complicated by the infiniteness and mathematical complexity of midisuperspace. By including such features, study of RPM's brings into sharp focus a number of issues outside of those found and studied by use of minisuperspace. Uses include unveiling

operator ordering problems, toy modelling structure formation, further investigation of effect of linear constraints on various Problem of Time strategies.

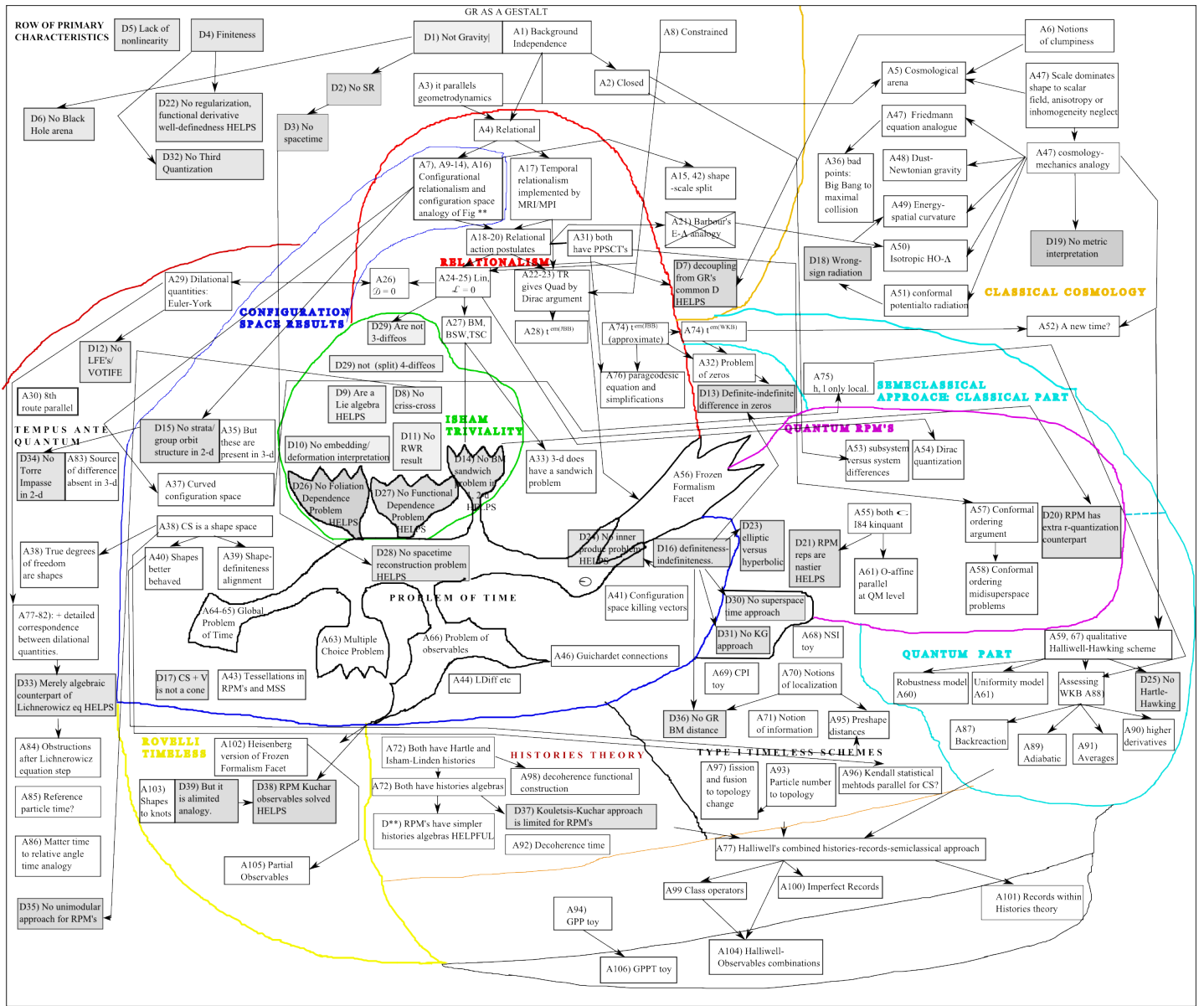


Figure 56: The inter-relationships between the 106 analogies and 39 differences between RPM's and GR. A superceded analogy is crossed out, differences are in shaded boxes and labels can then be pdf-searched for in the text for full statement of each. To explore this figure, blow up its pdf version by around 200 percent. Note what is left of the Ice Dragon here: frozen breath but no teeth, one leg (Multiple Choice), a tail, only small wings and no scales. The lower parts of the figure reflect what strategies RPM models can be specifically geared toward. Arrows with no box at one end are from the entire region at that end.

Question analyze the other toy models in [400] and the Preface to this extent, represent them by similar figures and compare them.

16.6 Some asides

16.6.1 Aside 1: is scale meaningful? With consequences and uses.

Scale is possibly undesirable from the relational perspective via being a single heterogeneous addendum to the shapes [85, 97]. The cone structure makes scale especially extraneous in RPM's – it has the same sort of dynamics irrespective of what the shapes are. It may then seem quite conceptually undesirable for scale to play so prominent a roles in Cosmology. However, 1) there are currently no credible alternatives to this as regards explaining Cosmology. 2) I hope my following 'heterogeneity can be useful by alignment with choices that would be harder to make without it' argument renders scale more presentable from a conceptual perspective.

It is the scale contribution that gives the indefiniteness in GR kinetic term, a feature not found elsewhere in physics and which causes a number of difficulties. (E.g. invalidation of the usual Schrödinger interpretation of the inner product, and a whole new theory for the dynamical meaning of zeros in the potential factor.) I also note that RPM's cannot model this particular feature; minisuperspace does.

Note 1) Use of the indefiniteness to pick out a ‘superspace time’ fails.

Note 2) Scale as internal time fails (at least globally), and I argue above that use of dilational quantity conjugate to scale as internal time carries bad connotations.

Note 3) However, h–l alignment with scale–shape removes ambiguity of how to allot h and l roles. Then the scale part contributes an approximate timestandard with respect to which the shape part runs according to usual positive-definite kinetic term physics (‘scale-indefiniteness alignment’ exploited by aligning h with both). This can then be fed into more promising Histories-Records-semiclassical combination strategies.

16.6.2 Aside 2: further directions in RPM modelling

I showed how the Barbour–Bertotti 1982 and Barbour 2003 theories transcend formulation in a way in which the Barbour–Bertotti 1977 theory fails to, which provides a further strong reason to favour the first two (and readily leads to formulation of various other theories with similar limitations to Barbour–Bertotti 1977).

This article also laid out the beginnings of the study of alternative mirror-image-identified and indistinguishable-particle RPM’s up to the level of configuration space geometry. The configuration spaces for these are quotients of the preceding by \mathbb{Z}_2 and by permutation groups, which include \mathbb{RP}^{n-1} , so-called weighted projective spaces, spherical primes and generalizations thereof, and the 3-*d* half-space with and without edge. The mirror-image-identified case has applications as regards quantizing manifolds with edge and being a toy model of affine geometrodynamics. The indistinguishable particle case is tied to arrow structure and arrow statistics (I use the word ‘arrow’ instead of ‘spin’ so as to not carry any angular momentum connotations in models that do not possess any notion of angular momentum in physical space).

This article has considered lucid and useful coordinatizations of shape space, the momenta conjugate to shape coordinates. The shapes split irreducibly into relative angle and relative distance (non-angular ratio) quantities, and this is paralleled by their conjugate momenta splitting into relative angular momenta and relative distance momenta. Finally, I interpret RPM conserved quantities in these terms.

16.7 Tempus ante Quantum

Jacobi–Barbour–Bertotti (JBB) time is a time before quantization but not one which resolves the Frozen Facet Formalism. I also provide a more detailed and lucid account of its properties. It has value as *lii*) the now-emergent counterpart of Newtonian, proper and cosmic time in various contexts. I also discussed its limitations (a complicated definition in general, precluding its computation in general and requiring insights elsewhere obtained in the case of recovering the usual kind of solar system ‘ephemeris time’; I also point out gaps in its ‘accounting for clocks marching in step’. I discuss its relation to the JBB time as I formulate it also contains an ambiguity: it is (M, W, t) triples that are invariantly defined. It is a whole-universe LMB notion of time in principle, but not as regards practical and sufficiently accurate computations (taking these into account it is a LMB-A notion of time, which then plays a substantial role in this article’s Tempus Post Quantum and Tempus Nihil Est treatments too).

Scale time has similar limitations in both RPM’s and GR as regards a lack of monotonicity. In both cases this is improved upon by passing the time interpretation to the dilational quantities canonically-conjugate to the scales. For GR one instance of this is York time; I previously wrote of an *Euler time* analogue for this in RPM’s [20]. The monotonicities in these cases are guaranteed in a number of sectors by the *CMC-lapse-fixing equation* and the *Lagrange–Jacobi equation* respectively. However, in this article I pointed out *these two are not exact analogues, each being embedded rather in a family of models*. There is one per scale variable, each with its own dilational conjugate which nontrivially affects 1) the form of the subsequent monotonicity-guaranteeing dilational time propagation equation. 2) the computation for finding the true Hamiltonian by arranging to form $p_{t^{\text{true}}} = H_{\text{true}}$ which straightforwardly gives a time-dependent Schrödinger equation upon quantization. This is a useful clarification, it avoids the logarithm problems of Euler, and makes clear a further level to the Multiple Choice Problem. There is an algebraic equation in place of GR’s Lichnerowicz–York quasilinear elliptic partial differential equation that needs explicit solving so as to promote the ‘true Hamiltonian’ to operator form. However [contrary to 1.0)], I demonstrate that even algebraic equations lead to terrible ill-definedness and operator-ordering ambiguities, and frankly hard-to-interpret equations even for cases in which the physics is known elsewhere to be simple.

Viewing scaled RPM as a relational recovery of Newtonian Mechanics in the senses of Appendix 5.C, I have found that substantial portions of Newtonian Mechanics stripped of its absolute structure nevertheless possesses a (nonunique) notion of internal time. This should serve as a warning to those that consider privileged notions of time to be unnatural in relational theories (including GR) on aesthetic or philosophical grounds [1.0]: **theories that are relational can nevertheless provide a privileged notion of time from within their own structure.**

On the other hand, toy models for many of GR’s technical complexities will need to be more advanced than this article’s. Time-dependent potentials are hard to interpret here [1.0]; such are usually held to correspond to non-conservative physics, but, unlike in that setting, the model is a whole-universe so there is nothing for energy to dissipate into.

I gave a relative angle time example as an analogue of matter time. Whether one can use reference particles in RPM’s in a similar way to reference fluids in GR, I leave as an open question for now. **Matter time programs have difficulties**

with meeting tangibility and physicality [1.0]). Finally, I also revealed the **non-relationalness of the unimodular approach to the Problem of Time**, which could be viewed as a reason to question this approach.

16.8 Tempus Post Quantum

I mostly considered the Semiclassical Approach: there is no Klein–Gordon approach for RPM’s, no third quantization (at most the analogue would be second quantization, though there are ample reasons not to believe in such a scheme as a Problem of Time approach [400, 335]). From here down, all approaches considered are universal, thus meeting 1.ii) well.

Here, I consider $\text{scale} = h$ and $\text{shape} = l$ in analogy with Quantum Cosmology. I find that emergent WKB time is a recovery of approximate JBB time. **This gives emergent WKB time a Machian interpretation.** I obtain an approximate emergent time dependent Schrödinger equation. Using scale alone for a time is somewhat LMB-like in globablity but not in using the totality of the universe’s contents.

I showed that in the cosmologically-relevant setting with 1 h degree of freedom, in general the l -subsystem is *not* governed by a time-dependent Schrödinger equation, but rather by a more general time-dependent wave equation.

Back-reaction is particularly significant; by allowing the l -degrees of freedom to now participate as a small perturbation in the h -dominated determination of the time-standard, **the Semiclassical Approach’s timestandard gains in LMB-A relational character** (Relationalism 6 and 7, or, more precisely 7M). I.e., the l -subsystem back-reacting on the h -subsystem and so itself contributes (a bit) to the timestandard. The then h -subsystem plays a perturbative role as well as a chroniferous one in the l -physics. This in turn opens the possibility that such an emergence of time for the l -subsystem may on occasion be betrayed by noticeable deviation from the naïve l -physics due to this perturbation term, although the Semiclassical Approach works just as well if this perturbation’s effects are unobservably small. A perturbative framework for investigating these is tractable enough in RPM arena set up in this article. Inclusion of scale among 7M)’s variables amounts to using a combination of knowledge of local shapes and the estimate of a global averaged scale variable.

The applicability of the WKB ansatz is truly important in the sense that if this does not apply, one no longer gets anything like a time-dependent Schrödinger equation for the l -physics. I showed it does not hold in all regions of configuration space [against 1.0]). I also study models for which the exact solution is known and thus comparable region-by-region with the approximate solution following from making the WKB ansatz and approximation. In situations in which this ansatz is applicable, then its means of working through retaining and manipulating a small term has further qualitative, and perhaps quantitative, repercussions through the equations containing further similar small terms which are nevertheless habitually neglected in the literature. I view this modelling as a useful arena for making qualitative tests of the Halliwell–Hawking scheme. I have also pointed out other relations in the sizes of various small terms, which restrict how freely one can neglect some but not other small terms.

The 2-body in 3-body, Euler in Navier–Stokes and emergent-time-dependent Schrödinger equation within an emergent-time-dependent Hartree–Fock scheme issues substantiate a qualitative suspicion of some of the approximations being made in them [against 1.0]). As such it may be worth questioning for the moment whether one actually believes in semiclassical quantum cosmological calculations.

RPM’s are also open to checks of the Semiclassical Approach’s assumptions and approximations against exact solutions: the system is simple enough that it can be solved without making semiclassical assumptions; then in which cases and in which regions does the Semiclassical Approach give a good approximation to the exact solution? E.g. I calculated this for the 3-stop metroland with HO-like potential. Alternatively, a wider range of exactly soluble models including a triangleland one, might be comparable against the small-time approximation solution of their not-exactly-soluble semiclassical schemes (perturbative treatments with and without back-reaction) .

Finally, **there is an absolute versus relative phase issue in the Semiclassical Approach** [190] [this may also impinge on 1.0)].

16.9 Type 1 timeless strategies

Here, the configuration space is all, though we do need to supplement its study with that of subconfigurations and grainings.

Records Theory is set up highly relationally and q -centrically. I included Naïve Schrödinger Interpretation examples for RPM’s as demonstration of computibility/application of paper’s exact QM solutions for simple models. This includes computations of quantities qualitatively significant to Quantum Cosmology. These include investigating Prob(universe is uniform) and Prob(universe is contents-homogeneous). On the other hand, the more promising, if more complicated investigation of the Conditional Probabilities Interpretation and its recently proposed extension by Gambini–Porto–Pullin [244, 522] has not yet been studied for RPM’s. I do not envisage RPM’s to be able to help in practise with the lines along which Page himself [499, 500] has proposed to extend that scheme.

I consider records to be localized subconfigurations of a single instant that are useful as regards reconstruction of a semblance of dynamics. Notions of localization in space and in configuration space are useful here. I discuss what criteria a notion of distance should satisfy if it is to apply to a relational theory: axioms of distance, q -invariance, sufficient generality, naturality (dynamical rather than just kinematical, compatible with the physical laws of that system), perhaps the possession of a QM counterpart, or a structural analogue in QM or Statistical Mechanics). Notions of distance include configuration space metric

based ones such as the RPM action itself. However, these are indefinite in the case of GR, therefore not at all serving as a notion of distance in that case [against 1.0)]. One possible resolution of that is that the pure-shape part of an object of that nature serves for both RPM's and GR however – a new interpretation of the conformal gravity action [48]. This resolution has the problem of incompatibility with the laws of nature, which come in terms of the DeWitt supermetric and not this definite object. However, in the Semiclassical Approach with h-l scale-shape aligned, the l-physics *does* possess the matching supermetric. I additionally consider various further options, some of which serve for both GR and RPM (more desirable from a universal perspective). For just GR, these include the Matzner [449] and Seriu [570, 571] spectral ‘distances’ (they lack various axioms) and for GR or RPM, the Hosoya–Buchert–Morita (HBM) [324] distance. I do not as yet know whether a form of the Gromov–Hausdorff distance carries over to GR configuration spaces. I note that the Gromov–Hausdorff [277], embedding point, curvature invariant and spectral measure approaches perhaps overly favour geometry over matter. The Kendall and Barbour comparers treat the two on an equal footing, whilst the HBM object is based on the matter content.

A suitable notion of localization in space and in configuration space may be hard to come by and/or to use for Quantum Gravity in general. I.e., ‘where’ particular records are can be problematic to quantify, and the records can be problematic to access and use too, since the relevant information may be ‘all over the place’.

One then requires a suitable notion of information. This can be based on knowing the RPM wavefunctions via the microcanonical ensemble if one assumes no fusions/fissions. Relative/mutual information and correlations are then useful subsystem notions. Information may additionally in practise be problematic firstly as it may be of too poor a quality to reconstruct the history. Secondly, because a suitably general notion of information is missing from our current understanding of classical gravity (ref Sec 14.9.1 for references), never mind Quantum Gravity with its unknown microstates. (Mechanical toy models are useful in not having this last obstruction). Finally, the further Records Theory notions of significant correlation patterns and how one is to deduce dynamics/history from them looks to be a difficult and unexplored area even in simpler contexts than gravitation.

As regards semblance of dynamics [crucial for 1.0) to succeed], the two purely atemporal alternatives known are 1) Barbour’s conjecture based on configuration space shape somehow bringing about bubble chamber track-like phenomena with sufficient frequency to account for the richness of records around us. 2) Page’s in-principle but not in-practise approach involving a single configuration containing ‘memories of ‘previous configurations’’. I provided in particular a step-by-step critique of the former (the main problem with the latter is that one knows very little as to how to calculate, or even toy-model, with such a scheme). See also the combined strategy section for non-purely atemporal ways round the problem of whither the semblance of dynamics.

16.10 Histories Theory

Histories theory is set up highly relationally; it replaces \mathfrak{q} -primality by Hist-primality. It was the first place to have the expansion of primality by propositioning. I have shown that one can study simple RPM models as Histories Theories in both the Gell-Mann–Hartle and Isham–Linden formulations. A global problem surfaces in the Dirac approach to triangle-land. I view this article’s work on Histories Theory mostly as a useful prequel to combined strategies in subsequent work.

16.11 Observables and Type 2 ‘Rovelli’ timeless approaches for RPM’s

Rovelli’s timeless approaches are set up along the lines of contextual relationalism. For the indirect formulation of scaled RPM, observables alias evolving constants of the motion alias perennials are functions $O(\rho, \pi)$ with vanishing Poisson brackets with the constraints,

$$\{O, \mathcal{L}_\mu\} = 0 , \quad (1001)$$

$$\{O, \mathcal{E}\} = 0 . \quad (1002)$$

Justification of the name ‘constants of the motion’ now follows from the Hamiltonian taking the form $H[\dot{I}, B^\mu] = \dot{I} \mathcal{E} + B^\mu \mathcal{L}_\mu$ so that (1001,1002) imply that

$$\frac{dO}{dt}[\rho(t), \pi(t)] = 0 . \quad (1003)$$

Thus, observables are automatically constants of the motion with respect to the evolution associated with any choice of strut function \dot{I} and point-identifying map B^μ . The quantum counterpart of these then straightforwardly involves some operator form for the canonical variables and commutators in place of Poisson brackets. [The partial observables concept has the virtue of carrying straight over to the quantum level, which is strong from perspective 1.iii)] The operator-and-commutator counterparts of the above are then another manifestation of the Frozen Formalism Problem in the RPM context [Analogy 102)]. For the indirect formulation of pure-shape RPM, all is as above except that one has additionally

$$\{O, \mathcal{D}\} = 0 , \quad (1004)$$

and the form of the Hamiltonian is $H[\dot{I}, B^\mu, C] = \dot{I} \mathcal{E} + B^\mu \mathcal{L}_\mu + C \mathcal{D}$ so that (1001,1002) and the point-identifying map now involves both B^μ and C . For the r-versions, observables are functions $O(S, P_S)$ in the pure-shape case and $O(\sigma, S, P_\sigma, P_S)$ in

the scaled case. Now only the r-versions of (1002) apply, the Hamiltonian is $H = \dot{\mathbf{I}}\mathcal{E}$ and there is no point-identifying map required.

The corresponding Kuchař observables are those quantities whose brackets with the linear constraints vanish. It is then clear from this condition being empty for the r-formulation that the set of Kuchař observables for pure-shape RPM is precisely the set of all functions of the shape variables and the shape momenta. Likewise, the set of Kuchař observables for pure-shape RPM is precisely the set of all functions of the scale and shape variables and the scale and shape momenta. It then follows that these, as functions of ρ and π , are also the solution to the problem of what the Kuchař observables are in the indirectly-formulated case. **Thus Sec 3 and the parts of Sec 4 that find and interpret the r-formulation's momenta can also be viewed as an exposition of the solution to the Problem of Kuchař observables** for 1- and 2- d RPM's (which include by the identity of indiscernibles the '3- d ' 3-particle problem). In other words, if one can solve the Best Matching Problem (generalized Thin Sandwich Problem), then one can formally solve the problem of finding the Kuchař observables for the indirectly-formulated version of the theory in question.

Difference 38) For this article's RPM's, the problem of Kuchař observables is resolved, unlike in geometrodynamics.

Analogy 103) There is a loose analogy [1.iii)] between pure-shape RPM's shapes or scaled RPM's scale-and-shapes (classical resolutions of the linear constraints) and LQG's knot states [246] (quantum resolutions of the linear constraints). Loops themselves are analogous to preshapes [i.e. the entities prior to the main part of the reduction – $\text{Rot}(d)$ for shapes or $\text{Diff}(\Sigma)$ for knots]. It is not hard to pass to a quantum resolution of the RPM linear constraints to render this analogy tighter: the mid-Hilbert space of wavefunctions that depend solely on the pure shapes or on the scale-and-shapes. One might then view the moves in passing to knot equivalence classes of loops as analogous to the transformations used in Secs 2 and 3 to unveil the relationalspace variables.

Difference 39) However Sec 2 and 3's RPM transformations are purely configuration space manoeuvres, whereas passing to Ashtekar variables is itself a canonical transformation. Thus RPM's fit better the philosophy of the central importance of the configuration space (as opposed to phase space or of any 'polarizations' of it that aren't physically configurations).

Attitude 1) If one adheres to Kuchař observables being the entirety of the legitimate story about observables, then by the above, one has overcome the Problem of Observables for this article's RPM's by 'disarmament treaty'; one then needs to face that one has less strategies available with which to try to overcome the flightless Ice Dragon.

Attitude 2) If one instead adheres to needing the more restrictive subset of complete observables, then one is to ask which functions of shape and shape momentum commute with the SRPM quadratic energy constraint, and which functions of scale, shape, scale momentum and shape momentum commute with the ERPM quadratic energy constraint.

A simple partial answer is that in a few cases these include (subsets) of the isometries, i.e. relative angular momenta, relative dilational momenta and linear combinations of these with certain shape-valued coefficients. E.g. this is a direct analogue of the angular momenta forming a complete set of commuting operators with the Hamiltonian operator, provided that the potential is central i.e. itself respecting the isometries of the sphere by being purely radial. Thus for the tower of very special problems for N -stop metrolands and triangle-land, and its more elaborate counterpart for quadrilateralland in [53], we have found some complete observables. These are not, however, generically present i.e. for arbitrary-potential models.

Another answer, at least in some simple models is that Halliwell's class operators comply with Quad.

For at least some simple \mathfrak{g} -trivial theories, Halliwell's class operators are complete observables, and they or their phase space generalization may provide the complete set of such.

Halliwell's class operators have not however yet been established to commute with \mathcal{L}_{inz} in \mathfrak{g} -nontrivial theories. Thus overall their current status as regards complete observables is that they are the *complement* of Kuchař observables. On this count, I name them **Halliwell proto-observables** in the \mathfrak{g} -nontrivial theory context; in the \mathfrak{g} -trivial context, they are **Halliwell observables**, as they would be also in the \mathfrak{g} -nontrivial theory context if they are additionally Kuchař.

Question. It would thus be interesting to see whether Halliwell's class operator construction technique can be applied to RPM's for which the Kuchař observables problem is resolved, so as to produce the set of complete observables. I.e. whether one can find the corresponding toy flighted Unicorn to counteract the RPM toy Ice Dragon of Fig 56. This is likely to work [51]. This is one particular further reason why the work described in the last 2 SSecs of Sec 15 is of value. This would involve applying Halliwell's class operator construction procedure using as building block only shapes and shape momenta for SRPM or only scale-and-shape for ERPM, from which sole use of building blocks the object constructed would lie within the set of Kuchař observables.

Question*/Analogy 104) More ambitiously [1.iii)], might one be able to extend Halliwell's class operator construction to LQG, so as to be able to, at least formally, write down a (perhaps partial) set of complete observables as a subset of the linear constraint complying knots? Either construct such a set or explain at what stage of relevant generality Halliwell's class

operator construction fails. [I.e. a tentative search for a fully-fledged Unicorn.] Note that this goes beyond the currently-known geometrodynamical minisuperspace scope of Halliwell’s class operators, but is an interesting direction in which to try to extend this scope.

Attitude 3) Note that partial observables are far more easy to come by. One can look into e.g. base, median and relative angle from the perspective of partial observables.

Analogy 105) RPM’s and GR can both be considered in terms of partial observables.

I comment that on the one hand Partial Observables as presented by Rovelli carry connotations of “*tot tempora quot motus*”, but, on the other hand, Rovelli showed us a possible means of demonstrating that the Ice Dragon has always been a flightless beast and thus less hard to vanquish.

Question. Relative information was tied to the partial observables approach in [540]. Can one more fully exploit this as regards being a specific object to compute/measure in the study of subsystem correlations?

Note 1) I recommend that the lattice of propositions structure also be incorporated into Type 2 ‘Rovelli’ timeless approaches. Indeed the partial observables notion would seem to fit in well with this as notions of correlation/mutual information corresponding to the physical questions that one can ask.

Note 2) Rovelli states that his approach is *required* to be in the Heisenberg picture [543]; it is a resolution of the Problem of Observables. Is this absolutely obligatory? If so, what does this imply anything about changes in the foundations/interpretation of QM? (Ordinary QM has equivalence between the Heisenberg and Schrödinger pictures; how is that lost in the passage to QG?)

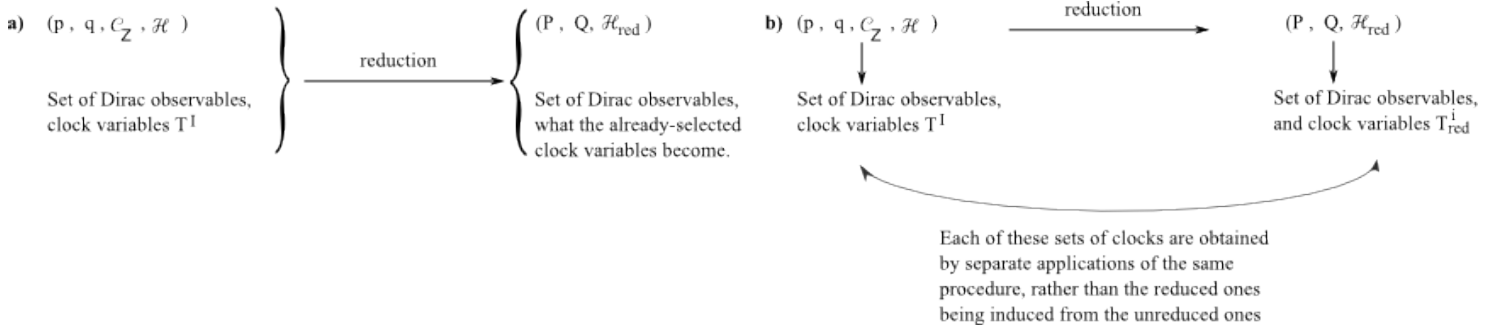


Figure 57: a) Thiemann’s notion of reduction versus b) the simpler and at least formerly standard notion, which logically precedes and mention of considering what to use for clock variables.

Thiemann’s pro-Dirac argument [605] (which is then used as a basis for arguments in e.g. [599]). It is in opposition to Kuchař’s pro-reduced argument of Sec 6. This is based on taking reduction along the lines of Fig 57 a), in which case Thiemann argues that extra freedom in clock choices in the Dirac picture has its fluctuations suppressed in the reduced case, rendering the reduced case less physical.

My counterargument. Reduction can simply be taken to be linear constraint elimination prior to any consideration of what the Dirac observables are or of what Rovelli–Thiemann style candidate clock variables are selected. Then, both the Dirac scheme, and the reduced scheme once already reduced (or indeed, taken de novo as a first-principles relationalspace scheme, which possible attitude strengthens the position of such not having to have its clock variables induced from an unreduced theory’s), have parallel ontological statuses as regards the selection of Rovelli–Thiemann style candidate clock variables for each. I.e. the scheme of Fig 57 b). I agree that the Dirac scheme having more variables means that it has a larger variety of such clock variables. However, the extra degrees of freedom in the Dirac approach’s variables prior to quantization relative to those in the reduced approach clearly have no physical content. Thus this extra variety in choice of clock variables should not be considered as an extra source of *physical* fluctuations. This is clear enough by applying Kuchař’s argument to go in the opposite direction. If one has the reduced system and adjoins auxilliary variables to it, it is not desirable for this unphysical adjunction to influence the physics, and this unphysical adjunction should not increase the variety of physical choices of clock variables in a way that is then to have physical consequences. Thus, at least by choice of ‘reduce and only then seek for clock variables and only then quantize’, Thiemann’s argument is bypassed. Kuchař’s argument then continues to apply as a good physical reason why to favour ‘reduce and only then seek for clock variables and only then quantize’ (i.e. what at least used to usually be called reduced quantization) over ‘seek for clock variables, quantize and then reduce’ (i.e. Dirac quantization). Thiemann’s argument, rather, favours Dirac quantization over a *third* approach: ‘seek for clock variables, then reduce, the new clock variables are then induced from the unreduced ones and only then quantize’, which he terms reduction but which does not in fact coincide with the simplest (and at least formerly standard) notion of reduction.

16.12 Comparisons of strategies

16.12.1 Internal time versus semiclassical time

1) Monotonicity and non-frozenness considerations indicate that emergent time can be more widely applicable than hidden dilational Euler time. Emergent time also exists for scale-invariant models [84], characterized by the constraint $E = 0$ which implies that the Euler quantity is frozen and thus unavailable as a timefunction. Although some portions of Newtonian Mechanics have guaranteed global monotonicity for t^{Euler} , solutions outside this portion may still possess intervals on which t^{Euler} is monotonic.

2) The internal time examples given above work as well for non-interacting hl-systems. Though this is not now a conceptual necessity, as h does not now impose a timefunction on l , but rather both contribute to a joint timefunction. Everything in the universe contributes in this species-by-species way. However, note the lack of role within for the potential, which makes it look more artificial or imposed, as the details of species from the potential plays no (direct) role in the construction of the timefunction.

3) Internal time is not universal, though scale and dilational time are universal within theories possessing scale. One insight afforded is that a theory possessing scale (and in GR the associated indefiniteness) is not necessarily conceptually poorly. In the internal time approach, the QM interpretation is formally standard and, unlike the Semiclassical Approach, there are no towers of approximations. But there are well definedness and ordering issues. And interpretational ones and as regards lack of a dissipatory interpretation. [Time-dependent potentials are hard to interpret here; such are usually held to correspond to non-conservative physics, but, unlike in that setting, the model is a whole universe so there is nothing for energy to dissipate into.]

4) Both give time-dependent Schrödinger equations but only the semiclassical one is interpretable in terms of dissipation into the h -system (need higher-order correction form in order to make that possible). This renders the status of $t^{\text{ante-dependent}} H_{\text{true}}$ mysterious, leaving open the possibility that hidden time is an ill-conceived scheme even at the conceptual level [against 1.0)]. Patching is tied to the Partial Observables approach ideas [Contextualism 3) and 4)] in Bojowald et al's work [131, 132] (see also the earlier [109]). Patching has a straightforward manifold coordinate patch type of interpretation at the classical level, but faces problems at quantum level as regards maintaining unitarity of evolution across patches, and as regards how to define observables. Some of these issues have semiclassical regime counterparts in the standard patching together of WKB regions using connection formulae. However the two situations as they currently exist in the literature are chrono-geometrically distinct. [This is partly because the scope of Bojowald et al's examples is for now minisuperspace, with patching in time rather than in space, so some parts of looking to tie together these two approaches will have to await the treatment of inhomogeneous-type models. On the other hand, the issue of how to define observables at the quantum level for models requiring patches could well still have some useful counterpart in the semiclassical WKB patching approach.]

16.12.2 Timelessness: Type 1 versus Type 2

1) Parallels between 'Rovelli'-type approaches and Page-Wootters' original Conditional Probabilities Interpretation [502] are as follows. Both involve notions of observables, and of subsequent pairings of such, with the one then serving as a clock for the other. However, Rovelli's scheme is less specific in not postulating a rigid formula like the Conditional Probabilities Interpretation one for the extraction of the information about the correlation between the two observables. Also, Rovelli insisted that only the Heisenberg picture remains meaningful for Quantum Gravity, by which Type 1 Approaches such as the Conditional Probabilities Interpretation/Records Theory could not be just the same physics as Type 2 'Rovelli' but in a different picture. [Though it is not clear, at least to me, whether this insistence is absolutely necessary; attempting to lift it might reveal further approaches to the Problem of Time, or further connections between existing approaches, such as between Type 2 'Rovelli' and Type 1 Timeless Approaches.] Also, Type 1 involves nonstandard interpretation of QM; is this also so of Type 2 'Rovelli'? [Does the Heisenberg picture alone remaining meaningful itself entail some interpretational changes? Thiemann's combination of Type 2 'Rovelli' with Histories Theory certainly involves bringing in nonstandardness of interpretation of Quantum Theory.] Finally, cautions about requiring the observables cast in the role of clocks/timefunctions to actually be (often hard to find) *good* clocks/timefunctions should then apply to both Type 1 and Type 2 schemes. See also the below comments as regards comparing and composing the Gambini-Porto-Pullin approach with Rovelli's.

2) Type 2 timeless approaches lies on the AMR notion of time, Barbour's Type 1 timeless approach on the LMB notion of time, other Type-1 timeless approaches like the CPI and Page's on the AMR notion of time also, and a number of schemes (including Halliwell's and A's schemes) lie best on the LMB-A notion of time, which was argued to be the most conceptually and practically satisfactory time out of the three.

3) The $t^{\text{em(JBB)}}$ choice of time is distinguished by its substantially simplifying both the momentum-velocity relations and the evolution equations. It amounts to an emergent recovery of other well-known notions of time: Newtonian, proper, cosmic and emergent WKB, in various contexts. However, emergent JBB time does no better as regards suggestion of by what prescriptions one might find an 'ephemeris-type' time (by which the advantage in regard 1.ii) of emergent JBB time over the Partial Observables approach is only modest.

4) Criterion 1.ii) applies to both Rovelli's approach and Barbour's approach: each allows for mechanics and GR to be formulated similarly, each in a distinct sense. Though then for Type 1, the situation is that it is mechanics and so the Newtonian time/emergent JBB time/ephemeris time will very often turn out to be useful, and this is an extra element that

the Partial Observables approach in no inherent way points to, so that 1.i) does not apply in this context. A useful clarification as regards using Celestial Mechanics examples for illustrative purposes as in [539], is that the mathematically-simple choice of time variable used there is specifically to illustrate partial observables in a context that is mathematically-familiar even to the newest students of physics, but *not* however how Celestial Mechanics (or time-keeping as a maximally-accurate pursuit) is actually done. More accurate timekeeping involves, rather, a generalized local ephemeris timekeeping/recalibrating set-up.

5) Thus relational formulations of this Barbour type *do* satisfy some of 1.ii) as regards providing further insights bearing some relation to ephemeris time in mechanics. They also satisfy 1.i): they smoothly carry over to GR (from the Jacobi-type action to the BFO-A and ABFKO actions as per Sec 2, from each of which an emergent JBB time notion of time follows). Additionally, they satisfy 1.iii), by suggesting further alternative theories [for all that some such like the theory in [48] are then questionable on logically-independent grounds]. Barbour's has more awareness of timekeeping, though LMB-A is required for a closer procedural match-up.

6) RPM's are not covered in Rovelli's book [545], in part because he does not there consider where in his 'order of things'/'space and categorification of theories' these toy models are to sit. The present article has amply demonstrated that, whilst RPM's are not special-relativistic, RPM models share a large number of other features with GR, thus laying out a particular arena the study of which helps patch up these gaps.

7) As regards 1.iii), one advantage of the Partial Observables approach is that this notion looks to remain useful at the quantum level, whereas emergent JBB time does not afford any particular methodology at the quantum level. (That is independent of whether quantum RPM's leads to useful insights. The point is that emergent JBB time is not used in getting at any of those insights, one can use most of the other approaches to quantization/the Problem of Time however. That includes partial observables as one possible method, alongside internal time, semiclassical time, Histories Theory, Records Theory, the Conditional Probabilities Interpretation and the Naïve Schrödinger Interpretation).

I next classify timeless approaches in a number of ways, suggesting a number of extra approaches in the process.

Classification 1) Freestyle (Type 1) versus insisting on Heisenberg picture and observables (Type 2 'Rovelli': column 1 in Fig 58).

Classification 2) By type of Machianity (rows in Fig 58). Thus we see that Rovelli's approach, the CPI, Page's approach and GPP's approach have a Rovellian attitude to time, whilst Barbour Records has LMB time and A-Records has LMB-A time; Halliwell records have LMB-A or LMB time natural to them.

Classification 3) By question type: 1-property (NSI) versus multi-property questions (CPI, Records, Rovelli partial observables).

Classification 4) by interpretation of QM, due to whole-universe and quantum-gravitational nonstandardnesses.

By 3) and 4) e.g. CPI and NSI are different.

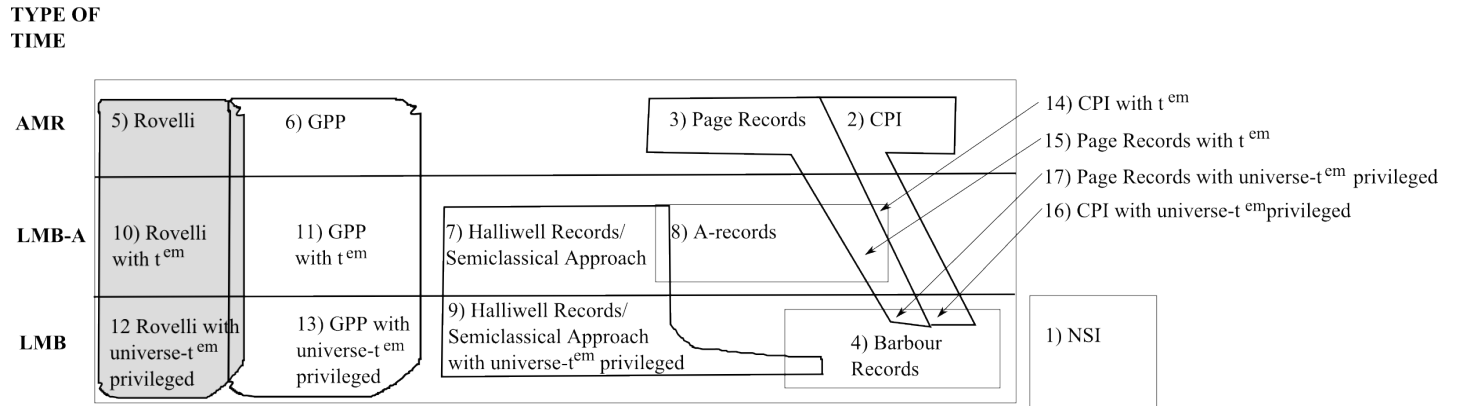


Figure 58: Classification of timeless strategies, including some new suggestions opened up by this point of view. The shaded region consists of observables-based approaches.

Note 1) A-records are localized, and with localized ephemeris-type timestandards

Note 2) 4, 7, 8, 10-17 are all approximate semiclassical approach or less approximate semiclassical approach compatible. The times involved are of course purely combos of position. The point of involving them is preferentially asking questions in terms of these. Moreover, I note that these are all compatible with the semiclassical approach. 10 to 17 represent hybrid approaches: structurally and interpretationally much else like the Rovelli, GPP, Page and CP approaches but, as regards time notion, using the LMB-A approach as argued for in the present article. E.g. one can consider partial observables where, specifically, the generalized local ephemeris is one of the quantities involved. Thus the present article unveils a further 8 variants of timeless approach/sharpens which questions to preferentially ask in 4 sorts of timeless approach.

16.12.3 Time functions versus timelessness

In practise, this is a case of choosing to face either Saint Augustin's "what is time" or the more recent alternative of "how to cope in physics without there being a time". Canonical quantum gravity hints at the latter, though there are of course

ways around it. I note that the latter often leads one to a nonstandard interpretation of QM, whereas the former allows for far more conventional interpretations of QM, at least in principle. This is because by possessing a time, which is then to be treated in a sui generis fashion, the former complies with the structural expectations of ordinary QM.

As regards matters of principle, this is an issue of economy (why use a time if it is not necessary) and of primality: the Machian ‘time is an abstraction from change’ versus the absolutist ‘time is a prerequisite for understanding change’. I argue that the former is more palatable, particularly as one generalizes one’s theory and passes to whole-universe models (for which, incidentally, the standard interpretation of quantum theory is moot on grounds other than the frozen formalism of canonical quantum gravity).

As regards more recent combined strategies involving both a timeless approach and a time/histories approach, in these for some purposes time is unnecessary, whilst the need to establish a semblance of dynamics from timeless premises is much less of a pressing issue in such schemes. This aspect of Halliwell’s program is likely to be the focus of the present Sec as regards time versus timeless, and is best left to combinations SSec rather than comparisons SSec. In any case that has its own SSecs (the last 2 of Sec 15), so I leave the reader to consult that.

16.12.4 Histories theory versus all the other approaches

Histories theory gives some of the internal time approach’s machinery a second chance by providing a histories canonical group to work with. Histories theory has a pre-existing framework by which an explanation of the semblance of dynamics is no longer necessary (one may however question the meaning of that framework, e.g. does it really make sense to base a canonical formulation on histories rather than, as was hitherto always done in physics, on configurations). Histories theory shares with Type-1 timeless approaches the need to interpret QM (or a generalization thereof) in a novel way (which also ties in with closed system/whole universe issues). Histories theory is then a common element in combinations that rid some of the other strategies of some of their problems (see below).

End-note: **insisting on q primality blocks hidden time and much of Histories Theory.**

16.13 Composing Problem of Time strategies has started to appear in the literature

Beware the conceptual danger in just pick and mixing useful features. On the other hand, a number of strategies are strongly related. These can be conceptually hard, and may in some cases be exposeable via conceptual meaninglessness. My first two examples are simple and yet as far as I know new to this article.

1) Concede making an approximation prior to quantizing tempus ante quantum’s true Hamiltonian, thus mixing that approach with a Semiclassical Approach. This does not necessitate a semiclassical time finding move; it gets round the impasse of exact solvability of the internal time approach.

2) The reverse combination of doing the Semiclassical Approach on top of the canonically-transformed Euler time-shape variables (i.e. without isolating a H_{true} , going for an H_1 instead).

As a more general principle, semiclassicality is composeable with whichever other strategies. E.g. with Histories Theory [294, 295], looking to provide the WKB ansatz via histories decohereing. Though the emergent semiclassical time move itself is only sometimes used in such combinations...

3) t^{em} is both the approximate JBB time and the WKB time; moreover it is useable in Histories Theory and in timeless approaches.

Semiclassical and records approaches are very natural successors of LMB-A relationalism These then support 4) and 5) below.

4) Halliwell’s [292, 294, 297, 298, 299, 295] is then a joint consideration of Histories Theory, the Semiclassical Approach and Records theory. This is an approach to which I presently also subscribe [28], the last 2 SSecs of Sec 15, forthcoming), as the main current thrust of RPM as toy model of Problem of Time problem. The present SSec then serves to compare that Sec with its current competitors for composite strategy, as well as to suggest further as-yet uninvestigated composite strategies. This strategy itself motivated as per Sec 11.10: in this combination the three strategies involved combine in a mutually supportive way, removing many but as far as I know for now not all of the individual faults of the strategies. RPM’s are used in further investigating this strategy in the last 2 SSecs of Sec 15. I consider these 2 SSecs to double as conclusion material as one of the more promising compositions at present.

5) Gambini–Porto–Pullin’s approach is an alternative to Halliwell that does not use (so many) semiclassical moves to get at its emergent time, but is still a Type 1 Timeless – Histories Theory combination. This is another way of getting to a semblance of dynamics.

6) Kouletsis’ [388] tie between Histories Theory and the Internal Time Approach. This approach has for now the setback of mostly only having been investigated at the classical level so far. I sincerely encourage further work at the quantum level for this strategy!

7) There are a number of ties between Rovelli-type Timeless Approaches and the Internal Time Approach.

7i) Internal time can provide the time/clock in schemes requiring a such if one is able or willing to pay for the usual inconveniences of a such (e.g. Dittrich [211]). Note that here these do constitute at least an emergent Schrödinger picture. How does this square with Rovelli’s insistence on solely the Heisenberg picture? Lawrie [419] has argued against combined approaches of this type (with matter or scale time, in the minisuperspace context). He gave the argument that volume at a given value of the scalar field does not make sense in the 2 naïve degrees of freedom model as regards meaningful (relational) physical content, with this invalidating the usual interpretation of matter time as a clock. [However, I note that one can get around this by considering models with further degrees of freedom.] He encounters ses of 2 different times give different results, and concludes neither can the be correct. However, that could well be just an ordinary manifestation of the Multiple Choice Problem, which could allow for one or both to be correct (in the case of both, with each referring to a different context). Another issue if two candidate times produce distinct results is whether one candidate is a better time than the other; one would then prefer to believe the *better* candidate (and which is better may depend on epoch, as per the next entry).

7ii) Bojowald et al. [131, 132] trade nonunitary issues in patching for a complex notion of time, and talk about fashionables in place of observables. Making time complex as in this program may interfere with this notion of time having the standard property list to be a time (or at least require extra work to demonstrate that such properties can be established in this less standard context).

Their ‘fashionables’ (only locally valid observables) represent development of Contextualism 4). Although they only consider minisuperspace for now, that ‘local’ is in principle in both space and time, making *fashionable* a good word for it (‘fashionable in the 1960’s’ and ‘fashionable in Italy’ both make good sense) Bojowald et al also address some (but not all) causes of the Multiple Choice Problem. All in all, it is an attack directed at the Ice Dragon’s tail and wings, with ties to Rovelli’s strategy to getting round the freezing breath. I view it as promising albeit not complete.

8) The above triangle of ties can be completed by combining Type 2 ‘Rovelli’ Timeless Approaches and Histories Theory as done by Thiemann [607], see [39] for some discussion of this. The main purpose of this is a more complete, if nonstandard, interpretation of the QM.

RPM modelling should be OK for 6), 7), but the main points of 5) are too specifically tied to properties of the diffeomorphisms, by which other toy models such as the bosonic string (used) or $2 + 1$ gravity (not used) would be more appropriate.

Questions: What of the Type 1 timeless counterpart of this triangle? And of the Type 2 timeless counterpart of Halliwell’s program? [there have been a few comments on crossover between Halliwell and type 2.] Or render the preceding two questions moot by establishing sufficient Type 1 – Type 2 equivalence?

9) Finally, Gambini–Porto–Pullin–Tortorolo [245] compose 4) with observables issues along the lines of Type 2 timeless approaches. They use an observables approach but differ from Rovelli in using a Conditional Probabilities Interpretation (or part-implicitly-histories Conditional Probabilities Interpretation). Kuchař argued that the physically-established propagator does not arise in this scheme; on the other hand, Page argued that such a propagator is not in fact meaningful in his scheme, which is based entirely on correlations within a single instant. [245]’s implementation has one, though this is perhaps not so surprising given that their approach partly implicitly rests upon Histories Theory. As this lies within the scope of what can be done for RPM’s, it is Analogy 106).

16.14 See also the separate Quantum RPM’s and further quantum-cosmological applications Conclusion

Peruse the separate conclusion of Part II if interested in these issues, including whether RPM’s give standard QM mathematics, and quantum-cosmologically relevant operator ordering, robustness to neglecting degrees of freedom, role of uniform states, structure formation. Part III’s semicl investigation doubles as a start on investigation of whether the approximations behind the Halliwell–Hawking scheme for structure formation is to be qualitatively trusted. For now, the RPM is argued to be a suitable theoretical lab for this, but many of the possible effects of not neglecting various terms remain uninvestigated.

16.15 Résumé of value of this article to Ashtekar variables/LQG

I argued in this article that the Ashtekar Variables formulation of GR is relational in Barbour’s sense too (at the usual metric level; whether background-independence and relationalism should be taken to apply at the topological level as well remains

a question largely for the future which may well take the line of thinking behind LQG beyond LQG into some new arena).

I remarked on how geometrodynamics has a solidity resting on primality of spatial geometry which already applies to mechanics, whilst Ashtekar variables has a distinct one resting on electromagnetism and Yang–Mills theory. Moreover deriving Ashtekar variables rests on extending the set of configurations and upon performing a canonical transformation. I provided moderate arguments for alternatives to each of these practises.

One should also take stock that a lot of uses of the word ‘relational’ in LQG are what this article terms ‘contextual’, which additionally explains a number of other senses in which the word ‘relational’ has been used over the years and to some extent fits various of the uses together. Thus e.g. GR in general and the Ashtekar variables formulation do have manifest many other senses of ‘relational’ outside of the more usually-ascribed contextual ones. I note that the present LQG literature does not systematically apply LMB(-A) relationalism (though there is some evidence for individual pieces of such thinking in various formalisms and techniques used). Doing so could give the subject more conceptual awareness. More generally, the extent and manner of the combineability of the Rovelli, Crane and LMB(-A) types of relationalism as expounded in Sec 16.3 is an interesting one and LQG is a reasonable arena in which to study this question. The nature of the Problem of Time facets and strategies covered in this article are generally relevant here too.

SSec 16.11 has already covered a number of further points valuable to this program. Additionally, Mackey’s Principle would seem to go well toward providing a further layer of structure to Rovelli’s timeless approach. Whether to take Rovelli’s timeless approach to Problem of Time in the setting of Ashtekar variables/LQG, or something else (or a mixture) is another topic of clear interest for LQG that is covered in this article.

I also comment that one source of good names for physical programs that purport to be relational and thus concern tangible entities, is to name the theories after the tangible entities rather than after convenient but meaningless parts of the formalism that happen to be in use in the mathematical study of that theory. By this criterion, ‘shape theory’/‘shape dynamics’ and ‘geometrodynamics’ are very good relational names, whereas ‘Loop Quantum Gravity’ leaves quite a lot to be desired. For, in this program the $SU(2)$ gauge group is put in by hand and it is only the subsequent removal of its significance that brings loops into the program’s formalism, but the lion’s share of the work in *any* canonical quantum GR program resides, rather, in removing $\text{Diff}(\Sigma)$ and in interpreting the Hamiltonian constraint. Even without the ‘put in by hand’ issue, the parallel naming for RPM’s would be ‘sphere theory’ or ‘sphere dynamics’ from this being the form the preshape space takes before the lion’s share of the work is done in removing $\text{Rot}(d)$ (and, in the dynamics case, interpreting the energy-type constraint), and this is rather clearly *not* enlightening naming because the preshape space spheres do *not* play a deep conceptual role in shape theory. As such, I would suggest names along the lines of ‘Knot Quantum Gestalt’ or ‘quantum nododynamics’. This is also particularly appropriate via the Gestalt conceptualization of GR being, at least at the level of metric structure, very much emphasized in the ‘LQG’ program itself, just not in the LQG program’s *name*. Thiemann’s alternative name ‘Quantum Spin Dynamics (QSD)’ is somewhat confuseable with the Ising model and its ilk, and also the spin connotations come from the $SU(2)$ gauge groups. On the other hand, letting the S stand for ‘spin-net’ disambiguates the nature of the theory, though a spin-net still does have but a similar ontological status to a loop. Taking relational primality further, then, the final suggestion is that the S should stand for S-knot (spin-knot).

Both by its relational meaning and by the present article’s arguments for reduced/relationspace formulations over indirect/Dirac ones, a case can be made that LQG should be placing the most value on the insights gained from the former.

16.16 This article and M-theory

Background-independence brings a Problem of Time, so it is exceedingly likely that there will be one of those here too. Geometrodynamics (very easily generalizable to arbitrary- d) could be seen as a first toy model of spatially 10- d M-theory. There may be better RPM’s and geometrodynamics models for this particular setting, e.g. spatially 10- d supergravity (which is a low-energy/‘semiclassical’ limit of M-theory). but many features of the Problem of Time are universal, so these will occur here too unless features such as supersymmetry and/or extended objects specifically conspire to eliminate facets. They were not however designed to do so, so this is a more stringent theoretical test for these notions/structures. Good physical theories work no matter how one approaches them (c.f. many routes to GR), rather than just have the aspects and properties they were constructed for. It would then be a pleasant surprise if, in addition to all the different usefulnesses of these structures, they also happened to cure the Problem of Time. But it is possible that they will not, or that the Problem of Time may require further or somewhat different structures. [39] listed a few approaches to time in M-theory, though such works are as yet few and far between.

Passing from studying particle mechanics to RPM’s is, in a sense, conceptually similar to passing from studying point particles to strings; however in the RPM case, one proceeds here by making less rather than more hypotheses. Technically they’re mostly very different, though, as this article has pointed out, orbifolds and weighted projective spaces occur in both. RPM’s produce a number of instances of so far unhandleable mathematics. Discovering new handleable mathematics is nice, albeit not directly essential to Theoretical Physics. But there is also value to trying to unlock unhandleable maths arising from well-motivated problems. I myself go for the case are with the most handleable maths case within RPM, so as to start to address old but not as yet fully understood foundational questions that can be asked of (almost) any theory. This is the right choice for me due to these questions being well-motivated.

In this article, considering string theory led to a refinement of how Barbour-type-relationalism should be defined. Note also how dimension ceases to be a primary input in RPM's, for all that this only so for particle number. Finally, supersymmetry [45] and the situation with extended objects that occupy/furnish a multiplicity of notions of space are interesting frontiers in which to further explore the relational postulates.

16.17 Epilogue: Problem of Time as only a part of the GR-QM inconsistency

Closed universe issues, as in Secs 6.13.1 and 7, can be viewed from this perspective. These include the idea of the notion of beables replacing that of observables; one should bear in mind the interpretational subtlety that Bojowald et al's 'fashionables' could also be taken to be local beables rather than observables. In fact, the two situations would be different in the detail of the interpretation of QM entailed, so I coin the distinct name 'degradeables' for localized beables. Much as 'fashionable' implies local appeal to an observer, *degradeable* implies mere temporary being rather than any sort of involvement of observers. And degradeables are, like I previously argued for fashionables, appropriately local in both time and space (now 'not past its sell-by date' in time and 'things keep for longer in the fridge' as regards spatial variations in degradeability).

Some observer issues have already been touched upon this article through study of contextualism. Some further issues are as follows (I do not claim completeness, and these may in some ways reinforce the form of contextualism).

Observers are modelled in GR as negligible energy-momentum entities, whilst, on the other hand, in QM, these are usually held to be much larger than the system in question. Thus the theory of the large and of the small is not just a straightforward comparison but also a comparison of each with the sizes and sensitivities of the observers. GR observers are moreover idealized as regards their internal constitution is not posited; this is also the case for any clocks involved (this is also true in the below-mentioned case of rods in GR). As such one should believe little in these idealizations. Is the QM way of handling these things extendible to subsystems within the GR setting?

More generally, the quantum theory of the gravity/background-independence gestalt may be a *contextual theory* with both observers and their times being required in the posing and answering of questions.

Question. Aside from time, clocks and observers, are there any other kinds of conceptual clashes that are not readily attributable to mathematical differences?

Some answers to this are space, length, rods and frames. In this respect, I note that Kochen's [384] or Bene's [115] brands of contextual QM involve quantum reference systems.

Space [355] is in a number of ways rather less controversial than time. It is time and not space or length that has a peculiar role in QM. But then GR looks to put time and space on an equal footing. Choices then are to challenge the peculiarities of time in QM or challenge the equal footing of GR. Tempus ante quantum and Semiclassical Approaches go for the latter, whilst abolishing time goes for the former.

There is a Newtonian Mechanics to SR gap in the theory of **rods** (rigid rods are meaningless in relativity) and the actual nature of rods is ignored in setting up GR. These are arguably a less deep notion/apparatus clocks e.g. via Bondi's argument [134] that they are made out of quantum matter which is ultimately underlaid by frequencies. Also, by their nature and function, rods are necessarily macroscopic [553] and so interact with their environment in uncontrollable ways, whilst microscopic clocks are indeed possible.

Finally, paralleling the question of what is a good clock, what is a good rod? It should be at rest in an unambiguous frame. At a fixed temperature, with high control of the temperature/a low thermal expansion. To measure elsewhere, we copy the master standard and take it with us (with less control). It is worth mentioning also that in the Gambini–Porto–Pullin scheme [244, 522], decoherence effects due to use of real clocks are accompanied likewise by decoherence effects due to use of real rods.

17 Appendix A: Some Classical Dynamics results

17.1 The general mechanics problem in arbitrary dimension in Cartesian coordinates

We assume a Jacobi-type action (though that is not necessary so as to get all of these results). The kinetic term is

$$T = \sum_{A=1}^k \dot{X}^A{}^2/2 . \quad (1005)$$

Then the conjugate momenta are

$$P_A = X_A^* . \quad (1006)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} := \sum_{A=1}^k P_A^2/2 + V = E , \quad (1007)$$

the middle expression of which also serves as the classical Hamiltonian H for the system. The evolution equations are

$$X_A^{**} = -\partial V/\partial X^A , \quad (1008)$$

one of which may be replaced by the Lagrangian form of (1007),

$$\sum_{A=1}^k X^{A*}{}^2/2 + V = E . \quad (1009)$$

17.2 The general 2- d mechanics problem in polar coordinates

The kinetic term is

$$T = \{\dot{R}^2 + R^2\dot{\chi}^2\}/2 . \quad (1010)$$

Then the conjugate momenta are

$$P_R = R^* , \quad P_\chi = R^2\chi^* = \mathcal{S} , \quad \text{constant if } V \text{ is independent of } \chi . \quad (1011)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} := \{P_R^2 + P_\chi^2/R^2\}/2 + V = E , \quad (1012)$$

the middle expression of which also serves as the classical Hamiltonian H for the system. The evolution equations are

$$R^{**} - R\chi^{*}{}^2 = -\partial V/\partial R , \quad \{R^2\chi^*\}^* = -\partial V/\partial \chi , \quad (1013)$$

one of which may be replaced by the Lagrangian form of (1012),

$$\{R^{*}{}^2 + R^2\chi^{*}{}^2\}/2 + V = E . \quad (1014)$$

If \mathcal{S} is a conserved quantity, one can rewrite the last equation as

$$\{R^{*}{}^2 + \mathcal{S}^2/R^2\}/2 + V = E , \quad (1015)$$

which can be integrated to give the quadrature

$$t^{\text{em}} - t^{\text{em}}(0) = \int dR/\sqrt{2\{E - V(R)\} - \mathcal{S}^2/R^2} . \quad (1016)$$

One can also readily obtain a quadrature for the shape of the orbit (for the nontrivial case of $\mathcal{S} \neq 0$):

$$\phi - \phi(0) = \mathcal{S} \int dR/R^2 \sqrt{2\{E - V(R)\} - \mathcal{S}^2/R^2} . \quad (1017)$$

17.3 The general 3- d mechanics problem in spherical polar coordinates

The kinetic term is

$$T = \{\dot{R}^2 + R^2\{\dot{\alpha}^2 + \sin^2\alpha\dot{\chi}^2\}\}/2 . \quad (1018)$$

Then the conjugate momenta are

$$P_R = R^* , \quad P_\alpha = R^2\alpha^* , \quad P_\chi = R^2\sin^2\alpha\chi^* = \mathcal{S}_3 , \quad \text{constant, if } V \text{ is independent of } \chi . \quad (1019)$$

These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} := \{P_R^2 + \{P_\alpha^2 + P_\chi^2/\sin^2\alpha\}/R^2\}/2 + V = E , \quad (1020)$$

the middle expression of which also serves as the classical Hamiltonian H for the system. The evolution equations are

$$**R - R\{\{*\alpha\}^2 + \sin^2\alpha\{*\chi\}^2\} = -\partial V/\partial R , \quad (1021)$$

$$*\{R^2\sin^2\alpha*\chi\} = -\partial V/\partial\chi , \quad (1022)$$

$$*\{R^2*\alpha\} - R^2\sin\alpha\cos\alpha\{*\chi\}^2 = -\partial V/\partial\alpha , \quad (1023)$$

one of which may be replaced by the Lagrangian form of (1020),

$$\{\{*R\}^2 + R^2\chi^{*2} + \sin^2\alpha\alpha^{*2}\}/2 + V = E . \quad (1024)$$

If all three \mathcal{S}_Γ are conserved quantities, one can rewrite this last equation as

$$\{R^{*2} + \mathcal{T}_{\text{ot}}/R^2\}^2/2 + V = E , \quad (1025)$$

which can be integrated to give the quadrature

$$t^{\text{em}} - t^{\text{em}}(0) = \int dR/\sqrt{2\{E - V(R)\} - \mathcal{T}_{\text{ot}}/R^2} . \quad (1026)$$

Finding the form of the orbits is somewhat less straightforward in this case.

17.4 The general \mathbb{S}^2 mechanics in spherical coordinates

The kinetic term is

$$\mathbb{T} = \{\dot{\alpha}^2 + \sin^2\alpha\dot{\chi}^2\}/2 . \quad (1027)$$

Then the conjugate momenta are

$$p_\alpha = \alpha^* \quad \text{and} \quad p_\chi = \sin^2\alpha\chi^* . \quad (1028)$$

The $p_\Gamma = u^\Gamma{}^*$ conjugate to the u^Γ are also used below. These momenta obey as a primary constraint the quadratic ‘energy equation’

$$\mathcal{E} := p_\alpha^2/2 + p_\chi^2/2\sin^2\alpha + V(\alpha, \chi) = E , \quad (1029)$$

the middle expression of which also serves as the classical Hamiltonian H for the system.

The equations of motion for the general sphere are

$$\alpha^{**} - \sin\alpha\cos\alpha\chi^{*2} = -\partial V/\partial\alpha , \quad \{\sin^2\alpha\chi^*\}^* = -\partial V/\partial\chi , \quad (1030)$$

one of which can be replaced by the Lagrangian form of (1029),

$$\alpha^{*2}/2 + \sin^2\alpha\chi^{*2}/2 + V = E . \quad (1031)$$

The general conserved quantity is (for V χ -independent)

$$p_\chi = \sin^2\alpha\chi^* = \mathcal{S}_3 \quad (1032)$$

If V is α -independent also, $\mathcal{T}_{\text{ot}} = \sum_{\Gamma=1}^3 \mathcal{S}_\Gamma^2$ is conserved as well. Combining (1031) and (1032) gives

$$\alpha^{*2}/2 + \mathcal{T}_{\text{ot}}/2\sin^2\alpha + V = E , \quad (1033)$$

and, so, integrating, the quadrature for the shapes of the orbits,

$$\chi - \chi_0 = \int \mathcal{S} d\alpha / \sin\alpha \sqrt{2\{E - V(\alpha)\}\sin^2\alpha - \mathcal{T}_{\text{ot}}} , \quad (1034)$$

while integrating (1031) itself gives the quadrature for the time-traversal,

$$t - t_0 = \int \sin\alpha d\alpha / \sqrt{2\{E - V(\alpha)\}\sin^2\alpha - \mathcal{T}_{\text{ot}}} . \quad (1035)$$

18 Appendix B: Some QM results

These are presented generally, though in each case a very common physical interpretation is attached. This article makes use of physical interpretations other than the usual ones; these are in the main part of the article.

18.1 Commutation relations/kinematical quantization for \mathbb{S}^2

We have three U^A such that $\sum_{A=1}^3 U^A{}^2 = 1$ forming a $\mathfrak{v} = \mathbb{R}^3$ and $\mathfrak{g}_{\text{can}} = \text{Isom}(\mathbb{S}^2) = SO(3)$ whose objects I denote by \mathcal{S}_C . Thus, overall $\mathfrak{g}_{\text{can}} \otimes \mathfrak{v} = SO(3) \otimes \mathbb{R}^3$. The commutation relations are

$$[\hat{\mathcal{S}}_A, \hat{\mathcal{S}}_B] = i\hbar\epsilon_{AB}{}^C \hat{\mathcal{S}}_C, \quad [\hat{U}^A, \hat{\mathcal{S}}_B] = i\hbar\epsilon^A{}_{BC} \hat{U}^C. \quad (1036)$$

18.2 Commutation relations/kinematical quantization for \mathbb{R}^3

We have three U^A forming $\mathfrak{v} = \mathbb{R}^3$ three P_A conjugate to these and three $SO(3)$ objects \mathcal{S}_C , these last two triples forming $\mathfrak{g}_{\text{can}} = \text{Isom}(\mathbb{R}^3) = \text{Eucl}(3)$. Thus overall $\mathfrak{g}_{\text{can}} \otimes \mathfrak{v} = \text{Eucl}(3) \otimes \mathbb{R}^3$, the Heisenberg group. The commutation relations are

$$[\hat{U}^A, \hat{P}_B] = i\hbar\delta^A{}_B, \quad (1037)$$

$$[\hat{\mathcal{S}}_A, \hat{\mathcal{S}}_B] = i\hbar\epsilon_{AB}{}^C \hat{\mathcal{S}}_C, \quad [\hat{U}^A, \hat{\mathcal{S}}_B] = i\hbar\epsilon^A{}_{BC} \hat{U}^C, \quad [\hat{P}_A, \hat{\mathcal{S}}_B] = i\hbar\epsilon^{AC} \hat{P}_C. \quad (1038)$$

18.3 Commutation relations/kinematical quantization for \mathbb{R}^2

We now have two U^A forming $\mathfrak{v} = \mathbb{R}^2$ two P_A conjugate to these and one $SO(2)$ object \mathcal{S} , these last two items forming $\mathfrak{g}_{\text{can}} = \text{Isom}(\mathbb{R}^2) = \text{Eucl}(2)$. Thus overall $\mathfrak{g}_{\text{can}} \otimes \mathfrak{v} = \text{Eucl}(2) \otimes \mathbb{R}^2$. The commutation relations are (1038) with $\mathcal{S}_3 = \mathcal{S}$, (1037) and

$$[\hat{U}^1, \hat{\mathcal{S}}] = -i\hbar\hat{U}^2, \quad [\hat{U}^2, \hat{\mathcal{S}}] = i\hbar\hat{U}^1, \quad [\hat{P}_1, \hat{\mathcal{S}}] = -i\hbar\hat{P}_2, \quad [\hat{P}_2, \hat{\mathcal{S}}] = i\hbar\hat{P}_1. \quad (1039)$$

18.4 Schrödinger equation and Laplacian for \mathbb{S}^2

$$\Delta_{\mathbb{S}^2} \Psi = 2\{V - E\} \Psi / \hbar^2 \quad (1040)$$

for

$$\Delta_{\mathbb{S}^2} = \sin^{-1} \alpha \partial_\alpha \{ \sin \alpha \partial_\alpha \} + \sin^{-2} \alpha \partial_{\chi^2}. \quad (1041)$$

18.5 Schrödinger equation and Laplacian for \mathbb{R}^3 in spherical polar coordinates

$$\Delta_{\mathbb{R}^3} \Psi = 2\{V - E\} \Psi / \hbar^2 \quad (1042)$$

for

$$\Delta_{\mathbb{R}^3} = R^{-2} \{ \partial_R \{ R^2 \partial_R \} + \Delta_{\mathbb{S}^2} \}. \quad (1043)$$

18.6 Schrödinger equation and Laplacian for \mathbb{R}^2 in plane polar coordinates

$$\Delta_{\mathbb{R}^2} \Psi = 2\{V - E\} \Psi / \hbar^2 \quad (1044)$$

for

$$\Delta_{\mathbb{R}^2} = R^{-1} \partial_R \{ R \partial_R \} + R^{-2} \partial_{\chi^2}. \quad (1045)$$

19 Appendix C: Linear Methods used

19.1 The Bessel equation

The Bessel equation of order p ,

$$v^2 w_{,vv} + v w_{,v} + \{v^2 - p^2\} w = 0 , \quad (1046)$$

is solved by the Bessel functions. I denote Bessel functions of the first kind by $J_p(v)$. The family of equations

$$x^2 y_{,xx} + \{1 - 2\alpha\} x y_{,x} + \{\alpha^2 + \beta^2 \{k^2 x^{2\beta} - p^2\}\} y = 0 \quad (1047)$$

map to the Bessel equation under the transformations $w = x^{-\alpha} y$ and $v = kx^\beta$. The subcase of this with $\alpha = 1/2$, $\beta = 1$ and $p = 1 + 1/2$ for $l \in \mathbb{N}$ are the well-known spherical Bessel functions [1].

19.2 Spherical harmonics and the (associated) Legendre equation

The (3- d) spherical harmonics equation arising as angular part of e.g. hydrogen-type problem and of 3- d isotropic HO. Its equation is

$$\sin^{-1} \theta \{\sin \theta \mathcal{Y}_{,\theta}\}_{,\theta} + \sin^{-2} \theta \mathcal{Y}_{,\phi\phi} + \lambda \mathcal{Y} = 0 \quad (1048)$$

It is straightforwardly separable by the ansatz

$$\mathcal{Y} = Y(\theta) P(\phi) \quad (1049)$$

into simple harmonic motion for ϕ and

$$\sin^{-1} \theta \{\sin \theta Y_{,\theta}\}_{,\theta} + \{J\{J+1\} - j^2 \sin^{-2} \theta\} Y = 0 , \quad (1050)$$

which is equivalent, under the transformation $X = \cos \theta$, to the associated Legendre equation

$$\{1 - X^2\} Y_{,XX} - 2XY_{,X} + \{J\{J+1\} - j^2 \{1 - X^2\}^{-1}\} Y = 0 . \quad (1051)$$

This is solved by the associated Legendre functions $P_j^{[j]}(X)$ (and unbounded second solutions), for $J \in \mathbb{N}_0$, $j \in \mathbb{Z}$, $|j| \leq J$. We use the standard convention that

$$P_J^j(X) = \{-1\}^j \{1 - X^2\}^{\frac{j}{2}} \frac{d^j}{dX^j} \left\{ \frac{1}{2^J J!} \frac{d^J}{dX^J} \{X^2 - 1\}^J \right\} , \quad (1052)$$

by which

$$\left\{ \sqrt{\frac{2J+1}{2}} \frac{\{J-|j|\}!}{\{J+|j|\}!} P_J^{[j]}(X) \right\} \quad (1053)$$

is a complete set of orthonormal functions for $X \in [-1, 1]$. We also require the recurrence relations [270, 1]

$$X P_J^{[j]}(X) = \frac{\{J-|j|+1\} P_{J+1}^{[j]}(X) + \{J+|j|\} P_{J-1}^{[j]}(X)}{2J+1} , \quad (1054)$$

$$\sqrt{1-X^2} P_J^{j-1} = \frac{P_{J-1}^j - P_{J+1}^j}{2J+1} . \quad (1055)$$

19.3 Ultraspherical harmonics and the (associated) Gegenbauer (ultraspherical) equation

The $\{k > 3\}$ - d ultraspherical harmonics equation arising as angular part of higher- d problems (see Sec 8.2) is straightforwardly separable by the ansatz-cum-change-of-variables into simple harmonic motion and a sequence of Gegenbauer problems. The Gegenbauer equation

$$\{1 - X^2\} Y_{,XX} - \{2\lambda + 1\} X Y_{,X} + J\{J+2\lambda\} Y = 0 \quad (1056)$$

is solved boundedly by the Gegenbauer Polynomials $C_J(X; \lambda)$. Normalization for these is provided in e.g. [1, 270]; the weight function is $\{1 - X^2\}^{\lambda-1/2}$ between equal- λ Gegenbauer polynomials. These furthermore obey the recurrence relations [1, 270]

$$X C_J(X; \lambda) = \frac{\{J+1\} C_{J+1}(X; \lambda) + \{2\lambda + J - 1\} C_{J-1}(X; \lambda)}{2\{J + \lambda\}} , \quad (1057)$$

$$C_{J+1}(X; \lambda) = \frac{\lambda \{C_{J+1}(X; \lambda+1) - C_{J-1}(X; \lambda+1)\}}{J + \lambda + 1} . \quad (1058)$$

The associated Gegenbauer equation

$$\{1 - X^2\} Y_{,XX} - \{2\lambda + 1\} X Y_{,X} + J\{J+2\lambda\} Y - j\{j+2\lambda-1\} \{1 - X^2\}^{-2} Y = 0 \quad (1059)$$

is solved boundedly by the associated Gegenbauer functions $C_J^j(X; \lambda)$. These are re-expressible in terms of Gegenbauer polynomials via [551]

$$C_J^j(X; \lambda) \propto \{1 - X^2\}^{j/2} C_{J-j}(X; \lambda/2 - 1 + j) . \quad (1060)$$

With this conversion, the recurrence relations between Gegenbauer polynomials (1057, 1058) turn out to suffice for this article. For $\lambda = 1/2$, (1056) is the Legendre equation solved by $P_J(X) = C_J(X; 1/2)$, (1057) becomes (755), and (1059) becomes the associated Legendre equation (1051) solved by $P_J^{-j} \propto C_J^j(X; 1/2)$, by (1060) and [1]

$$C_J(X; \lambda) \propto \{X^2 - 1\}^{1/4 - \lambda/2} P_{J+\lambda-1/2}^{1/2-\lambda}(X) . \quad (1061)$$

19.4 (Associated) Laguerre equation

The associated Laguerre equation,

$$xy_{,xx} + \{\alpha + 1 - x\}y_{,x} + ny = 0 , \quad (1062)$$

is solved by the associated Laguerre polynomials $L_n^\alpha(x)$ (and unbounded second solutions). They obey [1] the orthogonality relation

$$\int_0^\infty x^\alpha \exp(-x) L_\beta^\alpha(x) L_{\beta'}^\alpha(x) dx = 0 \text{ unless } \beta = \beta' \quad (1063)$$

and the recurrence relation

$$xL_\beta^\alpha(x) = \{2\beta + \alpha + 1\}L_\beta^\alpha(x) - \{\beta + 1\}L_{\beta+1}^\alpha(x) - \{\beta + \alpha\}L_{\beta-1}^\alpha(x) . \quad (1064)$$

The 2- d quantum isotropic harmonic oscillator's radial equation for a particle of mass μ and classical oscillator frequency ω ,

$$-\{\hbar^2/2\mu\}\{R_{,rr} + R_{,r}/r + m^2 R/r^2\} + \mu\omega^2 r^2 R/2 = ER , \quad (1065)$$

maps to the associated Laguerre equation under the asymptotically-motivated transformations

$$R = \{\hbar x/\mu\omega\}^{|m|/2} e^{-x/2} y(x) , \quad x = \mu\omega r^2/\hbar \quad (1066)$$

and so is solved by

$$R \propto r^{|m|} e^{\mu\omega r^2/2\hbar} L_r^{|m|}(\mu\omega r^2/\hbar) \quad (1067)$$

corresponding to the discrete energies $E = \{|m| + 2r + 1\}\hbar\omega$ for radial quantum number $r \in \mathbb{N}_0$ [569, 531].

This equation is then mapped to in all of the (p -dimensional) isotropic HO and hydrogenic problems (in both spherical and in parabolic coordinates).

19.5 The Tchebychev equation

The Tchebychev equation

$$\{1 - x^2\}y_{,xx} - xy_{,x} + n^2 y = 0 \quad (1068)$$

is solved by the Tchebychev polynomials $T_n(\xi) = \cos(n \arccos(\xi))$.

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